Dottorato di Ricerca in Matematica e Fisica
XXIV Ciclo

Università di Udine

3 novembre 2008

Tema A

Dissertations

Write one dissertation at your choice among the following.

1. The concept of energy and its conservation.
2. Action at a distance versus contact action in electromagnetism.
3. Describe a particle detector and discuss the physical principles of its operation.
4. Free energy and the nature of phase transitions.
5. Transport properties in semiconductors and application to circuit elements.
6. Conservation theorems in Lagrangian mechanics.
7. Fluids and solids: analogies and differences.
8. Variational formulation of the eigenvalues problem for self-adjoint compact operators in Hilbert spaces.
9. The theorem of minimal distance projection on a closed convex set in a Hilbert space, and some of its applications.
10. Uniform continuity and some of its applications in Mathematical Analysis.
11. The axiom of choice.

Problems

Solve as many problems as you can.

1. Consider the one-dimensional wave function for the ground state of the 1-dimensional infinite square well between 0 and a ($a > 0$). Calculate the average values for x, p and p^2 for a particle of mass m.

2. A dish antenna having a diameter of 20 m receives (at normal incidence) a radio signal from a distant source. The radio signal is a continuous sinusoidal wave with amplitude $E_{\max} = 0.200$ μV/m. Assume the antenna absorbs all the radiation that falls on the dish. (a) What is the maximum amplitude of the magnetic field in this wave? (b) What is the intensity of the radiation received by this antenna? (c) What is the power received by the antenna? ($\mu_0/4\pi = 10^{-7}$ S.I. units).

3. An object having mass 900 kg and traveling at speed $0.850c$ collides with a stationary object having mass 1400 kg. The two objects stick together. Find (a) the speed and (b) the mass of the composite object (there is no dissipation in the collision).

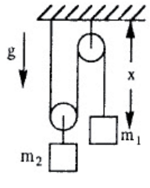
4. A point-like mass moves on the inner surfaces of a sphere, without friction. At the maximum height of the trajectory the point is at the same height of the centre of the sphere, moving with a velocity $V_0 = 1.00$ m/s, while at the lower one the point has a velocity $V_1 = 2.00$ m/s. Determine the radius of the sphere.

5. Two soap bubbles with identical equilibrium radius R_1 and surface tension σ come into contact and merge into a single bubble with equilibrium radius R_2 without any leakage of gas between the inside of the bubbles and the outside. Assuming that the temperature T_o and the external pressure P_o remain constant and that the gas within the bubbles can be regarded as ideal, find an equation to obtain R_2 as a function of R_1, σ and P_o.

6. Find the expression giving the equilibrium lattice spacing of a unidimensional ionic crystal constituted by atoms with alternate charge $\pm q$, and an interaction potential energy $V(r) = A/r^8 \pm q^2/r$, where the first term acts only between first neighbour atoms, and the sign of the second term is related with the attractive or repulsive character of the coulombian interaction between pairs of charges.

7. (a) Show that the de Broglie wavelength for a particle of mass m moving with the most probable velocity of a maxwellian distribution (like for an ideal gas) at temperature T is $\lambda = h/\sqrt{2mk_BT}$ (where $h = 6.63 \cdot 10^{-34}$ J$\cdot$ s and $k_B = 1.38 \cdot 10^{-23}$ J/K are the Planck and Boltzmann constants, respectively). (b) Calculate the de Broglie wavelength of a neutron ($m_n = 1.6747 \cdot 10^{-27}$ kg) moving with the most probable speed of a maxwellian distribution at $T = 20°$ C and its ratio with a typical interatomic distance in a solid.

8. A mass m is constrained to move without friction along a semicircular profile of radius R, whose end points belong to a vertical straight line, and which is rotating around that line with an angular speed ω. What is the greatest lower bound of the values of R that guarantee the stability of the equilibrium position (the one which is not on the rotation axis)?

9. In the system 

use d'Alembert's principle to find the acceleration of m_1.

10. Find the surfaces of revolution (around the x axis) with minimal area that pass through the points $(0, 1)$ and $(1, y_1)$ of the $z = 0$ plane. Show that if y_1 is small enough such surfaces are not regular.

11. Prove that in a finite field every element is the sum of two squares.

12. Write an explicit bijective map $f\colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$. Can f be chosen continuos?

13. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, periodic of period $T > 0$, and let $L > 0$. Prove, or disprove by a counterexample, the following claim: there exist $x_1, x_2 \in \mathbb{R}$ such that $x_1 - x_2 = L$ and $f(x_1) = f(x_2)$.

14. Exhibit an example of a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 0$ and for which $(0,0)$ is not a local minimum point, even though it is a (strict) local minimum for the restriction of f to any straight line through the origin.

15. Let $X = \mathcal{C}([0,1],\mathbb{R})$ with the norm $\|\cdot\|_\infty$. Consider the subset $E = \{f \in X : f(0) = 0,\ f(1) = 1,\ f \text{ nondecreasing}\}$. Decide whether E is open, closed, bounded, sequentially compact in X.

16. Let $K \subseteq \mathbb{R}^n$ be a closed convex set containing the origin, and let K^* be its *polar set*, that is, $K^* = \{w \in \mathbb{R}^n : w \cdot x \leq 1 \text{ per ogni } x \in K\}$. Prove that K^* is bounded if and only if $\dim K = n$ and the origin is in the interior of K.

17. Let $X_1, X_2, \ldots, X_n$ be independent real random variables with uniform distribution on the interval $[0,1]$. Let $Y := \min\{X_1, \ldots, X_n\}$. Find the probability distribution and the mean of Y.

Dottorato di Ricerca in Matematica e Fisica
XXIV Ciclo

Università di Udine

3 novembre 2008

Tema B

Dissertations

Write one dissertation at your choice among the following.

1. Action at a distance versus contact action in electromagnetism.
2. Macroscopic and microscopic interpretation of the concept of entropy in Physics.
3. Describe a particle detector and discuss the physical principles of its operation.
4. Example(s) of application of Monte Carlo methods in physics.
5. Defects in crystalline solids.
6. Conservation theorems in Lagrangian mechanics.
7. Equilibrium problems with constraints. The case of unilateral constraints.
8. Modes and eigenfrequencies in structural models in linear elasticity.
9. The theorem of minimal distance projection on a closed convex set in a Hilbert space, and some of its applications.
10. Some theorems and applications about the concept of Baire category.
11. Rectifiable curves and curvilinear integrals.
12. Permutation groups.

Problems

Solve as many problems as you can.

1. Consider the one-dimensional wave function for the ground state of the 1-dimensional infinite square well between 0 and a ($a > 0$). Calculate the average values for x, p and p^2 for a particle of mass m.

2. A dish antenna having a diameter of 20 m receives (at normal incidence) a radio signal from a distant source. The radio signal is a continuous sinusoidal wave with amplitude $E_{\max} = 0.200$ μV/m. Assume the antenna absorbs all the radiation that falls on the dish. (a) What is the maximum amplitude of the magnetic field in this wave? (b) What is the intensity of the radiation received by this antenna? (c) What is the power received by the antenna? ($\mu_0/4\pi = 10^{-7}$ S.I. units).

3. An object having mass 900 kg and traveling at speed $0.850c$ collides with a stationary object having mass 1400 kg. The two objects stick together. Find (a) the speed and (b) the mass of the composite object (there is no dissipation in the collision).

4. A point-like mass moves on the inner surfaces of a sphere, without friction. At the maximum height of the trajectory the point is at the same height of the centre of the sphere, moving with a velocity $V_0 = 1.00$ m/s, while at the lower one the point has a velocity $V_1 = 2.00$ m/s. Determine the radius of the sphere.

5. Two soap bubbles with identical equilibrium radius R_1 and surface tension σ come into contact and merge into a single bubble with equilibrium radius R_2 without any leakage of gas between the inside of the bubbles and the outside. Assuming that the temperature T_o and the external pressure P_o remain constant and that the gas within the bubbles can be regarded as ideal, find an equation to obtain R_2 as a function of R_1, σ and P_o.

6. Find the expression giving the equilibrium lattice spacing of a unidimensional ionic crystal constituted by atoms with alternate charge $\pm q$, and an interaction potential energy $V(r) = A/r^8 \pm q^2/r$, where the first term acts only between first neighbour atoms, and the sign of the second term is related with the attractive or repulsive character of the coulombian interaction between pairs of charges.

7. (a) Show that the de Broglie wavelength for a particle of mass m moving with the most probable velocity of a maxwellian distribution (like for an ideal gas) at temperature T is $\lambda = h/\sqrt{2mk_BT}$ (where $h = 6.63 \cdot 10^{-34}$ J$\cdot$ s and $k_B = 1.38 \cdot 10^{-23}$ J/K are the Planck and Boltzmann constants, respectively). (b) Calculate the de Broglie wavelength of a neutron ($m_n = 1.6747 \cdot 10^{-27}$ kg) moving with the most probable speed of a maxwellian distribution at $T = 20°$ C and its ratio with a typical interatomic distance in a solid.

8. Two fixed particles are at a distance d from each other. In between them there is a third particle, that interacts with the others with an attractive/repulsive force dependent on the mutual distance r with the law $F = \frac{r_0}{r} - \frac{r_0^2}{r^2}$. Find the equilibrium configurations and discuss the stability as a function of d.

9. A small meteorite is approaching the earth with eccentricity b and speed v_∞ at infinity. Show that the meteorite will hit the earth if $b < a\sqrt{1 + (v_0/v_\infty)^2}$, where a is the radius of the earth, and v_0 is the escape speed.

10. Suppose we dig a rectilinear tunnel joining two points on the surface of the earth, not necessarily antipodal to each other, and a weight of mass m slides without friction along the tunnel (ignore the rotation of the earth). Let the weight loose with speed zero at one end of the tunnel. Prove that the weight will show up at the opposite end after a time of about 42.2 minutes, independently of the length of the tunnel. Assume that the gravitational potential inside the earth at a distance r from the center is given by $\frac{1}{2}(g/R)r^2$, where g is the gravity acceleration at the surface, and $R = 6371$ Km is the radius of the earth.

11. Let A be a principal ideals domain that is not a field. Prove that A has infinitely many prime elements.

12. Let $B^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ be the unit disc and S^1 its boundary. Prove that there is no continuous map $f \colon B^2 \to S^1$ such that $f(x) = x$ for all $x \in S^1$.

13. Find the set of the values of the real parameter k for which the Cauchy problem $x' = (1 + x^2)t$, $x(0) = k$ has a solution defined on the interval $[0, 1]$.

14. Takethe linear differential form $\omega(x, y) := ((2x - y)dx + (x + 2y)dy)/(x^2 + y^2)$. Calculate the integral of ω along the straight line segment joining the point $(1, 0)$ to $(2, 0)$. Consider next the curve $\gamma(t) := (t \cos 2\pi t, t \sin 2\pi t)$, per $1 \leq t \leq 2$. Calculate $\int_\gamma \omega$.
(Hint: because of some properties of ω, the integral can be calculated in different ways).

15. Let $X = \mathcal{C}([0, 1], \mathbb{R})$ with the norm $\| \cdot \|_\infty$. Consider the subset $E = \{f \in X : f(0) = 0,\ f(1) = 1,\ f \text{ nondecreasing}\}$. Decide whether E is open, closed, bounded, sequentially compact in X.

16. Let $K \subseteq \mathbb{R}^n$ be a closed convex set containing the origin, and let K^* be its *polar set*, that is, $K^* = \{w \in \mathbb{R}^n : w \cdot x \leq 1 \text{ per ogni } x \in K\}$. Prove that K^* is bounded if and only if $\dim K = n$ and the origin is in the interior of K.

17. Let $X_1, X_2, \ldots, X_n$ be independent real random variables with uniform distribution on the interval $[0, 1]$. Let $Y := \min\{X_1, \ldots, X_n\}$. Find the probability distribution and the mean of Y.

Dottorato di Ricerca in Matematica e Fisica
XXIV Ciclo

Università di Udine

3 novembre 2008

Tema C

Dissertations

Write one dissertation at your choice among the following.

1. The concept of energy and its conservation.
2. Microscopic and macroscopic interpretation of the concept of entropy in Physics.
3. Describe an instrument for particle detection and discuss the physical principles of its operation.
4. Simulation techniques for particle systems far from equilibrium.
5. Defects in crystalline solids.
6. Conservation theorems in Lagrangian mechanics.
7. The wave equation and D'Alembert's solution.
8. Holonomic and anholonomic constraints.
9. Some theorems and applications about the concept of Baire category.
10. Convergence of power series.
11. Rings of formal power series.

Problems

Solve as many problems as you can.

1. Consider the one-dimensional wave function for the ground state of the 1-dimensional infinite square well between 0 and a ($a > 0$). Calculate the average values for x, p and p^2 for a particle of mass m.
2. A dish antenna having a diameter of 20 m receives (at normal incidence) a radio signal from a distant source. The radio signal is a continuous sinusoidal wave with amplitude $E_{\max} = 0.200$ μV/m. Assume the antenna absorbs all the radiation that falls on the dish. (a) What is the maximum amplitude of the magnetic field in this wave? (b) What is the intensity of the radiation received by this antenna? (c) What is the power received by the antenna? ($\mu_0/4\pi = 10^{-7}$ S.I. units).

3. An object having mass 900 kg and traveling at speed $0.850c$ collides with a stationary object having mass 1400 kg. The two objects stick together. Find (a) the speed and (b) the mass of the composite object (there is no dissipation in the collision).

4. A point-like mass moves on the inner surfaces of a sphere, without friction. At the maximum height of the trajectory the point is at the same height of the centre of the sphere, moving with a velocity $V_0 = 1.00$ m/s, while at the lower one the point has a velocity $V_1 = 2.00$ m/s. Determine the radius of the sphere.

5. Two soap bubbles with identical equilibrium radius R_1 and surface tension σ come into contact and merge into a single bubble with equilibrium radius R_2 without any leakage of gas between the inside of the bubbles and the outside. Assuming that the temperature T_o and the external pressure P_o remain constant and that the gas within the bubbles can be regarded as ideal, find an equation to obtain R_2 as a function of R_1, σ and P_o.

6. Find the expression giving the equilibrium lattice spacing of a unidimensional ionic crystal constituted by atoms with alternate charge $\pm q$, and an interaction potential energy $V(r) = A/r^8 \pm q^2/r$, where the first term acts only between first neighbour atoms, and the sign of the second term is related with the attractive or repulsive character of the coulombian interaction between pairs of charges.

7. (a) Show that the de Broglie wavelength for a particle of mass m moving with the most probable velocity of a maxwellian distribution (like for an ideal gas) at temperature T is $\lambda = h/\sqrt{2mk_BT}$ (where $h = 6.63 \cdot 10^{-34}$ J$\cdot$ s and $k_B = 1.38 \cdot 10^{-23}$ J/K are the Planck and Boltzmann constants, respectively). (b) Calculate the de Broglie wavelength of a neutron ($m_n = 1.6747 \cdot 10^{-27}$ kg) moving with the most probable speed of a maxwellian distribution at $T = 20°$ C and its ratio with a typical interatomic distance in a solid.

8. A point of mass m, subject to its weight, moves without friction on a sphere of radius R. Discuss the motion as a function of the initial velocity, given that it starts on a point on the equator of the sphere.

9. A point P of mass m is constrained to a horizontal plane π. The point P is subject to an elastic force $\mathbf{f} = -\kappa \overrightarrow{OP}$, with $\kappa > 0$, where the center O is fixed on π. This plane is itself rotating uniformly around a vertical straight line r that meets the plane at the point O'. Let $\vec{\omega}$ be the angular velocity of this movement, and let $\|\overrightarrow{OO'}\| = d$ be the smooth constraint imposed on P. (a) Find the equilibrium positions of P, if any, in the reference frame fixed on the plane π; (b) find the trajectories of the uniform motions, if any, of P in the reference frame fixed on the plane π; (c) assuming that $\kappa = 2m\omega^2$, calculate the values of the velocity $\vec{v}_0$ that must be given to P at an instant t_0, so that, starting at O' at that instant, the point moves henceforth of uniform motions in the reference frame fixed on the plane π.

10. Suppose we dig a rectilinear tunnel joining two points on the surface of the earth, not necessarily antipodal to each other, and a weight of mass m slides without friction along the tunnel (ignore the rotation of the earth). Let the weight loose with speed zero at one end of the tunnel. Prove that the weight will show up at the opposite end after a time of about 42.2 minutes, independently of the length of the tunnel. Assume that the gravitational potential inside the earth at a distance r from the center is given by $\frac{1}{2}(g/R)r^2$, where g is the gravity acceleration at the surface, and $R = 6371$ Km is the radius of the earth.

11. Petersen's graph Γ has ten vertices $\{A, B, C, D, E, a, b, c, d, e\}$ and fifteen edges $\{AB, BC, CD, DE, EA, Aa, Bb, Cc, Dd, Ee, ac, ce, eb, bd, da\}$. Prove that the group of automorphisms of Γ is S_5.

12. Let $X \subset \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$ be the set $X = \{((x, y), (\xi : \eta)) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : x\eta = y\xi\}$. Let $E \subset X$ be the set $E = (0, 0) \times \mathbb{P}^1(\mathbb{R})$. Prove that X is a surface, E is a curve in X. Consider the projection $X \to \mathbb{P}^1(\mathbb{R})$. Prove that (X, π) is a line bundle over $\mathbb{P}^1(\mathbb{R})$. Prove that X is diffeomorphic to the Moebius band, and that $X \setminus E$ is diffeomorphic to a cylinder.

13. Let $X = \mathcal{C}([0, 1], \mathbb{R})$ with the norm $\| \cdot \|_\infty$. Consider the subset $E = \{f \in X : f(0) = 0,\ f(1) = 1,\ f \text{ nondecreasing}\}$. Decide whether E is open, closed, bounded, sequentially compact in X.

14. Prove that the system of equations $2x - \sin(x + y) = 5,\ 3y + \arctan(x - y) = -3$ has a unique solution.

15. Let I be a real interval, and $f \colon I \to \mathbb{R}$ be a differentiable function. For $x, y \in I$ define $F(x, y) := (f(x) - f(y))/(x - y)$ if $x \neq y$, and $F(x, x) := f'(x)$. Prove that f is of class $\mathcal{C}^1$ on I if and only if F is continuous on $I \times I$.

16. Let $K \subseteq \mathbb{R}^n$ be a closed convex set containing the origin, and let K^* be its *polar set*, that is, $K^* = \{w \in \mathbb{R}^n : w \cdot x \leq 1 \text{ per ogni } x \in K\}$. Prove that K^* is bounded if and only if $\dim K = n$ and the origin is in the interior of K.

17. Let $X_1, X_2, \ldots, X_n$ be independent real random variables with uniform distribution on the interval $[0, 1]$. Let $Y := \min\{X_1, \ldots, X_n\}$. Find the probability distribution and the mean of Y.