UNIVERSITY OF UDINE - ITALY

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Ph.D. Thesis

REAL AND CONVENTIONAL ANISOTROPY, GENERALIZED LORENTZ TRANSFORMATIONS AND PHYSICAL EFFECTS

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Contents

Chapter 1

Introduction

1.1 Motivations

At present, there exist, apart from general relativity theory, a number of alternative metric theories of gravitation [1]. They all employ the Riemannian geometric model of spacetime borrowed from general relativity, and differ only by the field equations which describe the self consistent dynamics of spacetime and matter. Common feature to them is the fact that spacetime is Riemannian.

Rimannian space is locally isotropic in the sense that, at each point, its tangent space is a Minkowski space, that is a 4-dimensional pseudo-Euclidean manifold[1]. These theories assume spacetime to be locally isotropic everywhere and at any time.

Indeed, the following recent experimental discoveries:

- the absence of the the so called GZK cutoff in the spectrum of primary ultra-high energy cosmic protons [2, 3, 4, 5];
- an anisotropy of the relic background radiation filling the universe;

can find, to my opinion, a common explanation on the assumption that spacetime has a weak local anisotropy. Let us discuss these two experimental issues in some detail.

1.1.1 The GZK Cutoff

During the 1960's the primary cosmic-ray spectrum has been measured up to an energy of 10^{20}eV [6] and it was predicted [2, 3] that above this energy the primary spectrum will steepen abruptly, so the new generation of experiments

[1]This differ from the notion of isotropy based on Killing vectors field-isometry.

will at least observe it to have a cosmologically meaningful termination called GZK cutoff.

The cause of the catastrophic cutoff is due to the interactions of protons of extragalactic origin (as it is suggested by the apparently uniform distribution of their directions) with the photons of the cosmic microwave background radiation detected by Penzias and Wilson [7].

From the theoretical point of view, such result relies on the assumptions that the usual Lorentz transformations apply between inertial frames for all admissible velocities. In the case of uniformly distributed sources, the energy spectrum of primary cosmic protons should show a cutoff at an energy threshold of about 5×10^{19}eV, due to the inelastic collision of the protons with cosmic background radiations photons: protons colliding with these photons reach the threshold for pion photoproduction and cannot cover a long intergalactic distance without loosing most of their energy. Since isotropy of space is an essential assumption behind Lorentz transformations [8, 9], it is essential also for the existence of GZK cutoff.

Actually, it should be pointed out that one of the experiments measuring the ultra high energy cosmic ray spectrum, the AGASA experiment, has not seen the cutoff [4], while another experiment, HiRes, is consistent with the cutoff but a lower confidence level [5]. It is, of course, premature to speak in this situation of a real discrepancy between theory and experiment. The question should be answered in the near future by the AUGER observatory [10], GLAST space telescope [11] and MAGIC telescope [12].

Anyway several authors discuss the possibility of explaining the (few) observation of ultra-high-energy cosmic rays with energy above the GZK cutoff, keeping the relativity principle and introducing a "deformation" in the Lorentz transformation [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

We remark that Lorentz invariance imply that spacetime structure is the same at all scales; that is, Lorentz symmetry assumes a scale-free nature of spacetime. A departure from Lorentz invariance can therefore lead to discontinuous spacetime framework. Some authors study the implications of Einstein's principle of relativity when both a fundamental velocity scale and a fundamental length scale are postulated. Hence, in these framework there is a spacetime with a short-distance structure governed by an observer-independent length scale[2], these models are called double special relativity theories (DSR) [18, 19, 20, 21, 22, 23, 24, 25]. Along this line of thought the consistency of these postulates proves incorrect the expectation that modifications of the rules of kinematics involving the Planck length would *necessarily* require the

[2]It should be noted that the existence of a minimum length does not imply local Lorentz invariance violation [26]

introduction of preferred class of inertial observers [27, 28]. Indeed, DSR has not been shown to be consistent yet [29, 30].

Apart from the violation of Lorentz transformations, there exist also others possible causes of the absence of the GZK cutoff [25]. For example, cosmic rays might not come from the far intergalactic space or it may happen that collisions of the cosmic rays with the nuclei in the high terrestrial atmosphere, which have a center of mass energy much larger than the collisions studied in the terrestrial laboratories, have unexpected features, possibly a breakdown of the conservation laws. Obviously it is also possible that the relativity principle is not valid and there are privileged inertial frames. The fundamental laws may still be Lorentz covariant, but some long-range vector or tensor field may have a non vanishing expectation value that singles out the privileged frames.

Nevertheless, the assumption that the inertial frames could be linked by some new "generalized Lorentz transformations" markedly different from the usual Lorentz ones only at relative velocities extremely close to the velocity of light is in our opinion, the most fruitful. Thus, in what follow we will require that all the relativistic equations must be invariant with respect to the new transformations; that is, we will implement Einstein's principle of relativity with the generalized Lorentz transformations.

1.1.2 Anisotropy of The Relic Background Radiation and Preferred Reference Frame

According to the model of the hot Universe the temperature of relic radiation should not depend on the direction in which it is being measured. At the same time the temperature anisotropy, with dipole component, of relic radiation is already an experimental fact [31].

On the other hand, experiments looking for a preferred reference frame search for an anisotropy of physical processes; so the observed anisotropy of the cosmic background radiation have led to a renaissance of interest for this type of theories [32, 33].

In such models the old idea of an absolute "aether" is exploited with the only difference that the preferred frame is now identified with one in which the cosmic background radiation is locally isotropic. In fact, there is only one frame with this property, being all other frames experiencing the dipole anisotropy, and therefore distinguishable.

With respect to this reference frame the standard physical laws hold. In this way, attempts are sometimes undertaken to explain the experimental results by an *ad hoc* breaking of Lorentz invariance when going from the preferred frame to another (laboratory) inertial frame.

Investigators usually do not express a fundamental interest in such a dipole anisotropy because they believe that it arises from the fact that our laboratory frame accidentally moves at a certain velocity relative to the cosmic microwave background. This velocity w has been determined to be $w/c \approx 1.23 \cdot 10^{-3}$ by the measurement of the dipole term in the cosmic microvawe radiation by COBE [34].

Such an explanation would be more satisfactory if the corresponding anisotropy were also observed in the Hubble constant. Until now, studies of the angular dependence of the Hubble constant are neither precise enough nor covering a larger section of the sky. If a special analysis will show that there is no correlated dipole anisotropy in the Hubble constant then the dipole anisotropy of relic radiation might be an indication of a strong local anisotropy of space-time at an early stage of the evolution of the Universe. The point is that in a space with strong anisotropy there indeed exists a physically preferred frame.

With respect to this frame the hot background radiation was isotropic while the velocity distribution of massive relativistic particles was anisotropic. As a result, the Hubble constant became anisotropic. Therefore, by passage to another frame, a reversed situation becomes possible: the Hubble constant looses its dipole anisotropy while the background radiation picks it up.

The concept of a privileged reference frame grows up not only in the context of anisotropic propagation of light [33] or in the context of test theory of special relativity [32]. More recently, a new approach to the testing of quantum gravity effects was introduced. Based on the fact that the dispersion relations of photons in the vacuum does not have the usual covariant form, but the photon propagate with an energy dependent velocity [35]. This implies a breakdown of Lorentz invariance (since the statement $E_\gamma = c\,||\vec{k}||$ can be valid at most in a single reference frame), in this way a privileged reference frame is introduced in the theory [36, 37]. Indeed in [18] a dispersion relation which implies a wavelength dependence of the speed of light is described, but in this model such feature is not a manifestation of the existence of a preferred class of inertial observer.

The existence of a privileged reference frame can be a possible common explanation of both the experimental facts mentioned in the first section of this chapter.

The conclusion of these remarks seems to be that special relativity is not an ultimate theory and that some modification of it is needed. Einstein [38] stated as basis for his theory the following two postulates:

- The principle of relativity; that is, the existence and the equivalence of all inertial frames.
- The principle of constancy of light velocity: light always propagates

isotropically in empty space with a definite (one-way) velocity, independent of the state of motion of the emitting body.

Besides the two postulates, special relativity also uses other assumptions, concerning the *Euclidean* structure of gravity-free space and the homogeneity of gravity-free time.

By my opinion a generalization of special relativity to a relativistic theory of locally anisotropic spacetime, could be interesting to study. Indeed in this Ph.D. thesis we will relax only the spatial isotropy hypothesis. As demonstrated in [39] in such a situation the correct mathematical theory to use in building up a relativistic theory is Finslerian geometry [40], rather than Riemannian geometry.

1.2 Spatial Isotropy and Related Topics

In special relativity theory the spatial isotropy become a crucial issue in the synchronization procedure of two distant clocks. Einstein stressed [41] that the choice that light travels at equal speeds along the opposite directions of a particular path was "neither a *supposition nor a hypothesis* about the physical nature of light, but a *stipulation*" that can be freely made so as to arrive at a definition of simultaneity.

Let us briefly review this topic which leads us to the long standing debate between conventionalist and anticonventionalist thesis about the physical meaning of one-way speed, and to the test-theories of special relativity.

It is now useful to remember that Michelson-Morley like experiments [42, 43] ensure the constancy and isotropy only of the round-trip velocity of light, and only for light with wavelength much larger than Planck length. So is not avoid a possible dependence of velocity of light in vacuum from its wavelength [18], dependence experimental accessible only at large energy.

1.3 Conventionality of Synchronization

A critical point in Einstein's theory is the synchronization of distant clocks in an inertial reference system. Indeed, the definition of simultaneity, being intertwined with the concept of speed, became an issue some years before Einstein's work with the attempts to measure the one-way speed of light as a means to verify the existence of the aether [44].

The reanalysis of the concept of simultaneity formed one of the crucial and distinguishing elements of Einstein's special theory of relativity. Einstein deserves our admiration for recognizing that simultaneity is relative, an insight that lies at the foundations of his theory.

The absence of an *absolute* notion of simultaneity, has as consequence that the synchronization of distant clocks, the definition of 3-space, the spatial distance between events (at space-like separation) and the one-way velocity are all frame-dependent concepts.

The operational procedure adopted by Einstein was described some years before by H. Poincaré in several papers [44, 45, 46], and it is known that Einstein discussed these papers with his friends. Furthermore, Poincaré's procedure for synchronization was by no means original. The need for a precise procedure for the synchronization of clocks at different locations was recognized in antiquity in the context of the problem of the determination of geographical longitude (even today the precise determination of geographical position, by GPS, remains an important practical application of synchronization [47]).

Einstein proposed in the kinematic section of his paper [38] what is known as the "Einstein Synchronization", by which a global time can be defined relative to any inertial frame. This procedure is equivalent to assume that the clocks can be adjusted in such a way that the propagation (one-way) velocity of every light ray in vacuum, measured by means of the clocks, becomes everywhere equal to a universal constant. Hence, this operational method associates the synchronization within the frame with the velocity of light in the frame.

Using the "Einstein Synchronization method" the conventional nature of one-way velocity is manifest: one is prevented from measuring the one-way speed of light in a given direction because that would require the prior synchronization of clocks, and thus a prior knowledge of the speed to be measured. A persistent controversy surrounds the question of the synchronization of clocks at different locations, and the related question of whether an unambiguous and logically noncircular meaning can be attached to measurement of the one-way speed of light between two points (see for example [44] and references therein).

The controversy over the logical and physical significance of Einstein's synchronization began in the 1920's with H. Reichenbach [48] who introduced the debate between conventionalist and anticonventionalist thesis. The possibility in principle of postulating an anisotropic synchronization has been discussed extensively both in philosophical and physical contexts.

1.4 The Conventionalist Thesis

H. Reichenbach's contribution has been most significant in the history of the debate. He brought out and defended the definitionary nature of Einstein's treatment of simultaneity, and established a notation for expressing an anisotropy in the one-way speed of light.

The conventionalist thesis proposed by Reichenbach, states that quantities as the one-way speed of light are inherently conventional, and that do recognize this aspect is to recognize a profound feature of nature. Only proper time has "objective status in special relativity"[49]. This is because, one-way velocity's value for example, it is not a statement about the pattern of coincidences of events at a given space locations, but refers to the comparison of remote events, and so is inevitably conventional.

Following Reichenbach's viewpoint, it is possible to relax the second Einstein's postulate, abandoning the constancy of the one-way speed of light for the more realistic hyphotesis asserting the constancy of the round-trip speed of light. This results is an extension of the special theory of relativity, in which the two one-way speeds of light in the two senses of a round trip in empty space, respectively c_+ and c_-, are arbitrarily selected in such a way that their harmonic mean is the experimentally measurable round trip speed of light c.

Obviously, if one-way velocity is really a conventional quantity, all speed-dependent expressions, including the parameters of the Lorentz transformation and so time dilation factors and length contraction effects of a moving body as seen from an inertial frame, have irreducibly conventional elements and they are not directly observable formulas.

1.5 Physics

In order to determine the simultaneity of two events which take place at distant points A and B in an inertial frame, it is necessary to have a clock located at A and another identical clock at B, both at rest in the inertial frame. In addition, there must be a procedure to synchronize the two clocks so that the time coordinate of events occurring at A can be compared to the time coordinate of events occurring at B.

Let us consider a light signal, emitted from A at time t_1 and reflected at B, returning at time t_3 to clock A. Reichenbach said that clock B is synchronous with clock A if the arrival time t_2 of the light signal at B as registered by the clock is:

$$t_2 = t_1 + \epsilon_r (t_3 - t_1) \tag{1.5.1}$$

where $\epsilon_r \in (0,1)$, is called Reichenbach's synchrony parameter.

This is the Reichenbach's (non-standard) definition of simultaneity of distant events. The Reichenbach thesis of conventionality of simultaneity states that any choice for ϵ_r between 0 and 1 is equally valid. More precisely, Reichenbach claims that in the absence of any physical reason for choosing any particular value of ϵ_r, this choice is purely a matter of convention. Thus, synchronization is a matter of convention, and so is the value of any one-way speed and all related physical quantities.

Having a procedure to synchronize distant clocks, we can measure one-way velocities, and we can now see how the "measured" one-way velocities involve an element of convention (the synchrony parameter ϵ_r). If the one-way speed of light from A to B is c_+, by equation (1.5.1) we have

$$c_+ = \frac{d}{t_2 - t_1} = \frac{c}{1 - \varepsilon} \quad , \tag{1.5.2}$$

where d is the spatial distance from A to B, c is the isotropic two-way speed of light, for latter convenience we have defined $\varepsilon = 1 - 2\epsilon_r$, so $|\varepsilon| < 1$. Similarly, the one-way speed of light c_- from B to A is:

$$c_- = -\frac{d}{t_3 - t_2} = \frac{-c}{(1 + \varepsilon)} \quad . \tag{1.5.3}$$

According to the Michelson-Morely experiment we have:

$$\frac{1}{2}\left(\frac{1}{c_+} + \frac{1}{-c_-}\right) = \frac{1}{c} \quad . \tag{1.5.4}$$

We stress that the one-way speeds c_+ and c_- satisfying the conditions in (1.5.4), are determined by *convention* due to the arbitrarily fixed value of ϵ_r.

Some observations are useful. First of all the restriction on ϵ_r ensures the validity of the causality condition according to which the light signal arrives at B after it was emitted from A and before it arrives again at A. Secondly Reichenbach's definition of simultaneity reduces to the standard one of Einstein when $\epsilon_r = \frac{1}{2}$ or equivalently $\varepsilon = 0$. But as we already said every choices $\epsilon_r \neq \frac{1}{2}$ give an equally viable non-standard synchronization.

We also note that non-standard synchronization is anisotropic, since it identifies a preferred direction in which the one-way speed of light has a maximum value; however, since this anisotropy is merely the result of the choice of time coordinate, it is no more objectionable that the anisotropy that results when we adopt, say, cylindrical coordinates in space. Although such cylindrical coordinates identify a preferred direction in space (the direction of the cylindrical axis), this anisotropy is merely an artifact, and it leads to no demonstrably erroneous experimental consequences. Cylindrical coordinates permit a consistent description of physical phenomena (although in some respects more complicated than the description in rectangular coordinates). Likewise, Reichenbach's non-standard synchronization permits a consistent description of physical phenomena (although more complicated than the description in standard synchronization).

As we will see in the following chapters, in a more general framework this is what we will call *conventional* anisotropy to distinguish it from a *physical* anisotropy which we can not gauged away by a stipulation.

Finally the spatial distance from A to B can be measured either by a rod or by the travel time of a light signal emitted from A, reflected at B and returned to A. This measurement is based on the constancy of the round trip speed of light, and is performed by a single clock so that no clock synchronization is needed. Since the two-way speed of light is direction independent, so is the distance from A to B. Thus, the distance from A to B equals the distance from B to A and space appears isotropic.

1.5.1 Reichenbach's Special Theory of Relativity

J.A. Winnie in his paper [50] formulated relativity in the non-standard Reichenbach synchronization, and he constructed the "one-way velocity Lorentz transformations" between two reference frames in motion with each other, the explicit expression are

$$\begin{cases} X' = \Gamma\,(X - V\,T) \\ \\ T' = \Gamma\,[T\,(1 + P\,V) - Q^2\,V\,X] \end{cases} \tag{1.5.5}$$

where V is the one-way velocity of the primed reference system S' with respect S, and we have defined

$$\begin{cases} P = -\left(\dfrac{1}{c_+} + \dfrac{1}{c_-}\right) \\ \\ Q^2 = -\dfrac{1}{c_-\,c_+} \end{cases} \tag{1.5.6}$$

and

$$\Gamma = \frac{1}{\sqrt{\left(1 - \dfrac{V}{c_+}\right)\left(1 - \dfrac{V}{c_-}\right)}} \quad . \tag{1.5.7}$$

Equation (1.5.5) form in the $(1 + 1)$ dimensional case a one parameter transformation group that keeps invariant the following anisotropic relativistic pseudo-norm

$$S^2 = T^2 - \Big(\frac{1}{c_+} + \frac{1}{c_-}\Big) X\, T + \frac{1}{c_-\, c_+} X^2 \quad . \tag{1.5.8}$$

As a consequence of the group structure, from equation (1.5.5) it is a simple matter to gain the velocity V_- of S with respect S' reference frame

$$V_- = \frac{-\,V}{1 + P\,V} \quad , \tag{1.5.9}$$

this equation states the *one-way velocity reciprocity principle.*

By analogy with equation (1.5.4) we define the round trip velocity, v, associated with the one-way velocity $V = V_+$, by the equation

$$\frac{1}{v} = \frac{1}{2}\Big(\frac{1}{V_+} + \frac{1}{-\,V_-}\Big) \quad . \tag{1.5.10}$$

From equations (1.5.5) and (1.5.9), if we call respectively U' and U the one-way velocities of a body with respect S' and S we can write the one-way velocity composition law as follow

$$U' = \frac{U + V_- + P\,U\,V_-}{1 + Q^2\,U\,V_-} \quad . \tag{1.5.11}$$

One important feature in this theory is that expressions which represent observable effects are *independent* of the anisotropy parameter, ϵ_r, and are identical with their counterparts in special relativity when espressed in terms of associated round-trip velocities. In the Reichenbach theory the presence of the anisotropy parameter, ϵ_r, distinguishes between physically significant effects and physically insignificants ones: only effects independent of ϵ_r are synchrony free; hence the anisotropy parameter, ϵ_r, acts as a marker.

The independence of Reichenbach special relativity expressions representing observables of the anisotropy parameter, ε_r has a nice geometrical interpretation: it is straightforward to demonstrate that the one-way Lorentz transformation (1.5.5), with its associated nonstandard synchronization, is not a new transformation. It is merely the standard Lorentz transformation, with its associated Einstein synchronization, expressed in oblique spacetime coordinates (T, X). The anisotropy parameter ϵ_r, is a parameter that measures the "amount" of nonorthogonality of the oblique coordinates according to equations

$$\begin{cases} T = t - \dfrac{\varepsilon}{c}\, x \\ \\ X = x \end{cases} \tag{1.5.12}$$

substituting this equations and their primed counterparts in equations (1.5.5), we obtain these transformations in the variables t, x and v

$$\begin{cases} t' = \gamma\,(t - \dfrac{v}{c^2}\, x) \\ \\ x' = \gamma\,(x - v\,t) \end{cases} \tag{1.5.13}$$

where

$$\gamma = \Gamma\,\big(1 + \frac{\varepsilon}{c}\, V\big) \quad . \tag{1.5.14}$$

They have a *form* identical with that of the usual Lorentz transformation of special relativity. Indeed, quantities that appear in this equations have different interpretation in special relativity than in Reichenbach relativity: in Einstein theory parameters v and c are one-way velocities which are, by convention isotropic; while in Reichenbach theory they are round-trip velocities of equations (1.5.4) and (1.5.10), which are isotropic as an experimental fact.

Equivalently we can say that in Reichenbach framework Lorentz transformation (1.5.13) are obtained from the one-way Lorentz transformation (1.5.5) by expressing one-way quantities in terms of associated round-trip ones, while Lorentz transformation of Einstein theory are obtained from equations (1.5.13) by adopting Einstein's convention, $\varepsilon = 0$.

1.6 Philosophy

In the context of philosophy, many authors centre the discussion on the grounds upon which the natural choice of isotropy may be regarded as obligatory; that is, synchronization is not simply a physically meaningless *gauge* to be applied to clock settings, but is constrained by nature to be unique for inertial frames.

These authors state that theoretical considerations based on the context and symmetries of the causal structure of Minkowski spacetime are sufficient to force the choice of Einstein synchronization upon any reasonable theoretical formulation [49, 51, 52, 53]. Readhead [54] and Havas [55] criticized this approach and infer that their result does not confer any compelling status to Einstein synchronization, and therefore does not challenge the essential point of the conventionalist thesis.

Those who believe that Einstein's isotropy "convention" is not a matter of convention but rather a matter of experimental fact will have to find an experiment distinguishing between the Reichenbach theory and special relativity that fix the value of ϵ_r.

A.A. Ungar in his article [56] demonstrated that such an experiment does not exist if special relativity is a coordinate independent theory. As a consequence, from the experimental point of view, Reichenbach's theory and special relativity are indistinguishable and, hence, represent the same physical theory. From this point of view the value of ϵ_r is a gauge choice.

The conventionalist thesis in the Reichenbach theory of relativity, implies besides the well known notion of the relativity of simultaneity between *two* different inertial frames, the conventionality of simultaneity in *one* inertial frame of reference[3] [48, 58].

Following the conventionalist thesis, one-way speed of light and clock synchronization presuppose each other, this creates a circularity which does not allow any escape at the *kinematical* level.

H.C. Ohanian [45] claims that an examination of the laws of *dynamics* resolves all ambiguities in synchronization: the nonstandard Reichenbach synchronization introduces into the equation of motion extra terms that can be interpreted as pseudoforces, and these pseudoforces are fingerprints of the non Einstein synchronization. However, this approach was convincingly refused by R.D. Klauber [59].

1.7 The Role of Light in Special Relativity

We have to observe that in what we have explained above, having light a central role in the expositions (for example in establishing a convention for the synchronization, or determining the rate of moving clocks or the length of moving rigid rods), one seems to link special relativity to a restricted class of natural phenomena, namely, electromagnetism. But relativity can be developed without any reference to light or electromagnetic radiation. This theory does not derive from the use of electromagnetic signals for synchronizing clocks. It was proved in [9, 39] that the existence of an invariant speed actually follows

[3]Tangherlini [57], Grünbaum [58] and Mansouri and Sexl exploited the Reichenbach synchronization to modify the Lorentz transformation so as to eliminate the relativity of synchronization, that is, the term $\frac{-Vx}{c^2}$ that appears in the standard Lorentz equation for the time coordinate.

from the relativity principle and from a few and very general hypotheses about spacetime, and does not require an independent postulate. The actual value of such a speed is thus an experimental issue as its identification with the speed of light in vacuum. So we can build up special relativity also if there are no real-world effects that travel at the invariant speed[4].

The lessons to be drawn from almost a century is that special relativity up to now seems to rule all classes of natural phenomena, special relativity at present time stands as a universal theory describing the structure of a common spacetime arena in which all fundamental processes take place.

1.8 Test-Theories of Relativity

One generally has the feeling that Einstein's theory is in agreement with experiment to a high degree of accuracy, but it is difficult to express this "high degree" in specific numbers. This can be made with a "test theory", by which one means a theoretical framework which contains a continuum of theories, in which a particular set of parameter values specify a theory to be tested; all other parameter combinations give rise to alternative rival theories.

Special relativity theory was formulated before the theory of general relativity, and is assumed within the latter to be valid in the limit of negligible gravitational effects. Nevertheless the experimental testing of special relativity with test-theories is not as extensive as in the situation of general relativity, and none of the corresponding test-theories enjoy the same status as the PPN test-theory used in discussing general relativity [1].

Although the assumptions and postulates used in the theoretical derivation of the Lorentz transformation are based on experimental evidence, there has been great interest in performing further experiments devoted to test directly the Lorentz transformation. The two separate considerations of synchronization procedure and the validity of special relativity, which are involved in this area, are often confused. Formulations of special relativity usually begin with the invocation of the Lorentz transformation to relate any two frames in relative motion. It then follows that the Minkowski metric is an invariant in every frame and is the chosen spacetime metric.

The most popular test-theories of special relativity, the one formulated by Mansouri and Sexl [32], concentrates on kinematical considerations and the

[4]Using this approach to special relativity it is possible to demonstrate that faster than light signals are kinematically compatible with special relativity, because the latter requires only the existence of an invariant speed, not necessarily a maximum one. They do not lead to causal paradoxes, which can arise only from "particle" whose speed has no fixed (maximum) value in a given reference frame [60]

structure of spacetime, reflecting the importance of both these properties in the foundational aspects of special relativity.

In Mansouri and Sexl test-theory, a parameterised deviation from special relativity was postulated by relaxing the constraint that it is the standard Lorentz transformation which links any two reference frames. Departing from this standard is to deny the invariance of the Minkowski metric under a boost. This arbitrariness must be constrained in some way by imposing enough structure on the theory to allow useful experimental predictions to be made. Empirical conclusions must be draw only within the context of these initial assumptions and these set constrains the possible parameter's value.

Mansouri and Sexl discussed the debated question of the equivalence of special relativity and aether theories, this aether system is defined by the requirements that the Einstein and the transport synchronization of clocks agree and that light propagation is isotropic with respect the aether. In the framework of Mansouri and Sexl's test theory it is possible to demonstrate that a theory that maintain absolute simultaneity is kinematically equivalent to Einstein theory.

1.9 Goal of the Present Doctoral Work

In this Ph.D. thesis we consider the situation in which matter is so diluted that gravitational effects can be ignored, and deal with the problem of a possible connections in relativistic theories between:

- the role of convention in the definitions of clock synchronization and simultaneity (we had explain above that this item is inestricably linked with the long-standing debate whether the one-way velocity of light is a physically meaningful quantity or merely a conventional one [44, 45, 48, 56, 58, 59, 61]);
- a model of spacetime with a "real" spatial anisotropy and how we can grasp a clear distinction between *conventional* and *real* anisotropy;
- the approach where there is a preferred reference frame [32, 62, 63, 64].

Our task is to demonstrate that Finsler geometry is the correct mathematical language by which we can link all these physical issues in a common framework. If this is the case, from the mathematical point of view the Poincaré group, that is the group of isometry of usual Minkowskian spacetime, is no longer the isometry group: the Poincaré symmetry is only approximate.

As a consequence the space of events has a *geometry* different from that of Minkowski space even at the level of special relativity. This is equivalent to giving up the hypothesis of pseudo-euclidean geometry of gravity-free spacetime.

We also noted that being an intrinsic property of spacetime, anisotropy is independent of the magnitude of relative velocities. Therefore also non relativistic physics as a whole is different from the Newtonian case. Obviously either the anisotropic Newtonian dynamics and the anisotropic special relativistic one become the usual well known dynamics when the anisotropy vanishes [65].

Non-Lorentzian transformations were considered in several works. The Lorentz transformations and their modifications have been used over the last century to work-up the high-energy phenomenology, derive the fundamental physical field equations, and predict new relativistic effects.

Despite the general feeling of a high degree of accuracy between predictions and measurements, various modifications, including Mansouri-Sexl transformations and Tangherlini transformations, have been used. Really, in a sharp contrast to the approach followed in Einstein's work [38], in which the special theory of relativity began with two fundamental invariance principles to derive the required transformations, a lack of profound invariance motivation is a common feature for the approaches based on the non-Lorentzian transformations mentioned above. In fact, this transformations have been introduced primarily to reanalyze the role of synchronization procedure [44, 48, 58]. No metric function invariant under these non-Lorentzian transformations is known.

Since the alternative to a local anisotropy is a strict local isotropy of spacetime, and since in nature the validity of any strict symmetry can be established only to some degree of approximation, it seems reasonable to continue investigations into the physical manifestations of local anisotropy. This is not only equivalent to studying a possible violation of the Lorentz transformation: following this line of research we can generalize the Mansouri and Sexl test theory of special relativity and of Lorentz invariance.

The present work is structured in the following chapters. In chapter 2 we introduce the mathematical tools we need to depict a differentiable manifold which models an anisotropic spacetime: we begin with a brief historical introduction of the genesis of Finsler geometry, then we explain the concept of Finsler structure and the associated Minkowski norm to arrive at the fundamental physical notion of locally Minkowskian spaces. Finally we examine the connection and geodesics of a Finsler manifold and we study the problems, that are usually skipped over the literature, that arise in defining a pseudo-Finsler structure.

In chapter 3, postulating some general properties for spacetime and relaxing only the usual hypothesis of isotropy of space, we infer the so called "generalized Lorentz transformation". As a consequence, we provide a kinematical basis for a new physics which is not Lorentz invariant but for which the difference between special relativity and the anisotropic model become marked only at high energies. We stress that in our approach, following the pioneering work of Ignatowsky, we have no need to postulate the invariance of the velocity of light. The main result of this chapter is inspired to the work by Lalan [39], who demonstrated that the metric invariant under such generalized Lorentz transformations is of pseudo-Finsler type. By this approach we can also make a clear distinction between "conventional" and "real" anisotropy: this is linked, among others, to the debate about whether one way speed has a physical meaning, or only a conventional one.

In chapter 4 we introduce the anisotropic dynamics for one particle: we gather the new dispersion relation and the conservation law for an elastic scattering. Then we deduce the relation between the anisotropic energy and momentum variable (which in our interpretation are the *real* energy and momentum) with the isotropic analogue used in special relativity; we find that the physical energy and momentum are nonlinear functions of the fictitious pseudo-momentum one-form whose components transform linearly under the action of the Lorentz group.

Then, in chapter 5, with the aid of this new framework we will examine the problem of the observed absence of the GZK cutoff and we show how the threshold energy, for example in the first pion-nucleon resonance $p + \gamma \rightarrow \Delta$, which is the main channel involved in the photopion production, is modified with respect the special relativistic case.

Finally, in chapter 6 we will summarize the conclusions and the main results and we will give some remark about the future perspectives for this area of research.

Chapter 2

Mathematical Tools

In this chapter we are going to introduce all the mathematical tools about Finsler geometry that we will need in our effort to generalizing the Lorentz transformation to the case in which spacetime is not isotropic.

From now on we assume that the reader knows a small amount of tensor analysis and had some exposure to differentiable manifolds, connections on a manifold (that is, the notion of parallelism on a manifold with respect to a path), and Riemannian geometry, see for example [66, 67]

2.1 Historical Review

2.1.1 Bernhard Riemann: the First Idea of "Finsler" Geometry

The fundamental idea of a Finsler space can be traced back to the famous lecture of Riemann's 1854 habilitation address, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen", in which Riemann aimed to introduce the notion of a manifold and its structures.

Traditionally, the structure being focused on is the Riemannian metric, which is a quadratic differential form; put another way, it is a smoothly varying family of inner products, one on each tangent space.

The choice of a quadratic form is essentially due to the fact that the modern theory of Riemannian geometry developed from the elementary differential geometry of surfaces in Euclidean space, by the usual mathematical process of abstraction. Riemann proposed to study the geometry of spaces where points are characterized by n coordinates x^i $(i = 1, 2, .., n)$ varying in a given range, and in which the "infinitesimal distance" ds between two points with coordinate difference dx^i is given by the formula

$$ds^2 = g_{ik}(x^1, .., x^n)\, dx^i\, dx^k \quad . \tag{2.1.1}$$

Here $g_{ik}(x^1, .., x^n)$ are arbitrarily prescribed functions of the coordinates x^i. However, once being given and thus determining a geometry of the space, they are assumed to transform, under a transformation $x^i \leftrightarrow y^i$ of coordinates in such a way as to make ds^2 independent of the choice of coordinates used.

Riemann showed that the Gaussian differential geometry of surfaces could be extended in unchanged form and that concepts like curvature could be carried over into such general geometries. He also pointed out that the classical Euclidean geometry was a special case of the general theory, namely, that in which there exist coordinates such that

$$ds^2 = \delta_{ik}\, du^i\, du^k \tag{2.1.2}$$

where δ_{ik} is the usual Kronecker delta. Special non-Euclidean geometries, discovered by Bolyai and Lobachevsky, entered into his larger framework as well.

Once we have recognized the logical possibility of replacing the Pythagorean formula (2.1.2) by the much more general formula (2.1.1) and of developing a consistent differential geometry in such spaces, we are led necessarily to the question: what it determines the functions $g_{ik}(x^1, .., x^2)$ of our experience? Riemann conjectured that the particular choice of geometry in nature depend on the reality which created or determined space; that is, the distribution of matter and the forces acting through space should determine geometry. He ended his thesis with the statement that, at this stage, we are crossing from the field of geometry into the field of physics.

The problems raised by these deep considerations of Riemann were faced by Einstein in his development of the general theory of relativity and given a solution which is logically and aesthetically satisfactory. But from the above it seems justified to consider Riemann as one of the most important precursors of modern relativity.

After Riemann had shown that the metric (2.1.2) of Euclidean geometry is a very special one and may be replaced by the more general equation (2.1.1), the question arose whether even this general form could not be further generalized. Indeed, the quadratic form on the right-hand side of equation (2.1.1) arose, in the Gaussian theory of surfaces, only from the fact that a two-dimensional surface had been embedded in a three-dimensional space in which a quadratic metric form (2.1.2) was assumed to be valid. This argument does not hold for a general space.

During his research work, Riemann discusses various possibilities by means of which an n-dimensional manifold may be endowed with a metric, and payed particular attention to a metric defined by the positive square root of a positive definite quadratic differential form. Thus the foundations of Riemannian geometry are laid; nevertheless, it is also suggested that the positive fourth root of a fourth order differential form might serve as a metric function.

These functions have three properties in common: they are positive, homogeneous of the first degree in the differentials, and are also convex in the latter. It would seem natural, therefore, to introduce a further generalization to the effect that the distance ds between two neighboring points represented by the coordinates x^i and $x^i + dx^i$ be defined by some functions $F(x^i, dx^i)$. This leads us to Finsler spaces, in which at every point of the space with coordinates (x^i), the length and the differential increments are related by

$$ds = F(x^i, dx^i) \quad , \tag{2.1.3}$$

where F is an arbitrary function, subjected only to some natural request which we are going to explain later in the chapter.

It is remarkable that the first systematic study of manifolds endowed with such a metric was delayed by more than 60 years. It was an investigation of this kind which formed the subject matter of the thesis of Paul Finsler in 1918, after whom such spaces were eventually named.

2.2 From B. Riemann to P. Finsler

Riemann saw the difference between the quadratic case and the general case; however, the latter had no choice but to lay dormant when he remarked that: "*...The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically expressed....*"

Happily, interest in the general case was revived in 1918 by Paul Finsler's thesis, written under the direction of Carathéodory.

It would appear that this new impulse was derived almost directly from the calculus of variations, with particular reference to the new geometrical background which was introduced by Caratheodory in connection with problems in parametric form. The kernel of these methods is the so-called indicatrix, while the property of convexity is of fundamental importance with regard to the necessary conditions for a minimum in the calculus of variations. Indeed, the remarkable affinity between some aspects of differential geometry and the calculus of variations had been noticed some years prior to the publication of Finsler's thesis, in particular by Bliss, Landsberg, and Blaschke.

Being Finsler geometry closely related to the calculus of variations, as such its deeper study went back at least to Jacobi and Adolf Kneser; it has its genesis in integrals of the form

$$\int_a^b F\big(x^1, .., x^n, \frac{dx^1}{dt}, .., \frac{dx^n}{dt}\big)\, dt \quad , \tag{2.2.1}$$

where the function $F(x^1, .., x^n, y^1, .., y^n)$ is positive unless all the y^i are zero, and is also homogeneous of degree one in y. Let us single out some contexts in which this integral arises.

- In certain physical examples, x stands for position, y for velocity, t would plays the role of time, F has the meaning of speed. In these cases, the above integral measures distance traveled.

- Another physical application originates in optics, because in anisotropic media the speed of light depends on its direction of travel. In this case $F(x, dx)$ is the amount of time it takes light to travel from the point with coordinates x to the point with coordinates $x + dx$. The integral $\int_a^b F(x, dx)$ represents the total time it takes light to traverse a given, possibly curved, path in this medium.

- Other contexts in which this integral arises is in the framework of mathematical ecology [68], where for instance x stands for the state of a coral reef, and y is the displacement vector from the state x to a new state. The quantity $F(x, dx)$ represents the energy one needs in order to evolve from state x to the neighboring state $x + dx$. Hence the integral $\int_a^b F(x, dx)$ is the total energy cost of a given path of evolution.

2.3 Tensor Calculus and L. Berwald

A few years after Finsler's work, the general development took a curious turn away from the basic aspects and methods of the theory as developed by him. Finsler did not make use of tensor calculus, being guided by the notions of the calculus of variations; but in 1925 the methods of tensor calculus were applied to the theory independently but almost simultaneously by Synge, Taylor, and Berwald. It was found that the second derivatives of the half of the square of

$F(x, dx)$ with respect to the differentials served as components of a metric tensor in analogy with Riemannian geometry, and from the differential equations of the geodesics, connection coefficients could be derived by means of which a generalization of Levi-Civita's parallel displacement could be defined.

In particular, Berwald's work stemmed from the study of differential equations, and was deeply rooted in the calculus of variations. Nevertheless, he introduced a connection and two curvature tensors: the Berwald connection is torsion free, but is (necessarily) not metric-compatible.

Berwald developed his theory with particular reference to the theory of curvature as well as to two-dimensional spaces. The significance of his work was enhanced by the advent of the general geometry of paths (a generalization of the so-called Non-Riemannian geometry) due to Douglas and Knebelman; for the initial approach of Berwald was such as to establish a close affinity between these branches of metric and non-metric differential geometry.

2.4 E. Cartan

Again, the theory took a new and unexpected turn in 1934 when E. Cartan published his work on Finsler spaces. He showed that it was indeed possible to define connection coefficients and hence a covariant derivative, such that the preservation of Ricci's lemma[1] was ensured, so his connection is metric-compatible but has torsion. On this basis Cartan developed a theory of curvature, and practically all subsequent investigations concerning the geometry of Finsler spaces were dominated by this approach. Several mathematicians expressed the opinion that the theory had thus attained its final form. To a certain extent this was correct, but not altogether so, as we shall now indicate.

The above-mentioned theories make use of a certain device which basically involves the consideration of a space whose elements are not the points of the underlying manifold, but the "line-elements" of the latter, which form a $(2n-1)$-dimensional variety. This facilitates the introduction of what Cartan calls the "Euclidean connection" which, by means of certain postulates, may be derived uniquely from the fundamental metric function $F(x, dx)$.

The method also depends on the introduction of a so-called "element of support"; namely, that at each point a previously assigned direction must be given, which then serves as directional argument in all functions depending on direction as well as position. Thus, for instance, the length of a vector and the vector obtained from it by an infinitesimal parallel displacement depend on the arbitrary choice of the element of support. It is this device which led to

[1] In Riemannian geometry it implies the vanishing of the covariant derivative of the metric tensor.

the development of Finsler geometry in terms of direct generalizations of the methods of Riemannian geometry.

It was felt, however, that the introduction of the element of support was undesirable from a geometrical point of view, while the natural link with the calculus of variations was seriously weakened. The rejection of the use of the element of support, desirable from a geometrical point of view, led to new difficulties: for instance, the natural orthogonality between two vectors is not in general symmetric, while the analytical difficulties are certainly enhanced, particularly since Ricci's lemma cannot be generalized as before.

2.5 Modern Perspectives

The fundamental problem in local Finsler geometry, just like Riemannian geometry, is the equivalence problem, that is, to find a complete system of invariants or to decide when two Finsler metrics differ by a coordinate transformation. In the Riemannian case this problem was solved in 1870 by E. B. Christoffel and R. Lipschitz. In his solution Christoffel introduced a covariant differentiation, which Ricci developed into his tensor analysis, making it a fundamental tool in classical differential geometry.

It is not unreasonable to expect that the solution of the equivalence problem will again involve a connection and its curvature, together with the proper space on which these objects live.

The geometrical data in Finsler geometry consist of a smoothly varying family of Minkowski norms, one on each tangent space, rather than a family of inner products. This family of Minkowski norms is known as a Finsler structure. It is an amusing irony that although Finsler geometry starts with only a norm in any given tangent space, it regains an entire family of inner products, one for each direction in that tangent space. This is why one can still make sense of metric-compatibility in the Finsler setting.

Back in the torsion-free camp, the next progress came in 1948, when the Chern connection was discovered. In the generic Finsler case, none of the connection we mentioned operates directly on the tangent bundle TM over M. Chern realized in his solution of the equivalence problem that, by pulling back TM so that it sits over the manifolds of rays[2] SM rather than M, one provides a natural vector bundle on which these connections may operate.

[2] SM is defined as the quotient of $TM\backslash\{0\}$ under the following equivalence relation: $(x, y) \sim (x, \tilde{y})$ if and only if y and $\tilde{y}$ are positive multiples of each other, that is, every ray $\{(x, \lambda y)|\lambda > 0\}$ is treated as a single point. In other words, SM is the bundle of all directions or rays, and is called the projective sphere bundle. Moreover it is diffeomorphic to the indicatrix bundle $\{(x, y) \in TM : F(x, y) = 1\}$, which is a subbundle of $TM\backslash\{0\}$.

Although Finsler geometry is widely considered as the most natural generalization of Riemannian geometry it would be more appropriate to describe it as Riemannian geometry, without the quadratic restriction [69].

In a certain sense Finsler geometry has originated from two simple innovations in Riemannian geometry, namely:

- Supplementing the position parameter in geometric quantities with a new independent vector variable. Here, this is given the name "Finsler parameter".

- Using a of a norm, here called "Finsler fundamental function" (a scalar distance function of position and the Finsler parameter).

The range of Finsler parameter is usually assumed to be all non-zero tangent vectors and skipped over quickly. However, the subject merits more attention, particularly in the case of metrics with indefinite signature. The Finsler parameter can be present in geometric quantities such as connections components, which in a differential geometric context, all need to be differentiable, albeit, not infinitely. It is therefore necessary that this parameter takes only values for which all such quantities are well-behaved. The most natural and practical way to determine the range of Finsler parameter is evidently through Finsler fundamental function. Domain of differentiability of this function seems the best (and the only available) candidate for the purpose.

The range of Finsler parameter has to be a fibre bundle in order to obtain a vertical bundle, absolutely necessary in the modern formulation[3]. When the metric is positive definite this requirement is easily satisfied because the corresponding fundamental function is differentiable for all non-zero tangent vectors, which form a fibre bundle. However, for an indefinite metric, domain of differentiability of the fundamental function is more restricted and it is not clear if it forms a fibre bundle in general.

2.6 Finsler Structure

Let M be a n-dimensional C^∞ manifold, T_pM is the tangent space at $p \in M$ and $TM := \cup_{p\in M} T_pM$ together with an appropriate differentiable structure which will be explained in detail in this section, is called the tangent bundle of M. Each element of TM has the form (p, v), where $p \in M$ and $v \in T_pM$.

[3] modern formulation of Finsler geometry of a manifold M utilizes the equivalence between this geometry and the Riemannian geometry of VTM, the vertical bundle over the tangent bundle of M, treating TM as the base space.

The natural projection $\pi : TM \to M$ is given by $\pi(p, v) := p$, the dual space of T_pM is T_p^*M, called the cotangent space at p. The union $T^*M := \cup_{p\in M} T_p^*M$ is the cotangent bundle of M.

A Finsler structure globally defined on M is a function

$$F : TM \to [0, +\infty)$$

with the following properties:

1. **Regularity of the Finsler structure**: having defined the slit tangent bundle of M as $TM\backslash\{0\} := \cup_{p\in M} T_pM\backslash\{0\}$, the function F must be: $F \in C^\infty(TM\backslash\{0\}, \mathbb{R})$.

2. **Positive homogeneity**: $F(p, \lambda v) = \lambda F(p, v)$ for all positive λ. We are requiring that F is a positive homogeneous function of first degree.

3. **Strong convexity**: The n x n Hessian matrix called Finsler metric

$$(g_{ij}(x, y)) := \left(\frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j}(x, y) \right)$$

   is positive definite at every point of $TM\backslash\{0\}$.

Here $x = (x^1, .., x^n)$ are the coordinates assigned in a given chart to point p of M, and $y = (y^1, .., y^n)$ are coordinates of $v \in T_pM$ defined as follows: fix any basis e_i with $i = 1, ..n$ for T_pM, indeed we will always choose $e_i = \frac{\partial}{\partial x^i}|_p$ although this restriction is not necessary, and express v as $v = y^i e_i$. In the last equality the summation convention was adopted: whenever an index appears once as a subscript and once as a superscript in the same expression, it is automatically summed over all the values from 1 to n.

The lowering and the raising of indices are carried out by the Finsler metric $g_{ij}(x, y)$ defined above, and its matrix inverse $g^{ij}(x, y)$. We also observe that the fundamental tensor $g_{ij}(x, y)$ defined at all $(x, y) \in TM\backslash\{0\}$ is invariant under positive rescaling in y.

However, it must be remembered that different unlike for Riemannian metric, the Finsler metric $g_{ij}(x, y)$ depends also on *velocity vector* $v = y^i e_i$.

Let $\varphi = (x^1, .., x^n) : U \subset M \to \mathbb{R}^n$ be a local coordinate system on an open subset U of M (that is a chart on M); as usual, $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ are respectively, the induced coordinate bases for T_xM and T_x^*M. The x^i give rise to local coordinates (x^i, y^i) on $\pi^{-1}(U) \subset TM$ through the law

$$y = y^i \frac{\partial}{\partial x^i} \quad .$$

In the future we will make no distinction between the point $(x, y) \in TM\backslash\{0\}$ and its coordinate representation (x^i, y^i). So, when we fix a chart on M, the Finsler structure F can be locally expressed as a function of the $2n$ real variables $(x^1, .., y^n)$.

Indeed we have also seen that TM is itself a $2n$-dimensional differentiable manifold [70]. An atlas for TM may be constructed out of an atlas for M as follows: for a chart (O, φ) on M, let $\widehat{O}$ be the subset of TM consisting of those tangent vectors whose point of tangency lie in O, thus $\widehat{O} := \cup_{p\in O} T_pM$. Then if $v \in \widehat{O}$ it may be expressed in the form $v = v^i \frac{\partial}{\partial x^i}$ where $\frac{\partial}{\partial x^i}$ are the coordinate vector fields associated with the coordinates on O. The coordinates of a tangent vector v are taken to be the coordinates (x^i) of its point of tangency, as given by the chart (O, φ) on M, and the components (v^i) of v, that is we have the map

$$\widehat{\varphi} : \widehat{O} \subset TM \to \mathbb{R}^{2n}$$

such that $\widehat{\varphi}(x, v) = (x^1, .., x^n, v^1, .., v^n)$. It is a straightforward matter to check that all possible charts on TM constructed in this way form an atlas for TM.

At this point we have to do a cautionary remark about our notation: when evaluating at the point $x \in M$, the symbol $\frac{\partial}{\partial x^i}$ refers to a coordinate vector on M; when evaluated at the point $(x, y) \in TM$, the same notation $\frac{\partial}{\partial x^i}$ stands for a coordinate vector on TM. As such, it would be on the same footing as the $\frac{\partial}{\partial y^i}$, which are also coordinate vectors on the tangent bundle TM.

In short, we are using the same symbol $\frac{\partial}{\partial x^i}$ to denote objects that belong to two different spaces, and obviously they do not obey the same transformation law: let $x^i = x^i(x'^1, .., x'^n)$ be a local change of coordinates on M

- As a coordinate vector fields on M, the $\{\frac{\partial}{\partial x^i}\}$ transform like

$$\frac{\partial}{\partial x'^k} = \frac{\partial x^i}{\partial x'^k}\frac{\partial}{\partial x^i} \quad . \tag{2.6.1}$$

- As a coordinate vector fields on TM, the $\{\frac{\partial}{\partial x^i}\}$ transform like

$$\frac{\partial}{\partial x'^k} = \frac{\partial x^i}{\partial x'^k}\frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial x'^k \partial x'^r} y^r \frac{\partial}{\partial y^i} \quad . \tag{2.6.2}$$

Every function $F : TM \to [0, +\infty)$ which satisfies the second and the third condition listed above is said a Minkowski functional on T_xM for every $x \in M$ [71].

We close this section by observing that the tangent spaces at different points of M are identical, in the sense that each is isomorphic to $\mathbb{R}^n$ and hence to every other. On the other hand, the realisation of such an isomorphism between two tangent spaces depends on the choice of a basis for each space, and in general there will be no obvious candidates to choose: in this sense, the tangent spaces are distinct.

2.7 Minkowski Norms

The entity (M, F) where M is C^∞ finite dimensional manifold and F is a Finsler structure is called a Finsler manifold. In a Finsler manifold, the restriction of a Finsler structure F to any specific tangent space $T_x M$ gives what is known as a Minkowski norm on $T_x M$. Thus a Finsler structure of M may be viewed as a smoothly varying family of Minkowski norms.

Every n-dimensional vector space is linearly isomorphic to $\mathbb{R}^n$, whose elements y have the form $(y^1, .., y^n)$. We can confine our discussion to Minkowski norm on $\mathbb{R}^n$ without loss of generality. We point out that the notion of a Minkowski norm, common in the studies about Finsler geometry, has nothing to do with Minkowski spacetime of special relativity theory and is also a different concept from the *usual* norm studied in functional analisys. We warn the reader about this potentially confusing terminology, which is however well-established in mathematics.

2.7.1 Euler's Theorem

First of all we dispense a technical ingredient we will use in the following, it is known as Euler's theorem for homogeneous functions.

Theorem 2.1 (Euler's theorem). *Let $G \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ and let α be a positive real number. Then the following two statements are equivalent:*

- *G is positively homogeneous of degree α. That is,*

$$G(\lambda\, y) \;=\; \lambda^\alpha\, G(y) \qquad \text{for all } \lambda > 0$$

- *the radial directional derivative of G is α times G. Namely*

$$y^i\, \frac{\partial G}{\partial y^i}(y) \;=\; \alpha\, G(y) \quad .$$

In particular, if G is positively homogeneous of degree 1, as in the physical case we will study in next chapter, then it holds

$$y^i\, \frac{\partial G}{\partial y^i}(y) \;=\; G(y) \tag{2.7.1}$$

and as a direct consequence

$$y^i\,\frac{\partial^2 G}{\partial y^i\,\partial y^k}(y)\;=\;0\quad. \tag{2.7.2}$$

Using this theorem in Finsler structure's framework, we can deduce from the definition of the Finsler metric $g_{ij}(x,y)$ the following equality:

$$g_{ij}(x,y)\;=\;\Big[F\,\frac{\partial^2 F}{\partial y^i \partial y^j}\;+\;\frac{\partial F}{\partial y^i}\,\frac{\partial F}{\partial y^j}\Big](x,y) \tag{2.7.3}$$

a straightforward result of these three last equation is

$$y^i\,g_{ij}(x,y)\;=\;F(x,y)\,\frac{\partial F(x,y)}{\partial y^j} \tag{2.7.4}$$

and from equation (2.7.1) we obtain the fundamental property which hold for every Finsler structure

$$g_{ij}(x,y)\,y^i\,y^j\;=\;F^2(x,y)\quad. \tag{2.7.5}$$

In next chapter's physical application, we will put $ds = F(x,y)$ where ds will be the infinitesimal spacetime distance between two event p (with coordinate representation x) and q such that the tangent vector $v = y^i \frac{\partial}{\partial x^i} \in T_pM$ "has his tip on q"; equally, we will write $ds^2 = g_{ij}(x,y)\,y^i\,y^j$.

Indeed, in the following we will be interested to the case in which $g_{ij}(x,y)$ will be a pseudo-Finsler metric, to this purpose we will have to change the first and the third properties we had required for a Finsler structure. We will study this issue in the last section of this chapter.

2.7.2 A Fundamental Inequality

We will see in this section that positivity of F and the triangle inequality are actually consequences of the defining properties of Minkowski norms, moreover the hypotheses of the following theorem define what one means by Minkowski norm on $\mathbb{R}^n$.

Theorem 2.2. *Let G be a nonnegative real-valued function on $\mathbb{R}^n$ with the properties:*

- $G \in C^\infty(\mathbb{R}^n\backslash\{0\},\mathbb{R})$.
- $G(\lambda\,y)\;=\;\lambda\,G(y)$ *for all* $\lambda\;>\;0$.

- *The $n\,x\,n$ matrix $(g_{ij}(y)) := [\frac{1}{2}\frac{\partial G^2}{\partial y^i \partial y^j}(y)]$, is positive definite at all $y \neq 0$.*

Then we have the following conclusions:

- *Positivity: $G(y) > 0$ whenever $y \neq 0$.*
- *Triangle inequality: $G(y_1 + y_2) \leq G(y_1) + G(y_2)$, equality holds if and only if $y_2 = \alpha\, y_1$ for some $\alpha \geq 0$*
- *Fundamental inequality: $w^i \frac{\partial G}{\partial y^i}(y) \leq G(w)$ at all $y \neq 0$ and equality holds if and only if $w = \alpha\, y$ for some $\alpha \geq 0$.*

The hypotheses of the above theorem define what one means by a Minkowski norm on $\mathbb{R}^n$: according to this theorem, there is no need to assume that G be positive at $y \neq 0$; it is necessarily so.

We point out here, that every *absolutely* homogeneous Minkowski norm on $\mathbb{R}^n$ (that is the second hypothesis of the above theorem is replaced by the more restrictive $G(\lambda\, y) = |\lambda|\, G(y)$ for all $\lambda \in \mathbb{R}$) is a norm in the sense of functional analysis (it is a simple matter to demonstrate that $G(0) = 0$). The simplest example of an absolutely homogeneous Minkowski norm on $\mathbb{R}^n$ is

$$G(y) = \sqrt{\delta_{ij}\, y^i\, y^j}$$

where δ_{ij} is the usual Kronecker delta. This norm is called the Euclidean norm of $\mathbb{R}^n$.

Moreover, at each point $(x, y) \in TM \backslash \{0\}$, the matrix $(g_{ij}(x, y))$ define an inner product on T_xM in this way:

$$(w \,|\, v)_{(x,y)} = g_{ij}(x, y)\, w^i\, v^j \tag{2.7.6}$$

with $w, v \in T_xM$ or equivalently, using the Finsler structure

$$(w \,|\, v)_{(x,y)} = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(x, y + s\, w + t\, v)\right]|_{s=t=0} \tag{2.7.7}$$

so it verifies the Cauchy-Schwarz inequality

$$|\, g_{ij}(x, y)\, w^i\, v^j \,| \leq \sqrt{[g_{ij}(x, y)\, w^i\, w^j]\,[g_{rs}(x, y)\, v^r\, v^s]} \quad . \tag{2.7.8}$$

Indeed, we can write for the Finsler structure a *generalization* of the Chauchy-Schwarz inequality: since we know that $F(x, y) > 0$ for $y \neq 0$, from the fundamental inequality we gain

$$w^i\, F(x, y)\, \frac{\partial F(x, y)}{\partial y^i} \leq F(x, w)\, F(x, y) \tag{2.7.9}$$

if we now use equation (2.7.4) we have

$$g_{ij}(x,y)\, y^i\, w^j \;\leq\; F(x,w)\, F(x,y) \tag{2.7.10}$$

and finally, with a straightforward calculation

$$|\, g_{ij}(x,y)\, y^i\, w^j \,| \;\leq\; F(w)\, F(y) \tag{2.7.11}$$

so, in the general case, we may view the fundamental inequality as a generalization of the Cauchy-Schwartz inequality, from inner products to Minkowski norms.

Note however that, when spelled out, equation (2.7.11) implies that

$$|\, g_{ij}(x,y)\, y^i\, w^j \,| \;\leq\; \sqrt{g_{pq}(x,w)\, w^p\, w^q}\, \sqrt{g_{mn}(x,y)\, y^n\, y^m} \tag{2.7.12}$$

we emphasize that in the first term on the right, it is $g_{pq}(x,w)\, w^p\, w^q$ and not $g_{pq}(x,y)\, w^p\, w^q$. As such, this last inequality is distinctly different from, and much more subtle than, the Chauchy-Schwarz inequality written in equation (2.7.8).

About equation (2.7.6) we have to point out the following observation: since $g_{pq}(x,y)$ is invariant under the positive rescaling $y \to \lambda y$, the inner products we assigned to the points $(x, \lambda y)$ for all $\lambda > 0$ are identical, that is

$$(w\,|\,v)_{(x,y)} \;=\; (w\,|\,v)_{(x,\lambda\, y)} \tag{2.7.13}$$

with $w, v \in T_x M$, as we already say, although Finsler geometry starts with only a norm in any given tangent space, it regains an entire family of inner products, one for each direction in that tangent space. So we have a redundancy in the above scheme, there is a simple way to restore economy: we can treat the ray $\{(x, \lambda\, y) | \lambda > 0\}$ as a single point in the projective sphere bundle as we mentioned above in footnote (2).

Different from Riemannian geometry where the metric tensor field can be use to establish a one-to-one correspondence between vectors and dual vectors, in Finsler geometry this identification can be made only if we also specify a "privileged direction" ν, a vector field on M ($M \ni p \to \nu_p \in T_p M \backslash \{0\}$):

$$T_p M \ni v = v^\alpha \partial/\partial x^\alpha \to g_{\alpha\beta}(p, \nu_p) v^\alpha dx^\beta \in T_p^* M \quad .$$

In physical application this "privileged direction" will be a time-light vector.

2.8 Minkowski and Locally Minkowski Spaces

Minkowski spaces are finite dimensional vector spaces equipped with a Finsler structure F invariant under translations: that is, given any tangent vector v based at an arbitrary $y \in \mathbb{R}^n$ (obviously now we are regarding $\mathbb{R}^n$ as a manifold, albeit a linear one), we can slide it without twisting (we are using the usual affine structure of the vector space), until it emanates from the origin O instead. Then we can evaluate F at the tip of this translated vector and it must hold

$$F(y, v^i \frac{\partial}{\partial y^i}|_y) := F(0, v^i \frac{\partial}{\partial y^i}|_0) \quad . \tag{2.8.1}$$

For a general Finsler space (M, F), each tangent space T_xM with $F(x, \cdot) := F|_{T_xM}$ is a Minkowski space. Thus to study the geometric structure of a Finsler space, we need to study Minkowski spaces first. A Finsler manifold (M, F) is said to be a locally Minkowskian space if, at every point $x \in M$, there is a local coordinate system $(x^1, .., x^n)$, with induced tangent space coordinates $(y^1, .., y^n)$, such that F has no dependence on the (x^i). Physically this is assured if we assume the homogeneity of spacetime.

In such a situation and with pseudo-Finsler metric (see the last section of this chapter), from the equality

$$g_{\alpha\beta}(v)v^\alpha v^\beta = F^2(v) \tag{2.8.2}$$

we obtain the generalization to the pseudo Finslerian case of the Special relativistic 4-velocity

$$y^\alpha = v^\alpha / F(v) \quad . \tag{2.8.3}$$

The one-form components are

$$\begin{aligned} y_\alpha &:= g_{\alpha\beta}(v) y^\beta \\ &= \frac{\partial}{\partial v^\alpha} F(v) \end{aligned} \tag{2.8.4}$$

in the last equality we have used equation (2.7.4).

2.9 Riemannian Manifolds From the Point of View of Finsler Geometry

Let M be an n-dimensional C^∞ (smooth) manifold. A smooth Riemannian metric g on M is a family $\{g_x\}_{x\in M}$ of inner products, one for each tangent space T_xM, such that the functions $g_{ij}(x) := g_x(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ are C^∞.

Since g_x is an inner product, the matrix $(g_{ij}(x))$ is positive definite at every $x \in M$. We can write

$$g_x \;=\; g_{ij}(x)\, dx^i|_x \,\otimes\, dx^j|_x \quad . \tag{2.9.1}$$

This g defines a symmetric Finsler structure F on TM by the law

$$F(x,y) \;:=\; \sqrt{g_x(y,y)} \quad . \tag{2.9.2}$$

So we can say that every Riemannian manifold (M,g) is a Finsler manifold. A Finsler structure F is said to be Riemannian if it arises from a Riemannian metric g in the manner we just described.

2.10 Connection and Geodesics

Now we consider general Finsler spaces. Geodesics are the first objects coming to a geometer's sight when he walks into an inner metric space. By definition, geodesics are locally length-minimizing constant speed curves which are characterized locally by a system of second order ordinary differential equations. Let (M,F) be a n-dimensional Finsler space.

For a C^1 curve $c : [a,b] \to M$, the length of c is given by

$$l(c) \;=\; \int_a^b F(c(t), \dot{c}(t))\, dt \quad . \tag{2.10.1}$$

A direct computation yields the Euler-Lagrange equations for a geodesic $c(t)$

$$\frac{d^2\, x^i}{dt^2} \;+\; 2\, G^i(x(t), \dot{x}(t)) \;=\; 0 \tag{2.10.2}$$

where $x(t) = (x^i(t)), i = 1,..,n$ denote the coordinates of $c(t)$ in a given chart, $\dot{x}(t) = (\dot{x}^i(t))$ are the coordinates of $\dot{c}(t)$ induced by the chart on M and G^i

is a globally defined vector field on TM that in the standard local coordinate system (x^i, y^i) in TM is given by

$$G^i(x,y) := \frac{1}{4}\, g^{il}(x,y)\,\big[2\,\frac{\partial g_{il}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\,\big]\, y^j\, y^k \tag{2.10.3}$$

where

$$g_{ij}(x,y) = g(x,y)[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] \tag{2.10.4}$$

and

$$g(x,y) = g_{ij}(x,y)\, dx^i \otimes dx^j \quad . \tag{2.10.5}$$

With the geodesic coefficients G^i in equation (2.10.3), we define a map $D_y : C^\infty(TM) \to T_xM$ for each $y \in T_xM$ by

$$D_yU(x) := \left\{ dU^i(x)[y] + U^j(x)\,\frac{\partial G^i}{\partial y^k}(x,y) \right\} \frac{\partial}{\partial x^i}|_x \tag{2.10.6}$$

where $U(x) = U^i(x)\frac{\partial}{\partial x^i}|_x \in C^\infty(TM)$. $D_yU(x)$ is called the covariant derivative of U in the direction y. We call the family $D := \{D_y\}_{y\in TM}$ the canonical connection of F.

With this connection D, we can define the covariant derivative $D_{\dot c}U(t)$ of a vector field $U(t)$ along a curve $c(t)$, $a \le t \le b$. $U(t)$ is said to be parallel along c if $D_{\dot c}U(t) = 0$. Clearly, a curve c is a geodesic if and only if the tangent vector field $\dot c(t)$ is parallel along c . The parallel translation $P_c : T_{c(a)}M \to T_{c(b)}M$ is defined by

$$P(U(a)) = U(b)$$

where $U(t)$ is parallel along c. From the definition, we see that P_c is a linear transformation preserving the inner products $g(c, \dot c)$. In general, P_c does not preserve the Minkowski functionals.

2.11 Non-Riemannian Curvatures

The canonical connection D has all the properties of an affine connection except for the linearity in y. Namely, $D_{y_1+y_2} \neq D_{y_1} + D_{y_2}$ in general. To measure the non-linearity, it is natural to introduce the following quantity

$$\mathbf{B}_y(u,v,w) := \frac{\partial^2}{\partial s \partial t}[D_{y+sv+tw}U]|_{s=t=0} \tag{2.11.1}$$

where $U \in C^\infty(TM)$ with $U(x) = u$. One can easily verify that $\mathbf{B}_y$ is a symmetric multi-linear form on T_xM. We call the family $\mathbf{B} := \{\mathbf{B}\}_{y \in TM\setminus 0}$ the Berwald curvature. A Finsler metric is called a Berwald metric if $\mathbf{B} = 0$. L. Berwald proved the simple fact that $\mathbf{B} = 0$ if and only if D is an affine connection.

For Riemannian metrics, $\mathbf{B} = 0$ and D is just the Levi-Civita connection. There are non-Riemannian Berwald metrics with $\mathbf{B} = 0$. Consider the following type of Finsler metric:

$$F(x,y) := a(x,y) + b(x,y) \tag{2.11.2}$$

where $a(x,y) = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is a Riemannian metric and $b(y) := \tilde{b}_i(x)y^i$ is a one-form with length $\|b\| = \sqrt{\tilde{a}^{ij}b_i b_j} < 1$. F is called a Randers metric.

M. Hashiguchi and Y. Ichiiyō first noticed that if b is parallel with respect to a, then $F = a + b$ is a Berwald metric. Later, they proved that if $db = 0$, then $F = a + b$ has the same geodesics as a and vice versa.

2.12 Riemann Curvature

In the context of Riemann's lecture, the restriction to a quadratic differential form constitutes only a special case. Nevertheless, Riemann saw the great merit of this special case, and he introduced for it the curvature tensor. The Riemann curvature tensor plays a major role in the "fundamental equivalence problem"; namely: how does one decide, in principle, whether two given Riemannian structures differ only by a coordinate transformation? This was solved in 1870, independently by Christoffel and Lipschitz, using different methods and without the benefit of tensor calculus. It was only 50 years later, in 1917, that Levi-Civita introduced his notion of parallelism (equivalent to a connection), thereby giving the solution a simple geometrical interpretation.

A. Einstein used Riemannian geometry to describe his general relativity theory, assuming that a spacetime is always Riemannian. For Riemannian spaces, there is only one notion of curvature: the Riemann curvature, that was introduced by B. Riemann in 1854 as a generalization of the Gauss curvature for surfaces. Since then, the Riemann curvature became the central concept in Riemannian geometry. Due to the efforts by L. Berwald in 1920's, the Riemann curvature can be extended to the Finslerian case.

Let (M, g) be a Riemannian space and D denote the Levi-Civita connection of g. The Riemann curvature tensor is defined by

$$R(u,v)\,w \;:=\; \{D_U\,D_V W \;-\; D_V D_U W \;-\; D_{[U,V]}W\}|_x \tag{2.12.1}$$

where U, V, W are local vector fields with $U(x) = u$, $V(x) = v$, $W(x) = w$ and $[U, V] = UV - VU$.

The core part of the Riemann curvature tensor is the following quantity:

$$\mathbf{R}_y(u) \;:=\; R(u,y)\,y \quad .$$

The Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is a self-adjoint linear transformation with respect to g and it satisfies $\mathbf{R}_y(y) = 0$. The family $\mathbf{R} = \{\mathbf{R}_{y\in TM\setminus\{0\}}\}$ is called the Riemann curvature.

Let (M, F) be a Finsler space. Given a vector $y \in T_xM\backslash 0$, extend it to a local nowhere zero geodesic field Y (i.e., all integral curves of Y are geodesics). Y induces a Riemannian metric

$$\widehat{g} \;:=\; g_Y \quad .$$

Let $\widehat{\mathbf{R}}$ denote the Riemannian curvature of $\widehat{g}$ as defined above. Define

$$\mathbf{R}_y \;:=\; \widehat{\mathbf{R}}_y \quad .$$

One can verify that $\widehat{\mathbf{R}}_y$ is independent of the geodesic extension Y of y. Moreover, $\mathbf{R}_y$ is self-adjoint with respect to g_y, that is

$$g_y(\mathbf{R}_y(u), v) \;=\; g_y(u, \mathbf{R}_y(v)) \tag{2.12.2}$$

and it satisfies $\mathbf{R}_y(y) = 0$. Let $W_y := \{u \in T_xM \,|\, g_y(y,u) = 0\}$. Then $\mathbf{R}_y|_{W_y} : W_y \to W_y$ is again a self-adjoint linear transformation with respect to g_y. Denote the eigenvalues of $\mathbf{R}_y|_{W_y}$ by

$$k_1(y) \le ... \le k_{n-1}(y) \quad .$$

They are the most important intrinsic invariants of the Finsler metric. We call $k_i(y)$ the i-th principal curvature in the direction y. The trace of $\mathbf{R}_y$ is denoted by $\mathbf{Ric}(y)$ which is called the Ricci curvature. $\mathbf{R}_y$ is given by

$$\mathbf{R}_y \;:=\; \sum_{ij=1}^{n} g^{ij}(y)\,g_y(\mathbf{R}_y(e_i), e_j) \;=\; \sum_{i=1}^{n-1} k_i(y) \quad . \tag{2.12.3}$$

2.13 Pseudo-Finsler Manifold

In this last section we want to study the general properties of “Finsler structure” we will use in the following chapters. More precisely we want to quit the strong convexity property of a generic Finsler structure, that is, we want to work with “Finsler structure” which give rise to a possible indefinite “Finsler-metric” as it is request in a relativistic physical framework.

In this case some problems arise because when there is an indefinite metric on M, the corresponding “Finsler structure”, that from now on we call pseudo-Finsler structure, would not be differentiable over $TM\backslash\{0\}$ but on a proper subset of TM. We want to investigate the property of such a subset, more precisely: for an indefinite metric, domain of differentiability of the fundamental function is restricted with respect $TM\backslash\{0\}$ and it is not clear if it forms a fibre bundle or not.

First of all we give the following definitions: Let N be some open submanifold of tangent bundle TM. A pseudo-Finsler fundamental function is defined as a map $F : N \to R$, satisfying a varying set of conditions. Naturally, first-degree homogeneity in y is nearly always among these conditions,

$$F(x, k\, y) \;=\; k\, F(x, y), \quad \forall k \,>\, 0, \quad \forall (x, y) \in N \tag{2.13.1}$$

where, it is implicitly assumed that if $(x, y) \in N$ then so is $(x, ky) \in N$, $\forall k > 0$.

Applying Euler theorem on homogeneous functions to F yields:

$$F^2(x, y) \;=\; \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)\, y^i\, y^j \tag{2.13.2}$$

the Finsler pseudo-metric tensor is defined as usual by:

$$g_{ij}(x, y) \;=\; \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) \tag{2.13.3}$$

where, the y-Hessian of F^2 is assumed to be of maximal rank and hence of fixed signature. The pseudo-metric $g_{ij}(x, y)$ may be definable only on a subset of N because F may be not differentiable on the whole of N.

Combining equations (2.13.2) and (2.13.3) yields the important relation

$$F^2(x, y) \;=\; g_{ij}(x, y)\, y^i\, y^j \quad . \tag{2.13.4}$$

If from the Finsler structure we obtain a metric tensor which has q negative eigenvalues and $m-q$ positive eigenvalues for all $(x, y) \in TM\backslash\{0\}$, then we say that (M, F) is a pseudo-Finsler manifold of index q, here m is the dimension of the manifold M. If, in particular, $q = 0$, (M, F) becomes a Finsler manifold, and if $m - q = 1$ we have a Lorentzian-Finsler manifold.

Alternative to the classical approach, given any arbitrary zero-degree y-homogeneous Finsler (or pseudo-Finsler) metric tensor, we can consider equation (2.13.4) as the definition of the Finsler (pseudo-Finsler) fundamental function corresponding to the given metric. We can now state an intriguing results [72]:

Theorem 2.3. *Given any differentiable Lorentzian metric $g_{\alpha\beta}$ on a smooth spacetime, the corresponding Finsler fundamental function is differentiable exactly on a fibre bundle over the spacetime defined by $LM := \{(x, y) \in TM \mid g_{\alpha\beta}\, y^\alpha y^\beta > 0\}$.*

Chapter 3

Generalized Lorentz Transformations

In this chapter we provide a kinematical basis for a new physics which is not Lorentz invariant at least at high energies. This might be needed in order to account for the overabundance of cosmic-ray TeV protons observed by HiRes, and for the missing GZK cutoff. We shall deduce the general form, and studying the mathematical properties, of what we have previously called "generalized Lorentz transformation", and indeed we will see that Lorentz invariance is (weakly) violated also at low energy scale due to a local space anisotropy. We will also see, as a consequence of the low level of space anisotropy, that testable non-Lorentz invariant effects are possible only at high energy scales.

From now on we consider the situation in which matter is so diluted that gravitational effects can be ignored. In such a situation, it is well known that Lorentz transformations can be derived imposing the constancy and invariance of the speed of light in every inertial frame, together with the linearity of the sought transformations [38, 73, 74].

The Lorentz transformation can be obtained using another original and more general approach. The alternative derivation of the Lorentz transformation, first discussed by the Russian mathematical physicist W. von Ignatowsky in 1910 [9], and later rediscovered and refined by numerous authors [8, 75, 76, 77] is based on the following hypotheses: the principle of relativity; the assumption that in every inertial frame space is homogeneous, isotropic, and euclidean and time is homogeneous; and the pre-causality condition must holds. The pre-causality conditions states that that if an event A takes place before an event B at a given point of an inertial frame, A takes place before B in every inertial frame [8, 39, 76].

We point out that following this approach the Lorentz transformation can be derived without assuming anything about the velocity of light or any other specific phenomenon. The parameter which occurs in the equations represents an invariant speed. Having demonstrated that there must be an invariant velocity, one can build up the rest of special relativity.

Our attempt is to generalized special relativity maintaining the relativity principle and the usual description with classical configuration variables such as position and velocity (in contrast with alternative approaches as doubly special relativity, in which the central role is played by energy-momentum space [18, 21, 22, 23, 24, 25, 81]). To this purpose we must relax one or more of the above hypotheses in the essay to construct a group of transformations that generalise Lorentz group. But in this effort very little freedom is allowed because of homogeneity of space and time is crucial in order to set up the fundamental notion of reference frame, and pre-causality condition is a fundamental requirement for causality to be defined. So our task is to demonstrate that if we assume to hold the following physical postulates [39]:

- The physical entity called spacetime is a metrically[1] structured four dimensional differentiable manifold. As a consequence we are not assuming the existence of a fundamental length scale, in contrast with double special relativity. Also, we assume the flatness of gravity free-spacetime. A point of the manifold is called "event", and it represents every "location" available for instantaneous punctual physical phenomenas.

- Homogeneity of spacetime: in particular this assumption imply homogeneity of space and time separately, they are crucial in order to define the fundamental notion of reference frame.

- Validity of the pre-causality principle: this is a minimal request if we want to build up a physical theory.

- Validity of the relativity principle.

From all these assumptions we can extract a new type of coordinate transformations which satisfies the following mathematical properties:

- linearity, so the principle of inertia holds good in all inertial frames[2];

[1] We will see in the following that by metrically structured manifold we mean in general a Finsler manifold. As we said in the previous chapter, Riemannian geometry is a "special case" of it.

[2] We know that the most general class of transformations which preserve the inertia principle are the group of projective transformations (affine maps belong on this group) [8]. One property of these transformations is that if two particles move with constant velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v_1} = \mathbf{v_2}$ in an inertial frame, applying a projective transformation they move in any other inertial frame yet along a straight line but in general with different, still constant velocities.

- possess group properties, we will see that this is a consequence of the implementation of the relativity principle at a kinematical level;
- they have a different *geometrical* meaning from the Lorentz ones: they serve as relativistic transformations of a flat *anisotropic pseudo-Finslerian* event space rather than of the Minkowski spacetime. So it is possible to conceive physical experiments imposing constrains to the parameters of the models (test-theory). Obviously they contain as particular cases both Galilei and Lorentz transformations.
- they lead to a generalized addition law of three velocities. This addition law is of Reichenbach's type (1.5.11).

We note that with respect the usual approach to the Einstein theory of special relativity we have dropped only the requirement of isotropy of space, and we have changed the hypotheses of homogeneity of space and time, with the request of homogeneity of spacetime.

3.1 Generalized Lorentz Transformation: Geometrical Approach

The two fundamental postulates stated by Einstein [38] as basis for his theory are:

1. The principle of relativity.
2. The constancy of the speed of light in all inertial reference frames.

Besides this two postulates, special relativity also uses additional implicit hypotheses: these other assumption concerns the homogeneity and *Euclidean* structure of gravity-free space and the homogeneity of gravity-free time [78].

In special relativity theory in a given inertial reference frame, spatial coordinates means the results of certain measurements with "rigid" motionless rods. A clock at rest relative to the inertial reference frame defines a local time and the local time at all points of space, indicated by synchronized clocks and taken together, give the global time of this inertial reference frame .

As a consequence each inertial reference frame is supplied in every space points with motionless, rigid, unit rods of equal length and motionless, synchronized clocks of equal running rate, then in each inertial reference frame an observer employ his own rigid rods and synchronized clocks to measure space and time intervals. With this method the observer can set up his own usual inertial coordinate system $(t, x^i), \quad i = 1, 2, 3, \quad x^1 = x,\ x^2 = y,\ x^3 = z.$

In the usual inertial coordinate system we can draw homogeneity of space and time, the isotropy and the Euclidean structure of space in the following way:

$$dL^2 = \delta_{rs}\, dx^r\, dx^s \tag{3.1.1}$$

$$dT^2 = dt^2 \tag{3.1.2}$$

everywhere and every time where dL and dT are respectively the spatial distance and time separation between two events with usual coordinates (t, x^i) and $(t + dt, x^i + dx^i)$ and $r, s = 1, 2, 3$.

We define a "Lorentz chart" as a differentiable function from the four dimensional differentiable manifold to the normed vector space $\mathbb{R}^4$ such that at each events it assigns a quadruple of real number which agrees with measurements performed using natural clocks and unstressed rods at rest in the frame to which the chart is adapted. Obviously for a given inertial frame these charts differ from each other at most by spatial rotations and translations and time translations.

The principle of relativity is a statement about formulae that express the laws of nature in terms of coordinate systems of a special kind (Lorentz charts). Such formulae represent the properties and relations of physical events by properties and relations of the real number quadruples assigned to the events by the said charts. It ensures that there exists an infinite continuous class of reference frames in spacetime which are physically equivalent, that is all laws of physics take the same form when referred to any one of these frames supplied with the adapted Lorentz chart, so no physical effects can distinguish between them. These reference frames are called "inertial frame" and a generic transformations connecting two arbitrarily inertial frame is called "inertial transformations". We stress that Einstein's relativity principle concerns Lorentz charts: the laws of physics are invariant when one such chart is substituted for another one; that is, in special relativity, when the formulae are subjected to a Lorentz transformation.

The existence of such equivalent reference frames corresponds to the validity of the principle of inertia, namely, that a physical object has no absolute state of motion. We observe that usual textbooks formulation of inertia's principle, that is the statement according to in an inertial frame, a force-free particle moves along a straight line at a constant speed contains, actually two implications very different in nature. One, that the motion takes place along a straight line, is a *physically* testable prediction, since the notion of a straight line is well defined in the Euclidean geometry that one presupposes when discussing the principle: this is the geometric content of the principle. The other, that motion is uniform, is a matter of *convention*. Of course, one could choose the "time" variable in such a way that motion is not uniform (this is the analogue of choosing a synchronisation different from Einstein's, or more generally different from Reichenbach's type synchronization), but this generalization will lead to no new physical phenomena only to a complication in the formulation of the laws of mechanics.

The main goal of this section is understanding whether or not the local structures of gravity-free space are Euclidean; let us consider for simplicity's sake the two dimensional case. Following Lalan [39], we are looking for the more general kinematics satisfying the postulates we made, that is we are going to gain the esplicity expression of the generalized Lorentz transformations.

After, we will prove that these new transformations are isometry of a pseudo-Finsler space rather than a Minkowski one. We point out that for the sake of generalization in what follows we do not specify what prescription has been adopted for the synchronization of distant clocks, we only require it to be compatible with the inertia and relativity principles.

We called "generalizzed Lorentz chart any differentiable function from the four dimensional differentiable manifold to the normed vector space $\mathbb{R}^4$ such that at each events it assigns a quadruple of real number which agrees with measurements performed using natural clocks and unstressed rods at rest in the frame to which the chart is adapted. The adjective generalized is added because we do not know if the time we are using agrees with Einstein time.

3.1.1 Linearity

First of all we have to explain what we exactly means by homogeneity of spacetime: physically we mean that the results of scientific experiments performed on isolated systems can be set up anywhere in an inertial frame (space homogeneity), be repeated at any time (temporal homogeneity) and the outcome should not depend both on experimental setup specific location and from the begining and finishing instants of the experiment. The mathematical content of spacetime's homogeneity is expressed requiring that the common domain of all Lorentz charts is a manifold transitively acted on by the additive group of $\mathbb{R} \oplus \mathbb{R}^3$, that is we appeal to the existence of two special classes of inertial transformation, namely space and time translations. This implies that no point of the manifold is special; however, it is compatible with a structure in which certain directions are favored over others.

Amounting to a simple displacement in space and time, they are supposed to leave the laws of physics invariant; using this freedom, we restrict our attention from now on to the class of inertial frames with common spacetime origins. More over we require that spacetime translations are an invariant abelian subgroup of the full relativity group we are looking for. If (M, ϕ_1) and (M, ϕ_2) are two global charts on the manifold M, we wish to know whether $\phi_2 \circ \phi_1^{-1}$ is a linear permutation of $\mathbb{R}^4$. We will see that homogeneity of spacetime as defined above is a sufficient condition.

If we consider two arbitrarily inertial frames, spacetime homogeneity requirement implies that *all* points of an inertial frame S' stand at *all* times in one and the same relation to another inertial frame[3] S. If all points of S always move in S' with the same velocity, the coordinate transformations must be indifferent to the choice of the particular origin and must be invariant if that origin is changed. In other words if before performing a change L of reference system from S to S' by which the origin of spacetime is conserved we made a translation, it is always possible to recover the original form L by choosing an appropriate translation in S'. In formulae:

$$\forall\, T_0,\ L,\ \exists\, T_0' \mid \quad T_0' \, L \, T_0 \;=\; L \tag{3.1.3}$$

where T_0' and T_0 are two translations, the first made with respect the system S' and the second made with respect the system S.

We can express this property using a coordinate language in this way: because of translations in two dimensional spacetime form an Abelian group with two parameters [79], it is always possible to choose a generalized Lorentz chart on the differentiable manifold M (by which we describe the spacetime), such that the coordinates (t, x) transform under an arbitrary change of origin of spacetime in the following way:

$$\begin{cases} t' = t - t_0 \\ \\ x' = x - x_0 \quad . \end{cases} \tag{3.1.4}$$

Using these coordinates we can write the operation L as:

$$t' = g(t, x), \qquad x' = f(t, x) \tag{3.1.5}$$

[3]Obviously this does not follow from the homogeneity of the spaces and the times associated at each inertial frame alone, but also depends on the way how the inertial frames move in each other; thus the linearity of the sought for coordinate transformations, as in the case of Lorentz transformation, is essentially a question of kinematics.

and equation (3.1.3) becomes:

$$\begin{cases} f(t-t_0, x-x_0) - x'_0 = f(t,x) \\ \\ g(t-t_0, x-x_0) - t'_0 = g(t,x) \quad . \end{cases} \tag{3.1.6}$$

From equation (3.1.6), that is, from the required homogeneity and from the continuity of f and g at $(0,0)$, we immediately deduce the linearity of the two functions with respect both variables x and t; so the generic change of reference system which preserve the origin of spacetime can be written

$$t' = a(v)\,x + b(v)\,t, \qquad x' = d(v)\,(x - vt) \tag{3.1.7}$$

where v is the velocity of the origin O' of the reference system S' measured by two observers in S. The coefficients $b(v)$ and $d(v)$ are pure numbers, the coefficient $a(v)$ has physical dimension as an inverse of velocity. Different values of these coefficients are obtained from different clock-synchronization conventions.

To conclude this subsection it is useful emphasizing the stringency of the homogeneity of spacetime hypothesis, especially as it concerns time, it does not hold in every physical theory, a simple example about evolutionary models of the universe (de Sitter spacetime) is given in [76].

3.1.2 Group Properties

Next step is to establish the explicit mathematical form of the functions $a(v)$, $b(v)$ and $d(v)$. This will be done by imposing that equations (3.1.7) are those of a group, in fact the physical equivalence of the inertial frames stated by the principle of relativity implies a group structure for the set of transformations we are looking for. In this way Lalan demonstrated in his article [39] the existence of three admissible kinematics, we report here the main results found by him.

First of all he imposed the natural condition that equations (3.1.7) for $v = 0$ were group's identity transformation, so transformation's coefficients can be expanded for low velocity in the following Maclaurin series:

$$\begin{aligned} d(v) &= 1 + \lambda\, v + o(v) \\ a(v) &= \mu\, v + o(v) \\ b(v) &= 1 + \rho\, v + o(v) \end{aligned} \tag{3.1.8}$$

with λ, μ, ρ real constant numbers. Dimensionally λ and μ are inverse of velocity, and μ is an inverse of a square velocity.

He constructed the group's generator: by straightforward calculation this is equivalent to impose the following algebraic equation:

$$r^2 - (\lambda + \rho)\, r + \mu + \lambda\, \rho = 0 \quad . \tag{3.1.9}$$

At this point Lalan distinguished three cases according to equations's root, in this way he found the three following admissible classes of kinematics (for the time being with respect relativity principle only).

3.2 First Possible Kinematics

The first kinematics we can obtain is characterized by two, distinct, real roots r_1 and r_2 of equation (3.1.9). It is straightforward, albeit with some tedium, to show that this kinematics has the following velocity addition law:

$$u' = \frac{u + v' - (\lambda - \rho) u\, v'}{1 - \mu u\, v'} \tag{3.2.1}$$

where v' is the velocity of S with respect the reference system S'.

We note that this is the same law, equation (1.5.11), found by Ungar in the framework of Reichenbach's special theory of relativity [56]. It seems that the composition velocity law is not able to single out real anisotropy from conventional one.

The most important feature of this addition velocity law is that there are some "notable" velocities: they satisfy the property that compose with any other velocity they still remain invariant. These invariant velocities are:

$$c_1 = \frac{1}{\lambda - r_1} \tag{3.2.2}$$

and

$$c_2 = \frac{1}{\lambda - r_2} \tag{3.2.3}$$

it is a simple matter to show that

$$\begin{aligned} c_1 + c_2 &= \frac{\lambda - \rho}{\mu} \\ c_1\, c_2 &= \frac{1}{\mu} \end{aligned} \tag{3.2.4}$$

physically this means that as in special relativity there is still an invariant speed, but its value can be different along different directions (in fact we will see in next subsection that if pre-causality principle holds, it must be $c_1\, c_2 < 0$, so we can consider for example the case with $c_2 > 0$ and $c_1 < 0$. We also observe that we have a constrain on μ parameter).

If both c_1 and c_2 are finite, real number the only admissible values of the parameter v are those that satisfy:

$$(1 - \frac{v}{c_1})(1 - \frac{v}{c_2}) > 0 \quad . \tag{3.2.5}$$

The transformations coefficients are:

$$\begin{cases} d(v) \;=\; \left(1 - \dfrac{v}{c_1}\right)^{-\frac{r_2}{r_2 - r_1}} \left(1 - \dfrac{v}{c_2}\right)^{\frac{r_1}{r_2 - r_1}} \\[2ex] a(v) \;=\; \dfrac{v}{c_1\, c_2}\, d(v) \\[2ex] b(v) \;=\; d(v)\left[1 - v\,(\dfrac{c_1 + c_2}{c_1\, c_2})\right] \quad . \end{cases} \tag{3.2.6}$$

Obviously, in contrast with Lorentz transformations, the generalized Lorentz transformations we can now write, that is

$$\begin{cases} x' \;=\; d(v)\,(x - v\,t) \\[2ex] t' \;=\; d(v)\left[\dfrac{1}{c_1\, c_2}\, v\, x + \left(1 - v\,(\dfrac{c_1 + c_2}{c_1\, c_2})\right) t\right] \end{cases} \tag{3.2.7}$$

do not leave invariant the Minkowsky pseudo-Euclidean metric. Therefore, the question arises as to what the metric invariant under such generalized Lorentz transformations is.

The rigorous solution to this problem is not a quadratic form but a homogeneous function of the coordinate differentials of degree one:

$$ds = (dx - c_1\, dt)^{-\frac{r_1}{r_2 - r_1}}\, (c_2\, dt - dx)^{\frac{r_2}{r_2 - r_1}} \quad . \tag{3.2.8}$$

The structure (3.2.8) falls into the category of pseudo-Finsler metrics. This is one of our model crucial point, indeed we have demonstrated that removing the isotropy space hypothesis, spacetime is representable as a pseudo-Finsler space and not as a pseudo-Riemannian manifold.

The metric (3.2.8) describes a flat but anisotropic event space, so in our model, at every length scale we have a different geometrical interpretation from special relativity.

We also observe that the only difference between equations (3.2.7) and (1.5.5), found in Reichenbach theory framework, that is a purely conventional theory, is contained only in the difference between multiplicative coefficients $d(v)$ and $\Gamma(V)$ of equations (3.2.7) and (1.5.7) respectively; or better it seems that the quantities

$$\frac{r_1}{r_2 - r_1} \qquad \text{and} \qquad \frac{r_2}{r_2 - r_1}$$

can capture the difference between real and conventional anisotropy. This observation explain our comment to equation (3.2.1), because in the construction of the velocity composition law, from equation (3.2.7) the factor $d(v)$ disappear. Indeed we will see that to grasp, in an easy way, this difference between real and conventional anisotropy we will have to introduce three new parameters linked to (λ, ρ, μ).

3.2.1 Special Cases

- Minkowski spacetime and Lorentz transformation

  For the first special case we require that:

  $$\begin{cases} r_2 + r_1 = 0 \\ c_1 + c_2 = 0 \end{cases} \tag{3.2.9}$$

  this case is particularly important because we recover Minkowski spacetime and Lorentz transformations as we easily verify replacing in equations (3.2.8) and (3.2.7) the above values.

  Using equations (3.1.9) and (3.2.3) we are able to express these occurrences imposing a condition on parameters λ and ρ

  $$\begin{cases} \lambda + \rho = 0 \\ \lambda - \rho = 0 \end{cases} \tag{3.2.10}$$

  this linear system has only one solution $\lambda = \rho = 0$. As a consequence we can interpreter the parameters λ and ρ as anisotropy's parameters because when $\lambda = \rho = 0$, we have seen that we reobtain special relativistic theory for which space's isotropy is an essential requirement. It is a simple matter and a useful task to see in a 3D parameters space $(\lambda,\, \rho,\, \mu)$ which points correspond to special relativity: $(0\,, 0\,, -1/c^2)$, where we have defined $c_2 = c$.

- The case with $c_2 = \infty$, that is $\mu = 0$ is also admissible if

  $$1 - \frac{v}{c_1} \geq 0. \tag{3.2.11}$$

where $c_1 = \frac{1}{\lambda - \rho}$, the group coordinates transformations are

$$\begin{cases} x' = \dfrac{(x - v\,t)}{\left(1 - \dfrac{v}{c}\right)^{\lambda\, c}} \\ \\ t' = \dfrac{t}{\left(1 - \dfrac{v}{c}\right)^{\rho\, c}} \quad . \end{cases} \tag{3.2.12}$$

Note that the second equation implies the existence of the *absolute simultaneity* that is, two events occurring in different spatial points and at the same instant with respect S reference frame are view as simultaneous in every reference frame S', since $\Delta\, t' = 0$ implies $\Delta\, t = 0$.

3.3 Second Possible Kinematics

The second possible kinematics, is given when the equation (3.1.9) has two equal real roots $r = r_2 = r_1$. There is only one invariant velocity $c = \frac{1}{\lambda - r} = \frac{2}{\lambda - \rho}$ not neccesary infinite (as in Newtonian physics), and the parameter v can vary all over the real numbers different from c. The coordinates transformations are those with the following coefficients:

$$\begin{cases} d(v) = \dfrac{e^{r\, v\,/\,(1 - \frac{v}{c})}}{1 - \frac{v}{c}} \\ \\ a(v) = \dfrac{v\, d(v)}{c^2} \\ \\ b(v) = d(v)\left(1 - \dfrac{2\, v}{c}\right) \quad . \end{cases} \tag{3.3.1}$$

The invariant pseudo-Finsler metric is in this case:

$$ds = (c\, dt - dx)\, e^{c\, r \frac{dx}{c\, dt - dx}} \quad . \tag{3.3.2}$$

The case with $\lambda = \rho$ and $\mu = 0$ that is $c = \infty$ become:

$$\begin{cases} x' = e^{v\, \lambda}\, (x - vt) \\ \\ t' = e^{v\, \lambda}\, t \end{cases} \tag{3.3.3}$$

the corresponding Finsler metric is

$$ds = dt\, e^{\lambda \frac{dx}{c\, dt}}. \tag{3.3.4}$$

Also for this kinematics time possess the *absolute simultaneity* and the principle of reciprocity hold true, but we reobtain the Galileo's kinematics and so the spatially isotropy only for $\lambda = 0$ (or equivalently $\lambda + \rho = 0$, as we see in the following). We note that this last observation confirm our previously physical interpretation for parameters λ and ρ.

3.4 Third Possible Kinematics

The last admissible kinematics obtained when equation (3.1.9) has two complex coniugate roots and as a consequence there are no admissible real invariant velocity, they are complex conjugate. In this case every velocity's values is permitted.

In these three last sections we saw that thee are three allowable group transformation's coordinate, for every one of these we can say that equation (3.1.3) or (3.1.6) imply that the group of change of origin is invariant under translations, or equivalently that spacetime translations are an invariant abelian subgroup of the full relativity group we have constructed, as it is required in a theory with homogeneity of spacetime.

3.5 Pre-Causality

Applying the pre-causality postulate, that is we require that if an event A takes place before an event B at a given point of an inertial frame, A takes place before B in every inertial frame (this is equivalent to impose that coordinates transformations satisfy $\frac{\partial t'}{\partial t} > 0$) we obtain that the only kinematics we can retain if c_1 and c_2 are finite, distinct, real or complex conjugates, are those that satisfy the conditions $c_1\, c_2 < 0$, that is $\mu < 0$. So we can drop out the third kinematic; in the first this condition imposes that there are two invariant velocities, one positive say c_2 the other c_1 negative, equation (3.2.5) states that they are also *limiting* velocities.

Indeed in the first kinematics also the case with $c_2 = \infty$ and c_1 real is acceptable. In the second kinematics the only admissible situation is the one with $c = \infty$, that is the "Galileo anisotropic case" (3.3.3).

We observe here, for future convenience, that if equation (3.1.9) has two real distinct roots, the pre-causality condition for two events A and B occurred in the origin of the reference system S at a different time $t_A < t_B$ can be write in a second inertial frame S' in the following way:

$$T'^2 - X'T'\frac{c_1 + c_2}{c_1 c_2} + \frac{X'^2}{c_1 c_2} > 0 \tag{3.5.1}$$

where (T', X') are values assigned to event B from a generalized Lorentz chart adapted to reference frame S', whereas the first event A is realized in spacetime origin of both coordinate reference systems.

This "anisotropic relativistic pseudo-norm" is the same found by Ungar as the invariant quantity for the one-way Lorentz group in Reichenbach special theory of relativity. It is easy to see that the anisotropic relativistic pseudo-norm (3.5.1) become the usual Minkowski pseudo-Euclidean metric if $c_1 + c_2 = 0$.

We conclude this section to point out that linearity of the laws that link arbitrarily inertial reference frame are consequence of homogeneity of spacetime only; the existences of invariant velocities is a consequence of homogeneity of spacetime and group properties of the generalized Lorentz transformations. Also we notice that in our derivation we have not appeal to any physical phenomenae, but only to a general spacetime's properties.

3.6 Generalized One Dimensional Composition Law: Properties

What we said in the three previous sections has as consequence that the most general velocity composition law compatible with spacetime homogeneity, relativity principle and pre-causality principle has the following form

$$\Phi(u, v') = \frac{u + v' + b\,u\,v'}{1 + a\,u\,v'}\ , \tag{3.6.1}$$

where a and b are arbitrary real constants, they encode the convention adopted in both reference frames for synchronising clocks, u is the velocity of a particle with respect to the reference frame S, v' is the velocity of S reference system with respect S' and $\Phi(u, v')$ is the particle velocity with respect to S'.

We may express the coefficients a and b using invariant velocities defined by the

$$\Phi(c, v) = c \tag{3.6.2}$$

for every admissible values of velocity v, so we obtain the followings two equalities

$$a = -\,\frac{1}{c_1\,c_2} \tag{3.6.3}$$

$$b = -\,\frac{c_1\,+\,c_2}{c_1\,c_2} \quad . \tag{3.6.4}$$

Equation (3.6.1) can be rewritten as

$$\Phi(u, v') = \frac{u\,+\,v'\,+\,(c_1\,+\,c_2)\,u\,v'\,/\,c_1\,c_2}{1\,-\,u\,v'\,/\,c_1\,c_2} \quad . \tag{3.6.5}$$

Obviously, using equations (3.2.2) and (3.2.3) we obtain again the formula (3.2.1).

Now we want to list some general features of the one-dimensional generalized composition law in $(1+1)$ dimension:

it is a real function of two real variables $\Phi : I \times I \to I$; this function satisfy the following properties:

- existence of a neutal element:

$$\Phi(u, 0) = \Phi(0, u) = u\;, \qquad \forall u \in I\;; \tag{3.6.6}$$

- existence of the inverse element:

$$\forall u \in I,\; \exists\, u' \in I \quad \text{such that} \quad \Phi(u', u) = \Phi(u, u') = 0\;; \tag{3.6.7}$$

  The inverse velocity is

$$u' \;=\; \frac{-\,u}{1\,-\,(\lambda\,-\,\rho)\,u} \quad . \tag{3.6.8}$$

  Note that u' is not necessarily equal to $-\,u$; we recover the so call "principle of reciprocity" [8] only if $\lambda\,-\,\rho\,=\,0$.

  It is nice noting that once again quantities involving the two parameters λ and ρ as in this case $\lambda\,-\,\rho$ are strictly related to space's isotropy; this is due to Berzi and Gorini's paper [80] in which they stated that the principle of reciprocity derived from the principle of spatial isotropy. Indeed in the following we have to distinguish between conventional (we see later that this is a case) and non conventional anisotropy. By conventional anisotropy we mean an anisotropy that is gauge eliminable or equivalently eliminable by a stipulation.

If we read the velocity in equation (3.6.8) as one-way velocity and use the notation $v_+ = v$, then from equation (3.2.1) the velocity v_- inverse to v_+ is given by the equation

$$v_- = \frac{-\, v_+}{1 + (\lambda - \rho)\, v_+} \quad . \tag{3.6.9}$$

Equation (3.6.9) states the *one-way velocity reciprocity principle* in Reichenbach's special theory of relativity, as a consequence we can interpreter the quantity $\lambda - \rho$ as a conventional one.

Obviously, in this framework the *round-trip* velocity v, associated with the one-way velocity v_+ is defined by the equation

$$\frac{1}{v} = \frac{1}{2}\left(\frac{1}{v_+} + \frac{1}{-\, v_-}\right) \tag{3.6.10}$$

- associative rule:

$$\Phi(\Phi(u,v),w) = \Phi(u,\Phi(v,w)) \;, \qquad \forall u,v,w \in I \tag{3.6.11}$$

- commutative rule

$$\Phi(u,w) = \Phi(w,u) \;, \qquad \forall u,w \in I \;. \tag{3.6.12}$$

  This equation states that the composition law among collinear velocities is commutative. This property is valid only in $(1+1)$ dimensions, it does not hold in general for the composition law of velocities along arbitrary directions in more than one spatial dimension [77]. In $(1+3)$ dimensions also in special relativistic theory this feature is false.

Hence, on writing $u \oplus v := \Phi(u,v)$, $\forall u,v \in I$, equation (3.6.1) defines the composition law of an Abelian group $(I,\oplus)$, with neutral element 0 and inverse u' of a generic element $u \in I$ defined by equation (3.6.8). We point out that the group structure is a consequence only of the relativity principle and the real form of equation (3.6.1) is a consequence of spacetime homogeneity. As Mermin demonstrated in [75] for such a composition law, there exists a differentiable real function h of a single variable defined on I, such that

$$h(0) = 0 \tag{3.6.13}$$

$$\Phi(u,v) \; = h^{-1}\left(h(u) \; + \; h(v)\right) \; . \tag{3.6.14}$$

or equivalently

$$h(\Phi(u,v)) \; = \; h(u) \; + \; h(v) \quad . \tag{3.6.15}$$

From the last equation written, we see that h function is not defined univocally, as a matter of fact if α is a real number and $\alpha \neq 1\,,0$, we can define a new function

$$H(u) \; = \; \frac{1}{\alpha}\, h(u) \tag{3.6.16}$$

such that

$$H(\Phi(u,v)) \; = \; H(u) \; + \; H(v) \quad . \tag{3.6.17}$$

We point out, as discussed by Mermin [75], the principle of relativity is compatible with a generalised kinematics, characterised by the function h that defines the composition law for velocities. Let us now define the function

$$\varphi(u) := \left.\frac{\partial \Phi(u,v)}{\partial v}\right|_{v=0} \tag{3.6.18}$$

on differentiating equation (3.6.14) with respect to v, and setting $v = 0$, we obtain

$$\varphi(u)\,\frac{dh(u)}{du} = \left(\frac{dh}{du}\right)_{u=0} \quad . \tag{3.6.19}$$

Since we can choose $(dh/du)_{u=0} = 1$ without loss of generality (this is a straightforward consequence of equations (3.6.16) and (3.6.17)), the function φ contains all the information needed to specify Φ.

The meaning of φ can be found by expanding u' to the first order in v:

$$u' = u + \varphi(u)\, v + \mathcal{O}(v^2) \tag{3.6.20}$$

this is the composition law between an *arbitrary* velocity u and a velocity v with *small* magnitude. Since equation (3.6.6) or equation (3.6.19) implies $\varphi(0) \, = \, 1$, at very small speeds one recovers Galilean kinematics.

The importance of the function $\varphi(u)$ is that if we assume only the relativity principle and the existence of a conserved total energy in elastic collision between asymptotically free particles, we are able to construct another conserved quantity

$$p(u) \; = \; \eta\, \varphi(u)\, \frac{d\,T(u)}{du} \; + \; \tau\, T(u) \; + \nu \tag{3.6.21}$$

where $T(u)$ is the kinetic energy of a particle with velocity u in an inertial frame and coefficients η, τ and ν are real constant. To avoid triviality we impose also $\eta \neq 0$.

In classical physics and special relativity theory if we choose the set of parameter

$$\begin{cases} \eta = 1 \\ \tau = 0 \\ \nu = 0 \end{cases} \tag{3.6.22}$$

the left hand member in equation (3.6.21) coincide respectively with momentum and spatial component of energy-momentum one-form. As a consequence we can identify this new physical observable as the "momentum" in a more general framework [82, 83].

Another interesting property of φ is that if some velocity, say C, is invariant, then $\varphi(C) = 0$. This follows immediately by applying equation (3.6.18) to the condition

$$\Phi(C, v) = C \ , \qquad \forall v \in I \ , \tag{3.6.23}$$

which expresses the invariance of C.

Indeed C might not belong to I, so, if for example $C = \sup I$ we define $\Phi(C, v) = \lim_{u \to C^-} \Phi(C, u)$ and $\varphi(C) = \lim_{u \to C^-} \varphi(u)$.

In the following we restrict ourselves to the case with c_1 and c_2 distinct real numbers such that $c_1 \, c_2 = 1/\mu < 0$. We assume $c_1 < 0$ so we have $c_2 > 0$ and the only admissible values for parameter v are those in the open interval $I := (c_1, c_2) \subset R$.

3.7 Conventional and Non Conventional Anisotropy

We know experimentally that the (two-way) speed of light (at least with wavelength much larger than Planck length scale) measured in a round trip does not depend on direction. However, this is not in contradiction with the possibility of having anisotropic propagation, provided one assumes that the one-way speeds of light are different, namely that the (one-way) speed of light when propagating from point P to point Q, say, differs from the one associated with propagation from Q to P.

Whether the one-way speed of light is a physically meaningful quantity or merely a conventional one, is the matter of a long-standing debate. We saw in chapter 1 that this issue is inextricably linked to another one, concerning the conventionality of clock synchronisation in special relativity. This creates a circularity which does not allow any escape at the kinematical level. However, Ohanian [45] claims that the dynamics of theories with different one-way speeds of light are not equivalent (we will deal with this problem in a way which is very different in nature from the approach of Ohanian in his article. Moreover Ohanian's point of view was convincingly criticized by Klauber [59] employing arguments that translated in our model language, are due to the fact that postulating, as in Reichenbach theory, the existence of anisotropic one-way speed one introduce a *conventional* anisotropy and not a physical real one). So no new physics can arise in such a model, but only more complicated (and equivalent) equations both in kinematics than in dynamics. Different from these assumptions, are the following two cases, the first with the existence of a preferred reference frame and so a preferred directions in velocity space; in the second we have to introduce a *real* anisotropy[4], thus we have a chance to test isotropy by means of dynamical experiments.

In this section we want to investigate this issue; to this aim it is convenient to introduce three new parameters instead of λ, ρ and μ. The first new parameter is the two-way velocity of light: in our anisotropic kinematics model we saw that there is still an invariant speed, but its value can be different along different directions. Although there is no fundamental reason why light should propagate at the invariant speed, there is excellent experimental evidence that any difference is very small. If as usual we identify both our one-way invariant speeds with the one-way speed of light in vacuum, it follows that consistency with experimental evidence requires

$$\frac{1}{c} = \frac{1}{2}\left(\frac{1}{c_2} - \frac{1}{c_1}\right) \tag{3.7.1}$$

where c represent round trip light velocity's experimental value.

This equation allow us to introduce a second addimentional parameter: the so called "Reichenbach's conventional[5] parameter" ε defined by

$$\begin{cases} c_1 = \dfrac{-\,c}{1 - \varepsilon} \\[2ex] c_2 = \dfrac{c}{1 + \varepsilon} \end{cases} \tag{3.7.2}$$

[4]We will see in the later what we mathematically mean by "real" anisotropy

[5]Conventional only for physicists which agree with the conventionalist thesis, of course!

with $|\,\varepsilon\,| < 1$.

It is now useful and instructive to understand the relations between old parameters (λ, ρ, μ) and the two new parameter c and ε. From second equation of (3.2.4) and definitions (3.7.1) we can easily find the relation between μ parameter and the two new variables ε and c (indeed, the value of this quantity is fixed by Michelson-Morely experiment; c different from ε is a fundamental physical constant of nature), so we have

$$\mu = \frac{\varepsilon^2 - 1}{c^2} \quad . \tag{3.7.3}$$

We can now rewrite the two real roots of equation (3.1.9): from equations (3.2.2) and (3.2.3) and using equation (3.7.3), we obtain

$$r_1 = \frac{1 + \lambda\, c + \varepsilon}{c} \tag{3.7.4}$$

and

$$r_2 = \frac{-1 + \lambda\, c + \varepsilon}{c} \quad . \tag{3.7.5}$$

If we compute the quantity $r_2 - r_1$ using equations (3.7.5) and (3.7.4) we find

$$r_2 - r_1 = -\frac{2}{c} \quad . \tag{3.7.6}$$

The Finsler structure (3.2.8) can be written as follows:

$$ds = \sqrt{\left(\frac{c}{1 + \varepsilon}\, dt + dx\right)\left(\frac{c}{1 - \varepsilon}\, dt - dx\right)} \left(\frac{\frac{c}{1+\varepsilon}\, dt + dx}{\frac{c}{1-\varepsilon}\, dt - dx}\right)^{\frac{\lambda\, c + \varepsilon}{2}} \quad . \tag{3.7.7}$$

From equation (3.1.9), (3.7.4) and (3.7.5) we have

$$r_1 + r_2 = \frac{2}{c}\,(\lambda\, c + \varepsilon) = \lambda + \rho \tag{3.7.8}$$

that is

$$\lambda - \rho = -\frac{2\,\varepsilon}{c} \tag{3.7.9}$$

so the linear combination $\lambda - \rho$ is proportional to a conventional parameter. As a consequence the anisotropy it lead in equation (3.6.8) is purely conventional even if in our model there is a real anisotropy.

Summarizing we have this situation

$$\begin{cases} \mu = \dfrac{\varepsilon^2 - 1}{c^2} \\ \\ \lambda - \rho = - \dfrac{2\,\varepsilon}{c} \end{cases} \tag{3.7.10}$$

so we can introduce a third new (addinensional) parameter σ linked with parameters λ and ρ in a compatible way with the second of the above equation, for example we can put

$$\begin{cases} \lambda = \dfrac{\sigma - \varepsilon}{c} \\ \\ \rho = \dfrac{\sigma + \varepsilon}{c} \quad . \end{cases} \tag{3.7.11}$$

We can definitely write the change in parameters as follows:

$$\begin{cases} \mu = \dfrac{\varepsilon^2 - 1}{c^2} \\ \\ \lambda = \dfrac{\sigma - \varepsilon}{c} \\ \\ \rho = \dfrac{\sigma + \varepsilon}{c} \end{cases} \tag{3.7.12}$$

and the inverse relations

$$\begin{cases} c = \dfrac{2}{\sqrt{\Delta}} \\ \\ \varepsilon = \dfrac{\rho - \lambda}{\sqrt{\Delta}} \\ \\ \sigma = \dfrac{\rho + \lambda}{\sqrt{\Delta}} \end{cases} \tag{3.7.13}$$

where $\Delta = (\lambda - \rho)^2 - 4\,\mu$, that is the discriminant of equation (3.1.9) which control coordinate transformations'group structure.

We observed that, if we read c parameter as round-trip velocity of light, then first equation in (3.7.13) is a relation by which we are able to express the fundamental c constant using the "old" three parameters related with coordinate transformation. As a consequence of the fixed experimental value of constant c, parameters $(\lambda\,,\rho\,,\mu)$ are not three independent number because they have to satisfy

$$(\lambda\,-\,\rho)^2\,-\,4\,\mu\,=\,\frac{4}{c^2}\quad. \tag{3.7.14}$$

Really this is not surprising if we remember that by our approach "à la Ygnatowsky" to pick out coordinates transformations, the existence of a invariant speed is a direct consequence of the hypothesis made about spacetime structure. So we can argue the existence of a relation between parameter related with coordinate transformation and the value of the invariant speed. We have only deduced the explicit form of this relation. Indeed we can also conclude that quantity $(\lambda\,-\,\rho)^2\,-\,4\,\mu$ is synchrony free because c is.

Moreover, from the second equation in (3.7.10), the usual Einstein synchronization procedure is expressed by old parameters imposing $\lambda\,-\,\rho\,=\,0$. If we note that from equation (3.7.11) we deduce $\sigma\,=\,\frac{\lambda\,c\,+\,\varepsilon}{2}$, we can now write in a handled way the Finsler pseudo-metric (3.7.7)

$$ds\,=\,\sqrt{\Big(\frac{c}{1\,+\,\varepsilon}\,dt\,+\,dx\Big)\Big(\frac{c}{1\,-\,\varepsilon}\,dt\,-\,dx\Big)}\,\left(\frac{\frac{c}{1\,+\,\varepsilon}\,dt\,+\,dx}{\frac{c}{1\,-\,\varepsilon}\,dt\,-\,dx}\right)^{\sigma} \tag{3.7.15}$$

Note that, even if $\sigma\,=\,0$, this is not just Minkowski spacetime in funny coordinates. In fact, we must keep in mind that t and x are directly related to outcomes of measurements. Otherwise, for the case $\sigma\,=\,0$, $c_+\,=\,\frac{c}{1-\varepsilon}$ and $c_-\,=\,\frac{-c}{1+\varepsilon}$ would be mere metric coefficients, devoid of any operational meaning.

We may equivalently write equation (3.7.16)

$$ds\,=\,c\,\frac{\big(1\,-\,\varepsilon\big)^{(\sigma\,-\,\frac{1}{2})}}{\big(1\,+\,\varepsilon\big)^{(\sigma\,+\,\frac{1}{2})}}\sqrt{\Big(dt^2\,+\,\frac{2\,\varepsilon}{c}\,dt\,dx\,-\,\frac{1\,-\,\varepsilon^2}{c^2}dx^2\Big)}\,\Big(\frac{dt\,+\,\frac{1\,+\,\varepsilon}{c}\,dx}{dt\,-\,\frac{1\,-\,\varepsilon}{c}\,dx}\Big)^{\sigma}. \tag{3.7.16}$$

The quantity under the square root is positive by pre-causality condition of equation (3.5.1) and we also observe that it is nothing else that the anisotropy pseudo-norm of equation (1.5.8) found by Winnie in Reichembach framework. As a consequence the difference between a purely conventional anisotropy theory from one which contains also a physical anisotropy is expressed in ds by the factor which contains the parameter σ. So we can interpreter this parameter as the parameter which measure real anisotropy.

If we perform the following change in coordinates variables

$$\begin{cases} T = t + \dfrac{\varepsilon}{c}\, x \\ X = x \end{cases} \tag{3.7.17}$$

the Finsler pseudo-norm (3.7.16) can be rewritten in a more easy way

$$ds = \frac{\left(1 - \varepsilon\right)^{(\sigma - \frac{1}{2})}}{\left(1 + \varepsilon\right)^{(\sigma + \frac{1}{2})}} \sqrt{\left(c^2\, dT^2 - dX^2\right)} \left(\frac{c\, dT + dX}{c\, dT - dX}\right)^{\sigma}. \tag{3.7.18}$$

A minimum request we have to do, is that the pseudo-Finsler metric is a continuous function on the entire slit tangent bundle $TM \setminus 0$, where zero here reppresents the zero-section of TM. According to this, the parameter σ is limited by the condition $|\sigma| < \frac{1}{2}$ or equivalently, using the old parameters we have to force the condition $|\lambda + \rho| < \frac{1}{c}$.

First of all we observe that special relativity theory is a particular case of our model: it is reacquire if we put both in (3.7.16) and in (3.7.18) $\varepsilon = 0$ and $\sigma = 0$. In these new coordinates the quantity ε appear only in a function which is a Finsler metric's conformal factor: as it must be if we remember that restriction $|\,\varepsilon\,| < 1$ was made to ensure a globally causal ordering of events. So we must have the same conformal structure in Finsler manifold for every arbitrary value of the Reichenbach's conventional parameter.

We conclude this section saying that the most general spacetime structure compatible with relativity principle is a pseudo-Finslerian, not a pseudo-Riemannian, one. What is really important in equation (3.7.18) is that the pseudo-Finslerian character is unavoidable when one wants to incorporate, through the parameter σ, a real physical space anisotropy. The other kind of anisotropy, linked to the parameter ε, is a matter of convention rather than a physical feature. One can always eliminate such an anisotropy by synchronising clocks according to the Einstein procedure (we can eliminate it by a stipulation, putting $\varepsilon = 0$, that is, we can always choose that light propagate isotropically in space even if there is a privileged direction!). On the other hand, a nonzero σ implies true chronogeometric effects (anisotropic time dilation and length contraction, modified dispersion relation), that cannot be gauged away by a stipulation, as we will see in next section.

3.8 What we Really Mean by Isotropy?

What happens if one ignores the possibility that there is anisotropy? That is, suppose one synchronises clocks according to the Einstein procedure, so

$\varepsilon = 0$, regardless of the fact that space might be kinematically anisotropic. Is not one "forcing" isotropy upon the theory? But then, in which sense can we claim that isotropy is a physical property, if one can implement it just by a stipulation?

The answer is that, although one can set ε to zero, σ is still around, and its value does not depend on the convention chosen in order to synchronise clocks. Now, if one considers two clocks moving at the same speed along opposite directions, these clocks will not delay but the same amount, if $\sigma \neq 0$ as we will see in next subsection. Hence, one has that anisotropy can manifest itself through physical effects, independent of the choice of synchronisation.

We can claim that σ is the parameter measuring the *real*, physical anisotropy, whereas ε is associated to a purely conventional anisotropy: one that can be gauged away simply by a stipulation. In fact, the value of ε merely reflects a choice of coordinates in spacetime.

There is a geometrical analogue of this situation. Imaging that someone wants to find out whether a two-dimensional surface is inhomogeneous. A possible way is to construct, at different places on the surface, identical triangles, measure the sum of the internal angles, and compare the results. On an inhomogeneous surface, the outcomes should disagree. Of course, there is a very easy way of making even a flat surface *appear* inhomogeneous with such kind of measurements: It is enough to choose rulers of different length at different places. Hence, in general, there will be two sources of inhomogeneity: a conventional one, linked to a funny choice of rulers, and a real one, related to the actual properties of the surface. These are the analogs of the parameters ε and σ, respectively. However, the conventional choice can be regarded merely as a transformation from coordinates "adapted" to the surface to arbitrary ones. In no way, the choice of "good" rulers (or coordinates) can affect the geometrical properties of the underlying surface.

The fact that one may not use Einstein synchronisation is a trivial one, and has no physical content. On the contrary, the fact that one *can* adopt it in any inertial frame is physically not trivial, as it reflects the validity of the principle of relativity.

There is another analogy: The principle of inertia. The statement that, in an inertial frame, a force-free particle moves along a straight line at a constant speed contains actually two implications, very different in nature. One, that the motion takes place along a straight line, is a physically testable prediction, since the notion of a straight line is well defined in the Euclidean geometry that one presupposes when discussing the principle. The other, that motion is uniform, is a matter of convention. Of course, one could choose the "time" variable in such a way that motion is *not* uniform (the analogue of choosing a synchronisation different from Einstein's), but this generalisation will lead to no new physical phenomena: only to a horrendous complication in the formulation of the laws of mechanics. Again, the relevant fact is not that one can make an absurdly complicated choice of time, but that one can make a choice that makes life simpler. Since a change of time variable does not entail new phenomena, we are confronted with a mere gauge, so the wisest choice is to use the simplest possible gauge. As Wheeler concisely and effectively wrote [1]: "Time is defined so that motion looks simple". We could paraphrase him saying: "Clocks are synchronised so that physics looks simple" .

3.8.1 One Special Case

We can now study the following simpler situation: we choose

$$\begin{cases} \lambda = \rho \\ \mu < 0 \end{cases} \tag{3.8.1}$$

that is, we are in the usual $\Delta > 0$ case with $c_2 = -c_1$, so if our identification between velocity of light and our model's invariant speed is right, we are in a occurrence by which light propagates isotropically; furthermore reciprocy principle holds, as we see in section (3.6).

From equation (3.7.13)

$$\begin{cases} c = \dfrac{2}{\sqrt{-\mu}} \\ \varepsilon = 0 \\ \sigma = \dfrac{2\,\lambda}{\sqrt{-\mu}} \end{cases} \tag{3.8.2}$$

the Finsler line element become

$$\begin{aligned} ds &= \sqrt{\left(c^2\,dT^2 - dX^2\right)}\left(\frac{c\,dT + dX}{c\,dT - dX}\right)^{\sigma} \\ &= \left(\frac{cdT + dX}{cdT - dX}\right)^{\sigma} ds_{sr} \end{aligned} \tag{3.8.3}$$

where ds_{sr} is the Minkowski line element; for future handling will be useful also the following form

$$ds = \Big(c\,dT - dX\Big)^{\frac{1}{2} - \sigma} \Big(c\,dT + dX\Big)^{\sigma + \frac{1}{2}}. \tag{3.8.4}$$

Mathematically this is equivalent to say that we are working with a locally Minkowskian pseudo-Finsler two dimensional manifold with a Finsler pseudo-metric

$$\begin{aligned} F(u,v) &= \Big(u - v\Big)^{\frac{1}{2} - \sigma} \Big(u + v\Big)^{\sigma + \frac{1}{2}} \\ &= \sqrt{u^2 - v^2}\left(\frac{u + v}{u - v}\right)^{\sigma} \\ &= (u^2 - v^2)^{\frac{1}{2} - \sigma}\,(u + v)^{2\sigma} \end{aligned} \tag{3.8.5}$$

where $(u,\, v)$ are coordinates of a tangent vector of $T_p\, M \backslash\, 0$ and p is a point in our two dimensional manifold M with coordinates, in the fixed generalized Lorentz chart, $(c\,t,\, x)$.

We can now easily write the fundamental quantity

$$g_{ik}(u,v) = \left(\frac{u + v}{u - v}\right)^{2\,\sigma} \eta_{ik} \tag{3.8.6}$$

we indicate by $g^{\alpha\beta}(y)$ the inverse matrix of $g_{\alpha\beta}(y)$, it is a simple matter to calculate this inverse pseudo-Finslerian metric and obtain

$$g^{\alpha\beta}(y) = \left(\frac{u + v}{u - v}\right)^{-2\,\sigma} \eta^{\alpha\beta} \tag{3.8.7}$$

by which, as we already saw in second chapter we can write

$$ds^2 = F^2(y) = g_{i,k}(y)\, y^i\, y^k \tag{3.8.8}$$

where we have put $y = (y^0 = u;\, y^1 = v)$.

The coordinate transformation become

$$\begin{cases} x' = \left(\dfrac{1 - \dfrac{v}{c}}{1 + \dfrac{v}{c}}\right)^{\frac{\sigma}{2}} \dfrac{x - v\,t}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\ \\ t' = \left(\dfrac{1 - \dfrac{v}{c}}{1 + \dfrac{v}{c}}\right)^{\frac{\sigma}{2}} \dfrac{t - \dfrac{v\,x}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \quad . \end{cases} \tag{3.8.9}$$

As a result of these equations, the Einstein addition velocities law is reobtained. It is a simple matter to show that despite of isotropically propagation of light, we are in a anisotropy framework for clocks and rods; in fact, for time dilation we find

$$t' = \left(\frac{1 - \beta}{1 + \beta}\right)^{\frac{\sigma}{2}} \frac{t}{\gamma} \quad . \tag{3.8.10}$$

In a first order MacLaurin expansion in β

$$t' = \big(1 - \sigma\,\beta + o(\beta)\big)\,t \tag{3.8.11}$$

where as customary we have defined

$$\begin{aligned} \beta &= \frac{v}{c} \\ \gamma &= \frac{1}{\sqrt{1 - \beta^2}} \end{aligned} \tag{3.8.12}$$

so we have an anisotropic time dilation factor that contains a term which is linear in v. Similarly, for rods contraction we have

$$l'_0 = \left(\frac{1 - \beta}{1 + \beta}\right)^{\frac{\sigma}{2}} \gamma\,l \tag{3.8.13}$$

where l'_0 is rod's length measured in rest reference frame S' and l is the value measured by an observer in S reference frame. We also infer that, differently from special relativity, the scales perpendicular to motion direction must be deformed.

If $\sigma \neq 0$ characterizing the magnitude of space anisotropy, is sufficiently small, then the factor, which distinguishes the generalized Lorentz transformations from the usual ones, becomes markedly different from unity only at relative velocities of the inertial frames extremely close to the velocity of light, as we can check from equation (3.8.11).

In the physics of ultra-high energy cosmic rays we deal with precisely such a situation. Therefore, the use of the "generalized Lorentz transformation" instead of the usual ones makes it possible, in principle, to remove the discrepancy between theory and experiment in this field; this may be regarded as a hint toward a local anisotropy of space.

By parallelism with pseudo-Riemannian geometry, we define in our Finsler framework, a physical light cone in a give spacetime events as the totality of trajectories passing through that point such that $ds^2 = 0$, where ds is given in equation (3.8.4). From this equation we can deduce another remarkable property of our anisotropic event space, infact it keeps the conformal structure (light cones) of Minkowski space, that is, light propagates according to the equation $c^2\, dT^2 - dX^2 = 0$ although we are dealing with a true anisotropic space.

Therefore, the velocity of light is independent of the direction of its propagation and is equal to c: this is not surprising because the assumption $\lambda - \rho = 0$ is equivalent to synchronize clocks using Einstein method (that is we are considering the choice $\varepsilon = 0$ in Reichenbach theory, as we state in equation (3.8.2).

We can conclude that light propagates along Minkowskian geodesics whereas free bodies move along pseudo-Finslerian ones. This feature of our model allow us to project cosmological experiment to test spacetime's pseudo-Finsler geometry for massive particle in order to obtain experimental estimations on σ.

3.8.2 Four Dimensional Generalization

Another remarkable consequence of line element (3.8.3) is that the corresponding 4D Finslerian pseudo-metric can be found if we note that from the 2D Finsler pseudo-metric (3.8.3) we can easily write

$$ds^2 = \eta_{\alpha\beta}\, y^\alpha\, y^\beta \left[\Big(\frac{\eta_{\mu\kappa}\, \nu^\mu\, y^\kappa\big)^2}{\eta_{\lambda\tau}\, y^\lambda\, y^\tau} \right]^{2\sigma} \tag{3.8.14}$$

where from the time being all indices run from 0 to 1, $(y^0,\, y^1)$ are coordinates in a given chart of an arbitrary slit fiber bundle point y, $\nu = (1,\, -1)$ is a constant vector and $\eta_{\alpha\beta}$ is the $(\alpha,\, \beta)$ component of Minkowski $(1+1)$ pseudo-metric

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.8.15}$$

we note that in this simple model the slit tangent bundle vector ν is light-like, that is

$$\nu^2 = g_{\alpha\beta}\, \nu^\alpha\, \nu^\beta = 0 \quad . \tag{3.8.16}$$

The four dimensional pseudo-Finsler metric is still equation (3.8.14) but with all indices running from 0 to 3; if equation (3.8.16) is still true in 4D we can write

$$ds^2 = (c^2\, dt^2 - d\vec{x}^2) \left[\frac{(c\, dt - \vec{\nu} \cdot d\vec{x})^2}{c^2\, dt^2 - d\vec{x}^2} \right]^{2\sigma} . \tag{3.8.17}$$

Spatial anisotropy is characterized by this pseudo-Finslerian metric which depends on two constant parameters, the scalar σ and the dimensionless unit vector $\vec{\nu}$. Equation (3.8.17) describes a flat spacetime with partially broken rotational symmetry. Instead of the 3-parameter group of rotations of Minkowski space, the spacetime (3.8.17) admits only the 1-parameter group of rotations about the unit vector $\vec{\nu}$, which indicates a preferred direction in 3D space. Of course, by our model construction, no changes occur for translational symmetry: spacetime translations leave the metric (3.8.17) invariant.

About σ parameter we can say that its value should characterize the degree to which Lorentz invariance is broken in nature. It seems that σ is not a combination of well-known fundamental physical constants, but instead, is meant to be a dimensionless fundamental one. This parameter is universal in the sense that it is of pure geometrical origin, that is, the corrections to the pseudo-Riemannian geometry of spacetime are introduced through this parameter. Merely, the parameter σ evaluates the degree of Finslerian non-Riemannianity of spacetime.

We also note that ds^2 is a regular and vanishing quantity in the limit $c^2\, dt^2 - d\vec{x}^2 \to 0$ because of the condition $|\sigma| < \frac{1}{2}$.

Moreover, since we already kwon that light propagates according to equation

$$c^2\, dt^2 - d\vec{x}^2 = 0 \tag{3.8.18}$$

that is, light velocity is independent of the direction of its propagation and is equal to c, it thus appears that the square of the distance dl^2 between adjacent points of 3D space, determined by means of exchange of light signals, is expressed by the formula $dl^2 \; = \; d\vec{x}^2$.

Thus, although in the 3D space there is a preferred direction, its geometry remains Euclidean. But, what does the anisotropy physically manifest itself in? For example, as we saw previously, it affects the dependence of proper time of a moving clock by including the direction of its velocity in addition to the magnitude.

Indeed, in four dimensional case, equation $ds^2 = 0$ has also solutions different from which define the usual light cone, these are

$$c\,dt \; - \; \vec{\nu} \cdot d\vec{x} \; = \; 0 \tag{3.8.19}$$

these solutions describe the trajectory with velocity such that $\vec{\nu} \cdot \vec{v} = c$, as a consequence $\|\vec{v}\| \geq c$, so they can violate conditions (3.2.5) and we have to reject they. We also observe that this solutions are the only solutions of equation $ds^2 = 0$ when $\sigma = \frac{1}{2}$ and as a consequence the notion of spatial extension disappearres. This is due to the absence of a light cone and, consequently, of the possibility of determining spatial distances using the exchange of light signals. According to (3.8.17), the interval $d\tau$ of proper time read by the clock moving with a velocity $\vec{v}$, is related to the time interval dt read by clocks at rest by the relation

$$d\tau \; = \; \frac{dt}{\gamma}\left[\left(1 \; - \; \frac{\vec{\nu} \, \cdot \, \vec{v}}{c}\right)\gamma\right]^{2\,\sigma} \tag{3.8.20}$$

in contrast to Minkowski space (for which the moving clock is always slow in comparison with the clock at rest), in the anisotropic space the time dilatation factor can take on values greater than unity. Therefore, at some of its velocities the clock moving in the anisotropic space is fast in comparison with the clock at rest. However, having returned to its starting point, it will necessarily run behind the clock at rest. Consequently, inertial motion is still uniform and along a straight line.

We stress that in this model, due to the presence of $\sigma \neq 0$ parameter, the possibility of a time *contraction* is a consequence of the existence of a real anisotropy; differently in Reichenbach theory it is merely a matter of convention (that is quantity has an ε-dependence) whether or not clocks moving along a one-way path go slower or faster.

Our last observation concern equation (3.8.14) when it is referred to the 4D case: we gain it by a straightforward generalization of 2D's case. With the aid of Euler's Theorem is easy to demonstrate that such an equation is not the more general we can work with using only $\eta_{\alpha\beta}\, y^\alpha\, y^\beta$ and $\eta_{\mu\kappa}\, \nu^\alpha\, y^\beta$. Every Finsler pseudo-metric of the form

$$ds^2 \;=\; \eta_{\alpha\beta}\, y^\alpha\, y^\beta\, F\Big(\frac{[\eta_{\mu\kappa}\, \nu^\mu\, y^\kappa]^2}{\eta_{\alpha\beta}\, y^\alpha\, y^\beta}\Big) \tag{3.8.21}$$

where F is an arbitrary real function, is an equally well defined homogeneous of degree two pseudo-Finsler metric.

3.8.3 Further Generalization: Curved Spacetime

One of the possible mechanisms of the appearance of a local anisotropy in spacetime is the induced phase transition in its geometric structure, caused by the breakdown of higher gauge symmetries and by the appearance of masses in fundamental fields of matter. This involves changes in the metric properties of spacetime manifold and it goes over from a state described by pseudo-Riemann geometry into a state described by Finsler geometry. Since Finslerian spacetime differs from pseudo-Riemannian spacetime by the anisotropy of its tangent spaces, in such a transition there occurs a violation of the local Lorentz symmetry. In the course of subsequent expansion of the Universe the initial strong local anisotropy of the Finslerian spacetime monotonically decreases and, on the average, tends to zero together with its curvature. Gradually the local Lorentz symmetry of spacetime is also restored.

In our framework scheme it is interesting to note, first of all, that we can generalize equation (3.8.14) in a easy way also to a curved spacetime. We should consider the parameters σ and $\vec{\nu}$ not as constants but as fields over spacetime with the matter-energy distribution as their source (this point of view produce a completely Machian theory, we stress here that such a result cannot be obtained if $\sigma = 0$).

So spacetime acquires a local anisotropy varying from point to point and we can straightforwardly generalize equation (3.8.14) in this way:

$$ds^2 \;=\; g_{\alpha\beta}\, y^\alpha\, y^\beta \left[\frac{\big(g_{\mu\kappa}\, \nu^\mu\, y^\kappa\big)^2}{g_{\lambda\tau}\, y^\lambda\, y^\tau}\right]^{2\,\sigma} \tag{3.8.22}$$

the given pseudo-Finsler metric is a function of three fields: $\sigma = \sigma(x)$, a scalar field determining the magnitude of local spatial anisotropy, $g_{\alpha\beta} = g_{\alpha\beta}(x)$, the field of a Riemannian metric tensor and finally $\nu^\alpha = \nu^\alpha(x)$ $(\alpha = 0, .., 3)$, a vector field of locally preferred directions in spacetime satisfying the condition $g_{\alpha,\beta}(x)\, \nu^\alpha(x) \nu^\beta(x) = 0$. Here $x = (x^0, .., x^3)$ are the coordinates of an event

P in a fixed arbitrary chart and $y = (y^0, .., y^3)$ is the induced coordinate representation of a tangent vector in $T_P M \backslash \{0\}$.

This Finsler pseudo-metric supply two significant properties: first of all at each manifold points, the curved pseudo-Finslerian spacetime (3.8.22) has its own locally anisotropic tangent space with a pseudo-Finsler metric given by equation(3.8.17) with its own values of the parameters σ and $\vec{\nu}$ which determine the local anisotropy. These values of the parameters are none others than the local values of the corresponding fields $\nu^\alpha(x)$ with $\alpha = 0, .., 3$ and $\sigma(x)$; that is at different manifold point we have different tangent spaces which differ by the values of $\sigma(x)$ and $\vec{\nu}$.

Secondly, the principle of correspondence with the pseudo-Riemannian metric of the curved locally isotropic spacetime of general relativity is satisfied, because when $\sigma = 0$ Einstein's theory of gravitation is reobtained.

We note here that equation (3.8.22) is the same law deduced by Bekenstein [84] in a wholly different framework with two geometries in a single gravitational theory: one geometry describes gravitation while the other defines the geometry in which matter plays out its dynamics. Obviously for all these two geometries theories the strong equivalence principle is violated, but they usually preserve weak equivalence. The analogy between our equation (3.8.22) and equation written by Bekenstein here reported for completeness

$$\begin{aligned} ds^2 &= g_{\alpha\beta}(x)\, y^\alpha\, y^\beta\, F(I, H, \psi) \\ I &\equiv L^2\, g^{\alpha\beta}\, \psi_\alpha\, \psi_\beta \\ H &\equiv \frac{L^2 \left(\psi_{,\alpha}\, y^\alpha\right)^2}{-\, g_{\alpha\beta}\, y^\alpha\, y^\beta} \end{aligned} \tag{3.8.23}$$

is clear if we consider a dimensionless scalar field $\psi = \psi(x)$ such that

$$\nu_\alpha = L\, \psi_{,\alpha} \equiv L\, \frac{\partial \psi}{\partial x^\alpha} \tag{3.8.24}$$

where L is a length scale and F is a fixed, unknown, dimensionless function one and the same for all coordinates systems. In our model we gave an explicit mathematical form to F. In such a situation we have the appearance of a "4-vector" which has the same form in all reference frames.

In two-geometries approach one invoke a Riemannian metric $g_{\alpha\beta}(x)$, then he built the Einstein-Hilbert action. In our case according to equation (3.8.22), the dynamics of Finslerian spacetime is completely determined by the dynamics of the gravitational field $g_{\alpha\beta}(x)$ and of the fields $\sigma(x)$ and $\nu^{\alpha}(x)$, responsible for local anisotropy. Since these three fields interact with each other and with matter, for a description of the dynamics it is necessary to construct equations which generalize the corresponding Einstein equations.

Our last observation has a topological character, in fact in our model the point $H = +\infty$ of the geometry is to be identified with $H = -\infty$, this is because the passage from one to another correspond to $g_{\alpha\beta}(x)y^{\alpha}y^{\beta}$ passing through zero from negative to positive values and the line element ds^2 is continuous as we see previously (because $|\sigma| < \frac{1}{2}$), when H jumps from $+\infty$ to $-\infty$.

Chapter 4

Single Particle Generalised Dynamics and Dispersion Relation

It is well known that in special relativity all fundamental equations are invariant under the transformations of the Poincaré group, the isometry group of Minkowski space. If the event space is described by the Finslerian pseudometric (3.8.17) of our simple example seen in the previous chapter, then the complete inhomogeneous group of its isometries turns out to be an 8-parameter group: along with spacetime translations (four parameters), the generalized Lorentz transformations (three parameters) and the group includes only a 1-parameter subgroup of rotations of 3D space about some preferred direction.

In this case, the fundamental relativistic equations must be modified in accordance with the requirement of invariance under this group. The requirement just formulated represents a generalization of the principle of relativity for the locally anisotropic spacetime.

4.1 General Aspects

The equations of relativistic mechanics, which satisfy the "special principle of relativity" for the locally anisotropic space, can be obtained if in the action integral

$$S = -m\,c \int ds \tag{4.1.1}$$

we replace the Minkowskian expression for ds by the pseudo-Finslerian expression (3.8.17) to obtain

$$S \,=\, -\,m\,c^2 \int \left(\frac{1 \,-\, \frac{\vec{\nu}\cdot\vec{u}}{c}}{\sqrt{1 \,-\, \frac{\vec{u}^2}{c^2}}} \right)^{\sigma} \sqrt{1 \,-\, \frac{\vec{u}^2}{c^2}}\, dt \tag{4.1.2}$$

more precisely we are saying that it is consistent with all our previous discussion to postulate that the classical trajectory of free particles are those which extremize the action (4.1.2).

As a result, the Lagrangian function corresponding to a free particle with rest mass[1] m in the locally anisotropic space, takes the form

$$\begin{aligned} L \,&=\, -\,m\,c\,F(v) \\ &=\, -\,m\,c^2 \left(\frac{1 \,-\, \frac{\vec{\nu}\cdot\vec{u}}{c}}{\sqrt{1 \,-\, \frac{\vec{u}^2}{c^2}}} \right)^{2\,\sigma} \sqrt{1 \,-\, \frac{\vec{u}^2}{c^2}} \end{aligned} \tag{4.1.3}$$

Using the general formulas for the momentum

$$\vec{p} \,=\, \frac{\partial\, L}{\partial\, \vec{u}} \tag{4.1.4}$$

and energy

$$E \,=\, \vec{p}\,\cdot\,\vec{u} \,-\, L \tag{4.1.5}$$

we can build up the “anisotropic” dynamics.

[1] Indeed, in the following we will give some observations about what in our model is the “rest mass” of a particle, the usual meaning we will change toward a Machian concept of mass.

4.2 One Dimensional Case

In one dimensional case we can use equations (3.8.4) and (4.1.4) to obtain for the momentum

$$\begin{aligned} p(u) &= m\,\gamma(u)\,\Big(\frac{1\,+\,\frac{u}{c}}{1\,-\,\frac{u}{c}}\Big)^{\sigma}\,(u\,-\,2\,\sigma\,c) \\ &= m\,c\,\frac{\big(1\,+\,\frac{u}{c}\big)^{\sigma-\frac{1}{2}}}{\big(1\,-\,\frac{u}{c}\big)^{\sigma+\frac{1}{2}}}\,(\frac{u}{c}\,-\,2\,\sigma) \end{aligned} \tag{4.2.1}$$

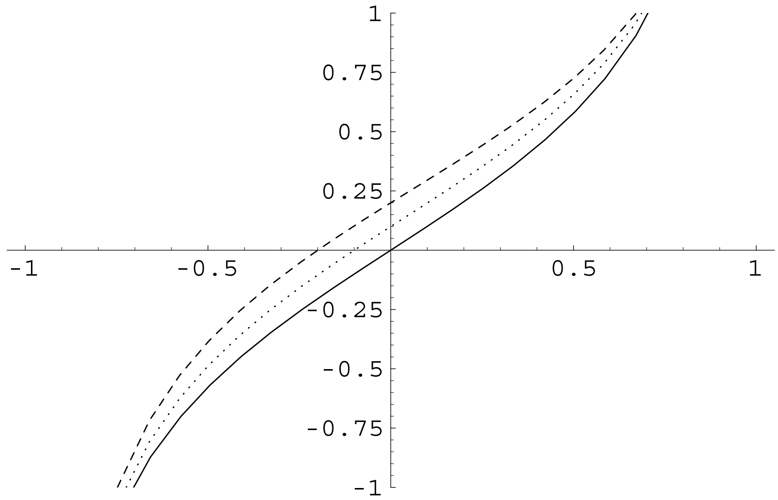

Figure 4.1: Plot of $p/m\,c$ as a function of u/c for different values of the anisotropic parameter σ. The special relativistic case, $\sigma = 0$, is the solid line. The dashed line is the anisotropic case with $\sigma = -0.1$, and the dot line is the anisotropic momentum with $\sigma = -0.05$.

From equation (4.1.5) we gain the energy

$$\begin{aligned} E(u) &= m\,c^2\,\gamma(u)\,\Big(\frac{1\,+\,\frac{u}{c}}{1\,-\,\frac{u}{c}}\Big)^{\sigma}\,(1\,-\,\frac{2\,\sigma\,u}{c}) \\ &= m\,c^2\,\frac{\big(1\,+\,\frac{u}{c}\big)^{\sigma-\frac{1}{2}}}{\big(1\,-\,\frac{u}{c}\big)^{\sigma+\frac{1}{2}}}\,(1\,-\,\frac{2\,\sigma\,u}{c}) \quad . \end{aligned} \tag{4.2.2}$$

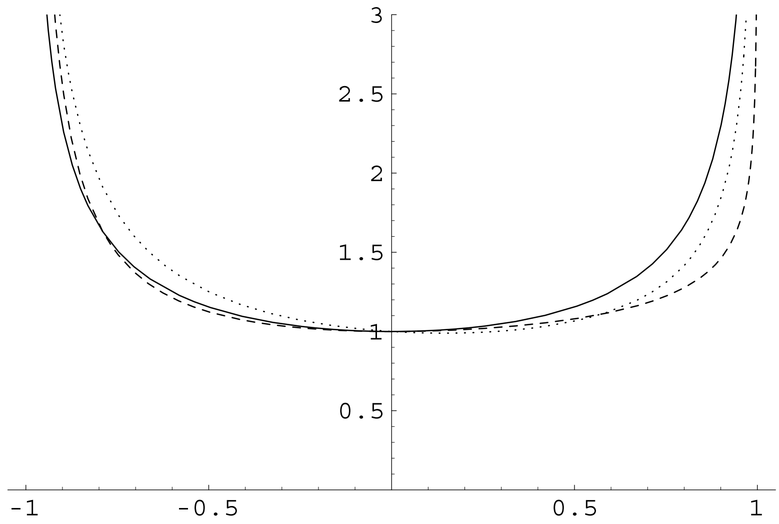

Figure 4.2: Plot of $E/m\,c^2$ as a function of u/c for different values of the anisotropic parameter σ. The special relativistic case, $\sigma = 0$, is show as solid. The dashed line is the anisotropic case with $\sigma = -0.3$, and the dot line is the anisotropic energy with $\sigma = -0.08$.

Equations (4.2.1) and (4.2.2) give the new relation between energy-momentum and velocity.

We first observe that conditions $|\sigma| < \frac{1}{2}$ and $|\frac{u}{c}| < 1$ assure positivity of single's particle energy. Obviously, when space is isotropic, that is when $\sigma = 0$, we gain the usual result of special relativistic theory, and for low velocity the Newtonian physics. Furthermore we observe that the energy is an even function on u (as both in classical physics and in special relativity) only if $\sigma = 0$ or for (the not permitted) values $|\sigma| = \frac{1}{2}$, in these case energy take the constant value $E = mc^2$.

We also observe from equations (4.2.1) and (4.2.2) that even in the case $u = 0$ the momentum of a particles does not vanish if $\sigma \neq 0$, there remains a "rest momentum" $p_0 := -2m\sigma c$, whereas energy reaches for every values of σ its absolute minimum "rest energy" $E_0 := mc^2$. The difference $p - p_0$ could be denoted the "kinetic momentum" in analogy with the "kinetic energy" $T = E - E_0$.

Moreover the momentum, if $\sigma \neq 0$ is not an even function. The MacLaurin series on u is

$$p(u) \,=\, -\,2\,m\,\sigma\,c \,+\, m\,u\,(1 \,-\, 4\,\sigma^2) + \\ +\, m\,\frac{u^2}{c}\,\sigma\,(1 \,-\, 4\,\sigma^2) + \\ +\, m\,\frac{u^3}{c^2}\,[\frac{1}{2} \,-\, 2\,\sigma^2\,(\sigma^2 \,+\, \frac{41}{4})] \,+\, o(u^3) \qquad (4.2.3)$$

as we already said for the value $\sigma = 0$ we have $p(u) \,=\, m\,u \,+\, \frac{m}{2}\,u^3 \,+\, o(u^3)$ which coincides with the power series of special relativity case and energy is not an odd function, it holds

$$E(u) \,=\, m\,c^2 \,+\, m\,u^2\,(\frac{1}{2} \,-\, 2\,\sigma^2) + \\ +\, m\,\frac{u^3}{c}\,\sigma\,(\frac{41}{4} \,-\, \sigma^2) \,+\, o(u^3) \quad . \qquad (4.2.4)$$

Therefore, from equations (4.2.3) and (4.2.4) we deduce that also nonrelativistic mechanics as a whole is different from the Newtonian case, this is not surprising because being an intrinsic property of space, anisotropy is independent of the magnitude of relative velocities.

Since within the framework of nonrelativistic mechanics the "rest mass" m is an additive quantity, the occurrence of the constant terms mc^2 and $-2\sigma mc$ in the above equations does not affect the conservation laws and the equations of motion. As a result, these terms can be omitted, and the kinetic energy and kinetic momentum, read off from equations (4.2.3) and (4.2.4) at the lowest order are

$$T(u) \,=\, \frac{m}{2}\,(1 \,-\, 4\,\sigma^2)\,u^2 \qquad (4.2.5)$$

$$p(u) \,=\, m\,(1 \,-\, 4\,\sigma^2)\,u \quad . \qquad (4.2.6)$$

Thus it seems that, by comparison with Newtonian's counterparts equations, the inertial properties of a nonrelativistic particle in anisotropic space is specified by the quantity $m\,(1 \,-\, 4\,\sigma^2)$ which depends openly on the σ parameter.

We infer that in 3D the usual Newtonian's inertial mass will be replaced by a "tensor of inertial mass" [13].

If now we write equation (3.6.21) of previous chapter using equations (4.2.2) and (4.2.1) we can estimate the values for parameters η, τ and ν in our anisotropic one dimensional model; it is straightforward to achieve

$$\begin{cases} \nu \,=\, 0 \\ \\ \eta \,=\, 1 \\ \\ \tau \,=\, -\,\dfrac{2\,\sigma}{c} \end{cases}$$

as a consequence the momentum p for a single particle defined in equation (4.2.1) is an additive and conservative quantity in any theory in which there are localised interactions between particles which do not change the total energy and a relativity principle holds [82].

Energy's asymptotic behavior is $E \approx 1/(1+\beta)^{1/2-\sigma}$ and $E \approx 1/(1-\beta)^{1/2+\sigma}$ for $\beta \to -1^{+}$ and $\beta \to 1^{-}$ respectively, different from special relativistic case. This feature will allow us to explain the absence of the GZK cut-off, as we will see in the following.

4.2.1 Dispersion Relation

The functional relation between kinetic energy and momentum is the particle version of a dispersion relation. In general, such a relation takes the form $g(E, p) = 0$ where g is a function that can be locally inverted to find the particle Hamiltonian $H(p)$.

In special relativity the dispersion relation of a particle is given by

$$s_i\, s^i \,=\, m^2\, c^2 \tag{4.2.7}$$

where m is particle's inertial mass, $s^0 = m\gamma(u)$ and $s^i = mu^i\gamma(u)$ with $u^i = \frac{dx^i}{dt}$; these relations have been tested for Lorentz factor values $\gamma \leq 10^4$, and were found to hold with a relative precision of $5 \cdot 10^{-4}$ [85]. However, at high γ, we cannot exclude the possibility that this dispersion relation might be violated.

The search for observable effects of quantum gravity has led researchers to focus mostly on modifications of the dispersion relations for elementary particles, leading to deviations from standard Lorentz invariance. Generically these modified dispersion relations can be cast in the form

$$\frac{E^2}{c^2} \,-\, p^2 \,+\, f(E, p, k) \,=\, m^2\, c^2 \tag{4.2.8}$$

where k denotes the mass scale at which the quantum gravity corrections become appreciable. Normally, one assumes that k is of order the Planck mass: $k \approx M_p \approx 1.22 \cdot 10^{19}\mathrm{GeV}/c^2$. Most interestingly, it was shown that several significant constraints can be put on the intensity of the Lorentz violating term f using current experiments and observations [86]. An open issue is the interpretation of the origin of such deformed dispersion relations. DSRs (deformed or doubly special relativity theories) attempt to "deform" special relativity in momentum space[2], by introducing non-standard "Lorentz transformations" that leave the modified dispersion relations invariant.

On the other hand, that Lorentz violating theories should generically predict the existence of privileged reference systems can be easily inferred if propagating particles have general functions of energy as dispersion relations. Consider for instance a photon: if its dispersion relation does not have the usual Lorentz covariant form, but propagate with an energy dependent velocity $v(E)$, the statement $E = ck$, can be at best valid in one specific inertial frame.

This selects a preferred frame of reference, where a particular form of the equations of motion is valid, and one should then be able to detect the laboratory velocity with respect to that frame. It is this fact that opens the possibility of detecting tiny violations of Lorentz symmetry.

In our framework the dispersion relation we are looking for takes the form

$$h(p^i) \;=\; m^2\, c^2 \tag{4.2.9}$$

where $h(p_i)$ is a homogeneous function of second degree in p_i. This conditions follows from the fact that the Lagrangian is a homogeneous function of first degree in dx^i/dt. We restrict ourselves to functions of the form

$$h(\sigma, p^i) \;=\; f(\sigma, p^i)\, [(p^0)^2 \;-\; (p^1)^2] \tag{4.2.10}$$

where $f(\sigma, p^i)$ is a homogeneous positive function of the momenta of zero degree; for small velocities, and with $\sigma = 0$ we must have $f(0, p^i) \cong 1$, as in the usual theory.

In our model it can be verified by direct substitution that energy and

[2]A brief note on terminology: the claim that Lorentz invariance is deformed (and not broken) states that the commutation relation among the boots and rotations are not altered and the DSR group acts on the momenta in a nontrivial manner.

momentum are linked by the formula

$$\left(\frac{E^2}{c^2} - p^2\right)\left[\frac{\frac{E}{c} - p}{\frac{E}{c} + p}\right]^{2\,\sigma} = m^2\,c^2\,(1 + 2\,\sigma)^{1+2\,\sigma}(1 - 2\,\sigma)^{1-2\,\sigma} \qquad (4.2.11)$$

this relation determines the square of the pseudo-Finslerian length of the $(1+1)$-momentum. So in our model, if we define for convenience

$$k(\sigma) = (1 + 2\,\sigma)^{1+2\,\sigma}(1 - 2\,\sigma)^{1-2\,\sigma} \qquad (4.2.12)$$

the "deviation function" f is

$$f(\sigma, p^i) = \frac{1}{k(\sigma)}\left(\frac{\frac{E}{c} - p}{\frac{E}{c} + p}\right)^{2\,\sigma}. \qquad (4.2.13)$$

We observe that the left hand members in equation (4.2.11) is well defined even if $\frac{E^2}{c^2} - p^2 = 0$ by the condition $|\sigma| < \frac{1}{2}$, but in this case we must fix $m = 0$. So, as in special relativity theory for massless particle we have the usual dispersion relation

$$\frac{E^2}{c^2} - p^2 = 0 \qquad (4.2.14)$$

but here, E and p are defined in a very different way with respect their counterparts in special relativity, we will investigate this issue in next section.

4.3 Three-Dimensional Case

Equations (4.1.3), (4.1.4) and (4.1.5), easily lead to the following expression for the momentum

$$\vec{p} = m\,c\,\gamma\left[\gamma\,(1 - \frac{\vec{u}\cdot\vec{\nu}}{c})\right]^{2\,\sigma}\left[(1 - 2\,\sigma)\,\frac{\vec{u}}{c} + \frac{2\,\sigma\,\vec{\nu}}{\gamma^2\,(1 - \frac{\vec{u}\cdot\vec{\nu}}{c})}\right] \qquad (4.3.1)$$

and energy

$$E = m\,c^2\,\gamma\left[\gamma\,(1 - \frac{\vec{u}\cdot\vec{\nu}}{c})\right]^{2\,\sigma}\left[1 - 2\,\sigma + \frac{2\,\sigma}{\gamma^2\,(1 - \frac{\vec{u}\cdot\vec{\nu}}{c})}\right] \qquad (4.3.2)$$

if $\sigma \neq 0$ the directions of velocities and, hence, the tracks of particles do not coincide with the directions of their momenta, so the conservation law of the total momentum does not lead to the fact that in a collision between two particles, one of which initially at rest, all the three tracks must necessarily lie in the same plane. The anisotropy of the event space (3.8.17) leads to nontrivial consequences even at the level of nonrelativistic physics; in fact in the nonrelativistic limits equations (4.3.1) and (4.3.2) give for $\vec{q} := \vec{p} - 2\,\sigma\, m\, c\,\vec{\nu}$ and $T = E - mc^2$

$$q^i \;=\; (1\,-2\,\sigma)\, m\left[\delta^i_j\;-\;2\,\sigma\,\nu^i\,\eta_{jk}\,\nu^k\right] v^j\;+\;o(\|\vec{v}\|) \tag{4.3.3}$$

and

$$T \;=\; \frac{1}{2}\,(1\,-2\,\sigma)\, m\left[\delta_{ij}\;+\;2\,\sigma\,\eta_{ik}\,\eta_{jp}\,\nu_k\,\nu_p\,\right] v^i\,v^j\;+\;o(\|\vec{v}\|^2) \tag{4.3.4}$$

In particular, the effective inertness of a particle of mass m turns out to be dependent on the quantities σ and $\vec{\nu}$, which characterize space anisotropy, and is determined by a tensor of inert mass

$$M_{\alpha\beta} \;=\; m\,(1-2\,\sigma)\,(\delta_{\alpha\beta}\;+\;2\,\sigma\,\nu_\alpha\,\nu_\beta) \tag{4.3.5}$$

thus Newton's second law takes the form

$$M_{\alpha\beta}\,a^\beta \;=\; F_\alpha \quad . \tag{4.3.6}$$

From this starting point it is possible, to our point of view, to build up a completely Machian theory. Equation (4.3.5) means that at $\sigma = \frac{1}{2}$ any massive particle loses its inertness.

4.4 Relation Between Physical "Anisotropic" Energy-Momentum and Special Relativistic Observables

Let $\phi(t) = (ct, \varphi(t))$ be the coordinate rappresentation of particle trajectory in $(1+1)$-dimensional spacetime, the tangent vector at an arbitrary point $\phi(t)$ is given by $v = c\frac{\partial}{\partial ct} + v^1\frac{\partial}{\partial x}$ with $v^1 = \varphi'(t)$. For simplicity, from now on, we will use the same symbol v to denote either the geometrical tangent vector and his space coordinate.

In special relativity usually one define the the two dimensional counterpart of what is called the energy-momentum four vector in this way

$$\pi \;=\; \frac{\partial L_{sr}}{\partial v} \tag{4.4.1}$$

where

$$L_{sr} \;=\; -\,m\,c^2\,\sqrt{1 - v^2/c^2} \tag{4.4.2}$$

and the energy is

$$\epsilon \;=\; (\pi\,v \;-\; L_{sr}) \tag{4.4.3}$$

finally we can summarize by

$$\begin{aligned} s^\alpha \;&=\; (\frac{\epsilon}{c}, \pi) \\ &=\; (m\,c\,\gamma, m\,\gamma\,v) \end{aligned} \tag{4.4.4}$$

and if we remember that the $(1+1)$-special relativistic velocity is defined as $u^\alpha = \gamma(v)(1, v/c)$, equation (4.4.4) can be rewritten in the following way

$$s^\alpha \;=\; m\,c\,u^\alpha \quad . \tag{4.4.5}$$

We can define, by analogy with special relativity, the $(1+1)$-anisotropic velocity vector w^α as follows

$$p^\alpha \;=\; m\,c\,w^\alpha \tag{4.4.6}$$

where $p^\alpha \;=\; (\frac{E}{c}, p)$, p and E are those of equations (4.2.1) and (4.2.2) respectively; the explicit expressions of $w^\alpha(\sigma, v)$ are

$$\begin{cases} w^0 \;=\; \gamma(v)\left(\dfrac{1\,+\,\frac{v}{c}}{1\,-\,\frac{v}{c}}\right)^\sigma \left(1\;-\;\dfrac{2\,\sigma\,v}{c}\right) \\ \\ w^1 \;=\; \gamma(v)\left(\dfrac{1\,+\,\frac{v}{c}}{1\,-\,\frac{v}{c}}\right)^\sigma \left(\dfrac{v}{c} - 2\,\sigma\right) \quad . \end{cases} \tag{4.4.7}$$

As a direct consequence we deduce the following two equalities

$$\frac{w^0 \,+\, w^1}{w^0 \,-\, w^1} \;=\; \Big(\frac{1 \,-\, 2\,\sigma}{1 \,+\, 2\,\sigma}\Big) \left(\frac{1 \,+\, \dfrac{v}{c}}{1 \,-\, \dfrac{v}{c}}\right) \tag{4.4.8}$$

$$\frac{v}{c} \;=\; \frac{2\,\sigma\, w^0 \,+\, w^1}{w^0 \,+\, 2\,\sigma\, w^1} \quad . \tag{4.4.9}$$

The $(1+1)$-anisotropic velocity one-form components are

$$\begin{cases} w_0 \;=\; \Big(\dfrac{w^0 \,+\, w^1}{w^0 \,-\, w^1}\Big)^{2\sigma}\, w^0 \\[2ex] w_1 \;=\; -\,\Big(\dfrac{w^0 \,+\, w^1}{w^0 \,-\, w^1}\Big)^{2\sigma}\, w^1 \end{cases} \tag{4.4.10}$$

that is

$$\begin{cases} w_0 \;=\; \Big(\dfrac{1 \,-\, 2\,\sigma}{1 \,+\, 2\,\sigma}\Big)^{2\sigma} \Big(\dfrac{1 \,+\, \frac{v}{c}}{1 \,-\, \frac{v}{c}}\Big)^{2\sigma}\, \gamma(v)\, \big(1 \,-\, \dfrac{2\,\sigma\, v}{c}\big) \\[2ex] w_1 \;=\; \Big(\dfrac{1 \,-\, 2\,\sigma}{1 \,+\, 2\,\sigma}\Big)^{2\sigma} \Big(\dfrac{1 \,+\, \frac{v}{c}}{1 \,-\, \frac{v}{c}}\Big)^{2\sigma}\, \gamma(v)\, \big(2\,\sigma \,-\, \dfrac{v}{c}\big) \quad . \end{cases} \tag{4.4.11}$$

The anisotropic analogue of the special relativity dispersion relation $u^\alpha\, u_\alpha \;=\; 1$ become in our framework

$$\begin{aligned} w^\alpha\, w_\alpha \;&=\; g^{\alpha\beta}(w)\, w_\alpha(\sigma, v)\, w_\beta(\sigma, v) \\ &=\; k(\sigma)\, \Big(\frac{w^0 \,+\, w^1}{w^0 \,-\, w^1}\Big)^{4\sigma} \end{aligned} \tag{4.4.12}$$

we can also write

$$\begin{aligned} k(\sigma) \;&=\; \Big(\frac{w^0 \,+\, w^1}{w^0 \,-\, w^1}\Big)^{-4\sigma}\, g_{\alpha\beta}(w)\, w^\alpha\, w^\beta \\ &=\; \Big(\frac{w^0 \,+\, w^1}{w^0 \,-\, w^1}\Big)^{-2\sigma}\, \eta_{\alpha\beta}\, w^\alpha\, w^\beta \end{aligned} \tag{4.4.13}$$

which is equivalent to the anisotropic dispersion relation we found before in equation (4.2.11).

From equation (4.4.12) we also deduce

$$F(w) \;=\; \sqrt{k(\sigma)}\,\Big(\frac{w^0\;+\;w^1}{w^0\;-\;w^1}\Big)^{2\sigma} \tag{4.4.14}$$

so every admissible $(1+1)$-velocities has constant "Finsler length"

$$\Big(\frac{w^0\;+\;w^1}{w^0\;-\;w^1}\Big)^{-2\sigma}\,F(w) \;=\; \sqrt{k(\sigma)} \tag{4.4.15}$$

which depends by the anisotropic parameter σ.

We found that special relativistic dispersion relation $\eta^{\alpha\beta}s_\alpha s_\beta = m^2c^2$ become the following

$$\Big(\frac{p^0\;+\;p^1}{p^0\;-\;p^1}\Big)^{-4\sigma}\,F^2(p^0,p^1) \;=\; m^2\,c^2\,k(\sigma) \tag{4.4.16}$$

where $k(\sigma)$ is defined in equation (4.2.12)

Using equations (4.2.1), (4.2.2) and (4.4.4) we can find the relation between the anisotropic momentum vector and special relativistic analogues

$$\begin{cases} p^0 \;=\; \Big(\dfrac{s^0\;+\;s^1}{s^0\;-\;s^1}\Big)^{\sigma}(s^0\;-2\,\sigma\,s^1) \\[2ex] p^1 \;=\; \Big(\dfrac{s^0\;+\;s^1}{s^0\;-\;s^1}\Big)^{\sigma}(s^1\;-2\,\sigma\,s^0) \end{cases} \tag{4.4.17}$$

that is, the physical energy and momentum are nonlinear functions of the special relativistic momentum vector, whose components transform linearly under the action of the Lorentz group. The inverse relations are

$$\begin{pmatrix} s^0 \\ \\ s^1 \end{pmatrix} \;=\; \frac{(1\;-\;2\,\sigma)^{\sigma\,-\,1}}{(1\;+\;2\,\sigma)^{1\,+\,\sigma}}\,\Big(\frac{p^0\;+\;p^1}{p^0\;-\;p^1}\Big)^{-\,\sigma}\,\begin{pmatrix} 1 & 2\sigma \\ \\ 2\sigma & 1 \end{pmatrix}\,\begin{pmatrix} p^0 \\ \\ p^1 \end{pmatrix} \tag{4.4.18}$$

obviously with the aid of this equation the modified dispersion relation (4.2.11) or (4.4.16) can be written

$$m^2\,c^2 \;=\; \eta_{\alpha\beta}\,s^\alpha(\sigma,p^\nu)\,s^\beta(\sigma,p^\nu) \quad . \tag{4.4.19}$$

If we define a velocity vector in pseudo-Finsler spacetime as $y^\alpha := dx^\alpha/ds$ and by analogy with special relativity, we define a "energy-momentum" (1+1)-vector $q^\alpha := mcy^\alpha$, then q^1 does not coincide with the particle momentum and q^0 does not coincide with E given by equation (4.2.2). We will now find the relations of q^0 and q^1 with $p^0 = E/c$ and p^1. The natural realization of the (1+1)-velocity vector for a particle in a locally Minkowskian pseudo-Finslerian spacetime is

$$y^\alpha \;:=\; \frac{v^\alpha}{F(v)} \tag{4.4.20}$$

from a general property of Finsler geometry holds

$$g_{ij}(v)\, y^i\, y^j \;=\; 1 \tag{4.4.21}$$

or, by the positive scale invariance of the pseudo-Finsler metric

$$g_{ij}(v) = g_{ij}(yF(v)) = g_{ij}(y)$$

we can write as in special relativity

$$g_{ij}(y)\, y^i\, y^j \;=\; 1 \tag{4.4.22}$$

where v is an physically admissible tangent vector, that is $F(v) > 0$ and F is the pseudo-Finsler structure.

Now we will give some general formulas we are going to use in the following; from the definition and equation (3.8.3), we gain

$$y^\alpha \;=\; \gamma(v)\left(\frac{1 + \dfrac{v}{c}}{1 - \dfrac{v}{c}}\right)^{-\,\sigma} \frac{v^\alpha}{c} \tag{4.4.23}$$

that is, writing explicitly by components

$$\begin{cases} y^0 \;=\; \gamma(v)\left(\dfrac{1 + \dfrac{v}{c}}{1 - \dfrac{v}{c}}\right)^{-\,\sigma} \\ y^1 \;=\; y^0\, \dfrac{v}{c} \end{cases} \tag{4.4.24}$$

We remember that the (1+1)-special relativistic velocity is $u^\alpha = \gamma(v)(1, v/c)$ then, equation (4.4.24) can be written as

$$y^\alpha = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{-\sigma} u^\alpha \tag{4.4.25}$$

we note that when $\sigma = 0$ the "anisotropic two-velocity" vector y^α become the usual special relativistic one. We can now arrive at the following equalities

$$\frac{y^0 + y^1}{y^0 - y^1} = \frac{1+\frac{v}{c}}{1-\frac{v}{c}} \tag{4.4.26}$$

$$\gamma(v) = y^0 \left(\frac{y^0 + y^1}{y^0 - y^1}\right)^{\sigma} \tag{4.4.27}$$

and as a consequence

$$\gamma(v)\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\sigma} = y^0 \left(\frac{y^0 + y^1}{y^0 - y^1}\right)^{2\sigma} \tag{4.4.28}$$

equations (4.2.1) and (4.2.2) can be summarized by

$$\begin{pmatrix} p^0 \\ \\ p^1 \end{pmatrix} = \left(\frac{q^0 + q^1}{q^0 - q^1}\right)^{2\sigma} \begin{pmatrix} 1 & -2\sigma \\ \\ -2\sigma & 1 \end{pmatrix} \begin{pmatrix} q^0 \\ \\ q^1 \end{pmatrix} \quad . \tag{4.4.29}$$

The relation between y_α and y^α is $y_\alpha = g_{\alpha\beta}(y)y^\beta$, that is

$$\begin{cases} y_0 = \left(\dfrac{y^0 + y^1}{y^0 - y^1}\right)^{2\sigma} y^0 \\ \\ y_1 = -\left(\dfrac{y^0 + y^1}{y^0 - y^1}\right)^{2\sigma} y^1 \end{cases} \tag{4.4.30}$$

or, equivalently from equation (4.4.27)

$$\begin{cases} y_0 = \gamma(v)\left(\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\sigma} \\ \\ y_1 = -\,y_0\,\dfrac{v}{c} \end{cases} \tag{4.4.31}$$

from equation (4.4.30) we can straightforward infer

$$\frac{y_0 \, + \, y_1}{y_0 \, - \, y_1} = \Big(\frac{y^0 \, + \, y^1}{y^0 \, - \, y^1}\Big)^{-1} \tag{4.4.32}$$

so, the inverse transformations of equations (4.4.30) are

$$\begin{cases} y^0 = \Big(\dfrac{y_0 \, + \, y_1}{y_0 \, - \, y_1}\Big)^{2\,\sigma} y_0 \\ \\ y^1 = - \Big(\dfrac{y_0 \, + \, y_1}{y_0 \, - \, y_1}\Big)^{2\,\sigma} y_1 \end{cases} \tag{4.4.33}$$

From the usual $q_\alpha = g_{\alpha\beta}(q^\nu)q^\beta$ using the positive scale invariance of the pseudo-Finsler metric we obtain $q_\alpha = mcg_{\alpha\beta}(y)y^\beta = mcy_\alpha$, in the same way, defining $p_\alpha = g_{\alpha\beta}(y)p^\beta$, we can find

$$\begin{cases} p_0 = \left(\dfrac{p^0 \, + \, p^1}{p^0 \, - \, p^1}\right)^{2\sigma} p^0 \\ \\ p_1 = - \left(\dfrac{p^0 \, + \, p^1}{p^0 \, - \, p^1}\right)^{2\sigma} p^1 \end{cases} \tag{4.4.34}$$

we arrive, by equations (4.4.29), at the following

$$\begin{pmatrix} p_0 \\ \\ p_1 \end{pmatrix} = \Big(\frac{1 \, - \, 2\,\sigma}{1 \, + \, 2\,\sigma}\Big)^{2\,\sigma} \Big(\frac{q_0 \, + \, q_1}{q_0 \, - \, q_1}\Big)^{-\,4\,\sigma} \begin{pmatrix} 1 & 2\sigma \\ \\ 2\sigma & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ \\ q_1 \end{pmatrix} \tag{4.4.35}$$

we observe that by the aid of the "anisotropy transformation matrix", written in the above equation, we are able to express anisotropy dependent quantities in terms of their "generalized special relativistic" counterparts.

4.5 More About Dispersion Relation and Momentum One-Form

Our dispersion relation (4.2.11) can be written in the following way:

$$\left(\frac{E^2}{c^2} - p^2\right) = m^2 c^2 k(\sigma) \left[\frac{\frac{E}{c} + p}{\frac{E}{c} - p}\right]^{2\sigma} \tag{4.5.1}$$

If we expand for small σ the right hands member we obtain

$$\frac{E^2}{c^2} - p^2 + m^2 c^2 f^{(1)}(\sigma, E, p) = m^2 c^2 + o(\sigma) \tag{4.5.2}$$

where $f^{(1)}(\sigma, E, p)$ is the first expansion term of $f(\sigma, E, p)$ for small σ: $f(\sigma, E, p) = 1 + f^{(1)}(\sigma, E, p) + o(\sigma)$. Indeed, we can write the anisotropic contribution to the dispersion relation in the following way

$$\begin{aligned} G(\sigma, E, p) &= -2\sigma m^2 c^2 \ln\left(\frac{E/c + p}{E/c - p}\right) \\ &= m^2 c^2 f^{(1)}(\sigma, E, p) \\ &= m^2 c^2 \ln f(\sigma, E, p) + o(\sigma) \quad . \end{aligned} \tag{4.5.3}$$

From now on we will always use the approximate dispersion relation

$$\frac{E^2}{c^2} - p^2 - 2\sigma m^2 c^2 \ln\left(\frac{E/c + p}{E/c - p}\right) = m^2 c^2 \tag{4.5.4}$$

we note that in this approximate dispersion relation we introduce a not invariant term (with respect our generalized transformation law), we stress that in our framework it appears a *privileged reference* frame only because we use an approximate relation.

This "anisotropic" dispersion relation (4.5.2) belongs also to the class of deformed dispersion relations proposed by Amelino-Camelia [18] in the context of doubly special relativity. Indeed, in [18] was proposed a deformation term of "power law" type $f \propto p^2 E$ instead of our transcendental function.

One of the crucial points of doubly special relativity model is the existence of a spacetime's fundamental length scale and a consequent dependence of light's velocity from wave length; this feature was based on the possibility that Quantum Gravity effects would modify the dispersion relations for particle propagation, such as photons so these modifications in turn would change the propagation velocity of photons, introducing delays for particles of different energies which could be detected if these particles would travel cosmological distances.

Indeed, in our model photons do not feel any pseudo-Finsler metric but for massive particles holds the special relativistic dispersion relation only if we make a "bad" choice of physical variables in an anisotropy space: in an anisotropic space we have not to use the "usual" (in special relativity) quantities s^α, but the "new anisotropic" ones defined in equations (4.2.1) and (4.2.2), and the "real" dispersion relation become equation (4.4.19) or equivalently equation (4.2.11).

We also point out that if we postulate the conservation of the total energy $E = E_{(1)} + E_{(2)}$ for a two particles system, where the single particle energy $E_{(k)}$ ($k = 1, 2$ is the numbers of particles) is defined as in equation (4.2.2), then, as demonstrated in [65] it is the quantity $p_{(1)} + p_{(2)}$ with $p_{(k)}$ ($k = 1, 2$) defined in equation (4.2.1) and not the special relativistic $s^1_{(1)} + s^1_{(2)}$ which is conserved in elastic collision between the two particles.

Obviously also the sum of the single particle special relativistic energy s^0 is not conserved in such a collision.

For future handling, we now writte the dispersion relation of equation (4.2.11) in the following way:

$$h(\sigma, p^\alpha) := f(\sigma, p^\alpha) \left[(p^0)^2 - (p^1)^2 \right] = m^2 c^2 \tag{4.5.5}$$

the function $f(\sigma, p^\alpha)$ is given in equation (4.2.13), if we define

$$z := \frac{p^1}{p^0} \tag{4.5.6}$$

we can rewrite equation (4.5.5) as

$$f(\sigma, z) \, (p^0)^2 \left(1 - z^2\right) = m^2 c^2 \, . \tag{4.5.7}$$

with

$$f(\sigma, z) = \frac{1}{k(\sigma)} \left(\frac{1 + z}{1 - z} \right)^{-2\sigma} \tag{4.5.8}$$

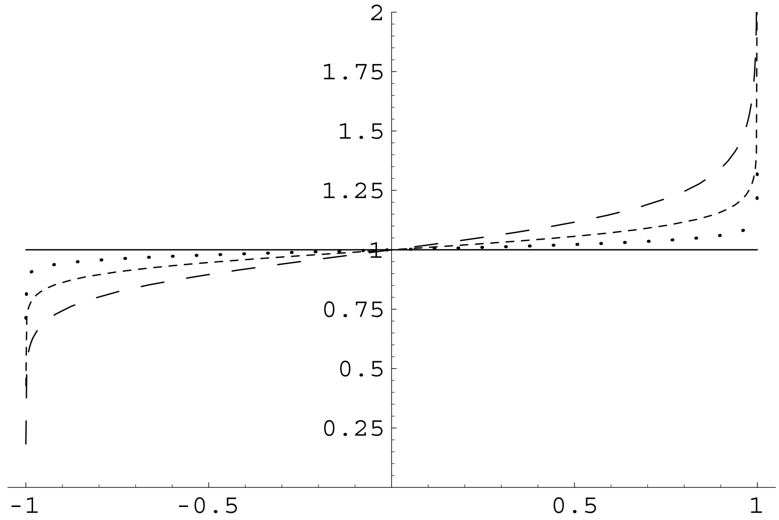

Figure 4.3: Plot of $k(\sigma)f(\sigma,z)$ as a function of z for different values of the anisotropic parameter σ. The special relativistic case, $\sigma = 0$, is represented by the solid line. The long-dashed line is the anisotropic case with $\sigma = -0.05$, the dashed line correspond to $\sigma = -0.025$, and the dot line is the case with $\sigma = -0.01$.

The implicit function theorem allow us to calculate

$$\begin{aligned} \frac{\partial p^0}{\partial p^1} &= -\frac{\partial h}{\partial p^1} \Big/ \frac{\partial h}{\partial p^0} \\ &= \frac{2\,\sigma\, p^0 \,+\, p^1}{p^0 \,+ 2\,\sigma\, p^1} \quad . \end{aligned} \tag{4.5.9}$$

The Lagrangian formalism ensure that $\partial p^0/\partial p^1$ is the velocity $\beta = v/c$, as we can directly check by substituting equations (4.2.1) and (4.2.2) in equality (4.5.9).

The new relation between energy-momentum and velocity is

$$\beta = \frac{2\,\sigma\, p^0 \,+\, p^1}{p^0 \,+ 2\,\sigma\, p^1} \tag{4.5.10}$$

or equivalently

$$\beta(\sigma, z) = \frac{2\,\sigma \,+\, z}{1 + 2\,\sigma\, z} \quad . \tag{4.5.11}$$

The inverse relation is

$$z(\sigma, \beta) \;=\; \frac{\beta \,-\, 2\,\sigma}{1 \,-\, 2\,\sigma\,\beta} \tag{4.5.12}$$

the first derivative with respect to β is

$$z'(\sigma, \beta) \;=\; \frac{1 \,-\, 4\,\sigma^2}{(1 \,-\, 2\,\sigma\,\beta)^2} \tag{4.5.13}$$

and from the condition $|\sigma| < \frac{1}{2}$ we deduce that $z(\sigma, \beta)$ is a monotonically strictly increasing function on β, we also easily obtain

$$z''(\sigma, \beta) \;=\; \frac{z'(\sigma, \beta)}{(1 \,-\, 2\,\sigma\,\beta)}\, 4\,\sigma \tag{4.5.14}$$

so the convexity-concavity is determined by σ's sign.

Using equation (4.5.7) we obtain an alternative expression for energy $E = c\,p_0$ and momentum p_1

$$\begin{aligned} E \;&=\; \frac{m\,c^2}{\sqrt{(1 \,-\, z^2)\, f(\sigma, z)}} \\ p_1 \;&=\; \frac{m\,c\,z}{\sqrt{(1 \,-\, z^2)\, f(\sigma, z)}} \quad . \end{aligned} \tag{4.5.15}$$

as $|\beta| \to 1$ and $|z| \to 1$ the energy and momentum tend to infinity (again we have to use condition $|\sigma| < \frac{1}{2}$), but according to a different law than the conventional one. This is the basis for a possible explanation of the anomalous behavior of the cosmic-ray spectrum at ultra-high energies, that is the absence of a cut-off of ultra-high-energy cosmic rays might be attributed to a breakdown of conventional relativistic theory at velocities near the velocity of light.

4.6 Transformation Law of Energy and Momentum

The law of transformation of energy and momentum on going from one reference frame to another can be obtained directly from the invariance condition of the dispersion law. By introducing the quantities $p^*_\alpha := p_\alpha \sqrt{f(\sigma, z)}$ and by considering the motion of system K' with velocity V along the x axis of system K, we get for p^*_α the usual Lorentz transformations

$$\begin{cases} p'^*_0 \;=\; \gamma(z^*)(p^*_0 \;-\; z^*\, p^*_1) \\ \\ p'^*_1 \;=\; \gamma(z^*)(p^*_1 \;-\; z^*\, p^*_0) \\ \\ p'^*_2 \;=\; p^*_2 \\ \\ p'^*_3 \;=\; p^*_3 \end{cases} \tag{4.6.1}$$

and obviously holds Einstein's conventional law of addition of velocities

$$z' \;=\; \frac{z \;-\; z^*}{1 \;-\; z\, z^*} \quad . \tag{4.6.2}$$

from the relativity principle we have to conclude that $z^* = V/c$ because for a particle at rest in K' system must hold $z' = -2\sigma$.

From equation (4.6.1) we obtain the following transformation law for the 4-momenta

$$p'_\alpha(\sigma, z') \;=\; [\Lambda^{-1}(z^*)]^\beta{}_\alpha\, p_\beta(\sigma, z)\, \sqrt{\frac{f(\sigma, z)}{f(\sigma, z')}} \tag{4.6.3}$$

where $[\Lambda^{-1}(z^*)]^\beta_\alpha$ is the usual Lorentz transformation matrix. It is interesting to note that within this scheme the transverse components of momentum change.

We stress that p and not p^* are chosen as the physical variables, this choice corresponds to the replacement of the usual pseudo-Euclidean space of the physical coordinates by the pseudo-Finsler space.

Chapter 5

Threshold Conditions

5.1 Possible Values for the Parameter σ

The first indication that nowadays the local Lorentz symmetry still remains slightly broken was obtained from the investigation of the spectrum of primary cosmic superhigh-energy protons. The point is that according to the calculations [2, 3], which substantially employ the local Lorentz symmetry of spacetime, the proton energy spectrum should be cutoff (due to the intense production of pions on relic radiation photons) at proton energies $E_p \approx 5 \cdot 10^{19}$eV.

The experimental data, however, are most likely indicative of the absence of such an effect. This situation induced the investigators to assume that the conventional Lorentz transformations become invalid for the Lorentz factors $\gamma > 5 \cdot 10^{10}$ and the correct relation between the various inertial reference frames at any values of γ is provided by other, so-called generalized Lorentz transformations.

Our generalized Lorentz transformations of equation (3.8.9), belong to a group of local relativistic symmetry of Finslerian spacetime, in which case the smaller the local anisotropy of pseudo-Finslerian spacetime, that is the closer is it to Riemannian one, the closer to the velocity of light tend the generalized Lorentz transformations to be markedly different from the conventional ones.

Therefore the use of these transformations in calculating the cut-off point of the primary cosmic proton spectrum enables one, in principle, to remove the emerged discrepancy between the theoretical predictions and the experimental data pertaining to the superhigh energy region. Taking into account that the conventional kinematics is correct with a relative precision of $5 \cdot 10^{-4}$ up to $\gamma \approx 10^4$ [64], one can conclude that the function f in equation (4.2.13) differs from unity by not more than 10^{-4} if $\gamma(\beta) \leq 10^4$, that is, we must impose

$$|f(\sigma, z(\beta)) - 1| \leq 10^{-4} \tag{5.1.1}$$

in this way we have a constrain on possibles values for σ: if $\gamma(\beta) \leq 10^4$ then $|\beta| \leq 1 - \frac{1}{2}\,10^{-8}$ and if we assume that $|\sigma| << 1$ and for example $\sigma < 0$ (that is $f(\sigma, z)$ is a monotonically increasing function) we can write

$$f(\sigma, z(\beta)) = \frac{1}{1 - 4\,\sigma^2}\left(\frac{1 + \beta}{1 - \beta}\right)^{-2\,\sigma}$$

$$\cong \left(\frac{1 + \beta}{1 - \beta}\right)^{-2\,\sigma}$$

by the monotonicity of $f(\sigma, z(\beta))$, and observing that

$$f(\sigma, z)\, f(\sigma, -z) = \frac{1}{k(\sigma)^2} \tag{5.1.2}$$

we have only to impose the following inequality

$$\left(\tfrac{1+\beta_s}{1-\beta_s}\right)^{-2\,\sigma} \leq 1 + 10^{-4} \tag{5.1.3}$$

where $\beta_s = 1 - \frac{1}{2}\,10^{-8}$ with straightforward calculation we obtain $|\sigma| \leq 2,5 \cdot 10^{-6}$.

5.2 Application To The Cosmic Ray Spectrum: The GZK Cut-Off

Let us investigate the influence of the correction term $f(\sigma, p^\alpha)$ in our dispersion relation on the computation of the stopping of ultra-high-energy protons by the background radiation.

Primary protons with energies greater than $5 \cdot 10^{19}$eV are expected to by strongly slowed down by interaction with the background thermal radiation, the origin of this cut-off comes from several processes that eat up the energy of primary cosmic rays, such as inverse Compton effect by the Cosmic Microwave Background photons. In spite of the theoretical prediction, quite a few cosmic rays of energies greater than the GZK cut-off have been found in several cosmic ray observatories, mainly AGASA. This is usually called the GZK anomaly.

Although it is premature to speak in this situation of a real discrepancy between theory and experiment, the absence of a convincing explanation of this complex situation justifies the question of whether the energy of the primary protons is not already large enough so that the observed discrepancy might be due to the breakdown of our present concepts. The essential point is that the primary protons have a uniquely large Lorentz factor $\gamma(\beta) \gtrsim 5 \cdot 10^{10}$, larger by many orders of magnitude than in any other experiment. At the same time γ enters in an essential way into the calculation of proton stopping. From this it follows that if models existed in which the deviation from conventional theory were determined by the product of γ^{-1} by some dimensionless critical parameter, one could hope to avoid contradictions with present data.

In the conventional theory the cut-off of the cosmic ray spectrum due to intense photoproduction of pions starts at proton energies for which the background photon has in the rest system of the proton an energy of the order of the pion mass (in the case of central collisions). Most important for the computation of the proton lifetime with respect to photoproduction is the statistical factor represented by the Planck distribution of background photons $H = exp(-\omega/k_B T)$, where ω is the energy of photons in the earth reference frame (supposed to be closed to the frame suggested by the microwave background radiation spectrum) $k_B = 8,6 \cdot 10^{-5}$eV K^{-1} is the Boltzmann constant and T is the temperature of background radiation.

This factor, for a central collision, written in the proton rest frame, takes the form $H = exp(-\omega_c/2\gamma_p K_B T)$, where γ_p is the Lorentz factor of the proton and ω_c is the photon energy in this system ($\omega_c \geq m_\pi c^2 \approx 140$)MeV. For $\gamma_p \geq \omega_c/2K_B T$ this factor rise sharply, leading to a rapid decrease in the proton lifetime.

If we want to attribute the absence of a break in the spectrum of the cosmic rays at an energy $E_k \approx 5 \cdot 10^{19}$eV to the deviation of f from unity, we have to assume that the difference $|f - 1|$ becomes of the order of the unity at this energy; we shall now repeat the above computation within the new scheme, restricting ourselves to corrections in only the statistical factor H, which contains the strongest (exponential) dependence upon the Lorentz factor. Taking into account the fact that in the earth system the photon distribution is again of the Planck type, we have to find the expression for the factor H in the rest system of the proton. To this purpose we have only to remember that the quantities $p^*_{\alpha,proton}$ and $p^*_{\alpha,photon}$ transform according to the conventional law, so that $p^*_{\alpha,proton}\, p^{*\alpha}_{photon}$ is an invariant. By writing this invariant in the two reference systems, we obtain the following relation between the photon energy in the earth system and in the proton system

$$\omega_c \;=\; 2\,\gamma_p\,\sqrt{f(\sigma, z_k)}\,\omega \tag{5.2.1}$$

where z_k is the value of z variable at the GZK cut-off. If $\sqrt{f(\sigma, z_k)} \;<\; 1$, the presence of the supplementary factor in the argument of the exponential function leads to a decrease in the effective temperature of photons and hence to an increase in the proton lifetime. In order to eliminate completely the cut-off in the cosmic-ray spectrum, it is necessary to have

$$e^{-\,\omega_c\,/2\,\gamma_p\,K_B\,T\,\sqrt{f(\sigma,z_k)}} \;<<\; e^{-1} \tag{5.2.2}$$

this inequality is satisfiable if $f(\sigma, z_k) \approx 0,1 - 0,01$ (we used $T = 2.7°K$ and $\gamma_p \approx 5 \cdot 10^{10}$), but within our framework of our generalized relativistic theory we have only $f(\sigma, z_k) \approx 1 - 2.46 \cdot 10^{-4}$: for cosmic-ray protons moving toward the earth holds $z \approx -1$ so condition

$$\lim_{z \to -1^+} f(\sigma, z) \;=\; 0$$

imply $\sigma < 0$. The first equation (4.5.15) can be written as

$$\frac{\gamma(z_k)}{\sqrt{f(\sigma, z_k)}} \;=\; \frac{E_k}{m\,c^2} \tag{5.2.3}$$

and being

$$f(\sigma, z_k) \;\approx\; [\gamma(z_k)\,(1 \;+\; |z_k|)]^{-\,4\,|\sigma|} \tag{5.2.4}$$

we infer

$$\gamma(z_k) \;\approx\; \left(\frac{E_k}{m\,c^2}\right)^{\frac{1}{1\,+\,2\,|\sigma|}} \tag{5.2.5}$$

inserting in equation (5.2.4) and assuming $|\sigma| = 2.5 \cdot 10^{-6}$, it holds

$$f(\sigma, z_k) \;\approx\; \left(\frac{E_k}{m\,c^2}\right)^{\frac{-\,4\,|\sigma|}{1\,+\,2\,|\sigma|}} \;\approx\; 1 - 2.46 \cdot 10^{-4} \tag{5.2.6}$$

so within our framework the GZK cut-off does not disappear, but it is only a few attenuate.

5.3 Two Particle Dynamics: Threshold Energies

The conservation law of total momentum manifests itself differently in isotropic and anisotropic spaces. As an elementary example, consider the elastic collision of two particles in isotropic space, one of which at first was at rest. The conservation law of total momentum then makes the tracks of the particle coplanar. For the same process but now in anisotropic space, where the directions of velocities and, hence, of the tracks of particles do not coincide with the directions of their momenta (see equation (4.3.1)), the conservation law of total momentum does not lead to the fact that all the three tracks must necessarily lie in the same plane.

However, since the amount of the deviation from coplanarity is a function of the magnitude of space anisotropy, possible effects of noncoplanarity should be searched for in regions where the magnitude of local anisotropy is significantly greater than its mean value. Such a situation may obtain in the vicinity of very large masses, for example, near the Sun. It seems reasonable to test this assumption with a corresponding detector on a space vehicle able to identify elementary events with nonstandard kinematics.

Moreover, if Lorentz symmetry is really violated the possible change for the dispersion relation can significantly modify the kinematical conditions for a reaction to take place, that is, the energy threshold for any reaction could be lowered, raised, or removed entirely, or an upper threshold where the reaction cuts off could even be introduced

If the law of energy-momentum conservation is valid then with the aid of the dispersion relation we can analyzing many Lorentz-violating phenomena, specially those related to threshold modifications. More precisely, conservation laws and dispersion relation are the basis of threshold analysis: the very powerful tool to discuss many astrophysical phenomena in the presence of (possible) Lorentz violation.

There is a host of phenomena which can be used to detect Lorentz Invariance violation. Among them, let us mention

- Breakdown of local rotational symmetry: “Aether wind effects”.
- Breakdown of Lorentz Boost Invariance “Kennedy-Thorndike experiments”.
- Dispersive processes in vacuum (such as energy dependent velocities or birefringence).
- Occurrence of “forbidden process” such as photon decay in vacuum.

There is a main group of observations with high enough sensitivity to probe a possible Lorentz invariance violation: cosmological or astrophysical tests of threshold analysis type, based on the modification of the dispersion relations of particles (the sensitivity requirements are indeed very strict).

5.3.1 Resonant Production $p + \gamma \to \Delta$

The most important reactions taking place in the description of ultra-high-energy cosmic ray propagation (and which produce the release of energy in the form of particles) are the pair creation $p + \gamma \to p + e^+ + e^-$ and the photopion production $p + \gamma \to p + \pi$. This last reaction happens through several channels (for example, the baryonic Δ and N and mesonic ρ and ω resonance channels, just to mention some of them) and is the main reason for the appearance of the GZK cutoff.

These processes degrades the initial proton energy with an attenuation length of about 50 Mpc. Since plausible astrophysical sources for ultra-high-energy particles are located at distances larger than 50 Mpc, one expects a cutoff in the cosmic ray proton energy spectrum, which occurs at around $5 \cdot 10^{19}$eV, depending on the distribution of sources.

All these processes have the same general structure in a Lorentz invariant picture: these reactions would be examples of a very low energy processes in the center of mass reference frame that appear boosted to very large Lorentz factors in the laboratory frame.

Many ideas have been put forward to explain the possible absence of the GZK cutoff [5, 24]. For example the cosmic rays might originate closer, in some unexpected way, by astrophysical acceleration or by decay of ultra-heavy exotic particles, or they may be produced by collisions with ultra high energy cosmic neutrinos. Indeed all of these explanations have problems.

Our interest in the following is to study the reactions involved in high energy cosmic ray phenomena through the threshold conditions because a change with respect the special relativity case in threshold energy offer a measure of how modified the kinematics is.

We now study the resonant production $p + \gamma \to \Delta$, the threshold energy is gained when the resonance Δ is created at rest with respect the center of mass reference frame; we have now to stress that in our anisotropic framework, in the reference system where $p = 0$ the particle will not be at rest. To understand this it is sufficient to verify that in general (different from the special relativistic case) the velocity follows

$$v = \frac{\partial E}{\partial p} \neq \frac{c^2 p}{E} \tag{5.3.1}$$

and therefore does not generally vanish at $p = 0$, more simply we can check this property directly from equation (4.2.1). There emerges, then, an important distinction between the "phase" velocity $c^2 p/E$ and the "group" velocity $v = \partial E/\partial p$ of the same particle (as we already noted in equation (4.4.9).

First of all, we state that energy-momentum one-form is an conserved quantity also in inelastic reaction, so the conservation energy-momentum law for the dominant process leading to the GZK cut-off, that is $p + \gamma \rightarrow \Delta$ are

$$\begin{cases} E \,+\, E_\gamma \;=\; M\, c^2 \\ \\ p \,+\, k \;=\; 2\,|\sigma|\, M\, c \end{cases} \tag{5.3.2}$$

where E is the proton energy, E_γ is the photon energy, p and k are momentum of proton and photon respectively; all this quantities are measured in the center of mass frame and M is Δ resonance's rest mass. Being interested in threshold energy we have assumed the resonance Δ created at rest in a central collision.

From conservation law we easily obtain

$$M^2\, c^2\, (1 \,-\, 4\,\sigma^2) = \left(\frac{E^2}{c^2} \,-\, p^2\right) \,+\, 2\left(\frac{E\,E_\gamma}{c^2} \,-\, p\,k\right) \tag{5.3.3}$$

we used photon dispersion relation, which is the same as in special relativity, so $||\vec{k}|| = E_\gamma/c$; if we now consider proton dispersion relation and in the last equation we omit second order's term on σ we gain

$$c^2\,(M^2 \,-\, m^2) \;=\; 2\left(\frac{E\,E_\gamma}{c^2} \,-\, p\,k\right) \,-\, 2\,|\sigma|\, m^2\, c^2\, \ln\left(\frac{\frac{E}{c} \,+\, p}{\frac{E}{c} \,-\, p}\right) \tag{5.3.4}$$

in a central collision holds $0 < -pk < E\,E_\gamma$ (we are assuming $k = E_\gamma/c$,), as a consequence

$$c^2\,(M^2 \,-\, m^2) \;\leq\; 4\,\frac{E\,E_\gamma}{c^2} \,-\, 2\,|\sigma|\, m^2\, c^2\, \ln\left(\frac{\frac{E}{c} \,+\, p}{\frac{E}{c} \,-\, p}\right) \quad . \tag{5.3.5}$$

From the conservation law in equation (5.3.2) we have

$$c^2\,(M^2 \,-\, m^2) \;\leq\; 4\,\frac{E\,E_\gamma}{c^2} \,-\, 2\,|\sigma|\, m^2\, c^2\, \ln\left(\frac{\frac{E\,-E_\gamma}{M\,c^2} \,+\, 2\,|\sigma|}{1 \,-\, 2\,|\sigma|}\right) \tag{5.3.6}$$

being the logarithmic function monotonically increasing we easily obtain the following threshold condition

$$c^2\,(M^2\,-\,m^2)\,\leq\,4\,\frac{E\,E_\gamma}{c^2}\,-\,2\,|\sigma|\,m^2\,c^2\,\ln\left(\frac{E\,-\,E_\gamma}{M\,c^2}\right)\quad. \tag{5.3.7}$$

We note that if $\sigma = 0$ we obtain again the special relativity threshold energy for the resonant production we are studying and if $E - E_\gamma < 0$ the process does not take place. As a consequence the anisotropic threshold energy E_a given by equation

$$c^2\,(M^2\,-\,m^2)\,=\,4\,\frac{E_a\,E_\gamma}{c^2}\,-\,2\,|\sigma|\,m^2\,c^2\,\ln\left(\frac{E_a\,-\,E_\gamma}{M\,c^2}\right) \tag{5.3.8}$$

differ from the special relativistic one, that is

$$c^2\,(M^2\,-\,m^2)\,=\,4\,\frac{E_{GZK}\,E_\gamma}{c^2} \tag{5.3.9}$$

and from equations (5.3.8) and (5.3.9) we can evaluate the difference

$$E_{GZK}\,-\,E_a\,=\,-\,\frac{m^2\,c^4\,|\sigma|}{2\,E_\gamma}\,\ln\left(\frac{E_a\,-\,E_\gamma}{M\,c^2}\right) \tag{5.3.10}$$

so special relativistic threshold energy is greater than anisotropic threshold, $E_{GZK} > E_a$, if and only if $E_\gamma < E_a < E_\gamma + Mc^2$, that is to say, $E_{GZK} < E_a$ if and only if $E_a > E_\gamma + Mc^2$.

In the same way for the case $p > 0$ and $k < 0$ we obtain for the threshold energy

$$c^2\,(M^2\,-\,m^2)\,=\,4\,\frac{E_a\,E_\gamma}{c^2}\,+\,2\,|\sigma|\,m^2\,c^2\,\ln\left(\frac{E_a\,-\,E_\gamma}{M\,c^2}\right) \tag{5.3.11}$$

and the relation with the analogue special relativistic energy is

$$E_{GZK}\,-\,E_a\,=\,\frac{m^2\,c^4\,|\sigma|}{2\,E_\gamma}\,\ln\left(\frac{E_a\,-\,E_\gamma}{M\,c^2}\right) \tag{5.3.12}$$

in this case special relativistic threshold energy is smaller than anisotropic threshold if and only if $E_a - E_\gamma > Mc^2$. We also note that, in our anisotropic model, the threshold energy for one reaction realized in opposite direction are different.

5.3.2 Pions Photo-Production

Reaction $p + \gamma \to \Delta$ is the dominant process leading to the GZK cutoff, as argued by Greisen, Zatsepin and Kuz'min. However, if $\Delta(1232)$ formation is not possible, a weakened version of the GZK cutoff may result from non-resonant photo-production of one or more pions.

For a single pion production:

$$p + \gamma \to p + \pi^0 \tag{5.3.13}$$

ordinarily, the threshold is $E_{GZK} = M_\pi(2M_p + M_\pi)/4E_\gamma \simeq M_\pi M_p/2E_\gamma$, according to this equation the Lorentz invariant threshold is proportional to the proton mass. Thus any Lorentz symmetry violating term added to the special relativistic proton dispersion relation will modify significantly the threshold if it is comparable to or greater than M_p^2 at around the energy E_{GZK}. As we already said, modifying the proton and pion dispersion relations, the threshold can be lowered, raised, or removed entirely.

Indeed in our framework we have only a small modification of order of $10^{-4}\, M_p^2\, c^2$, increasing the threshold if the proton is moving toward the earth and decreasing the threshold otherwise: in the first case, the anisotropic term in proton's dispersion relation is (at the GZK energy)

$$\begin{aligned} & 2\,|\sigma|\, m^2\, c^2\, \ln\left(\frac{\frac{E}{c} + p}{\frac{E}{c} - p}\right) = \\ & = -4\,|\sigma|\, m^2\, c^2\, \ln[(1 - z_k)\,\gamma(z_k)] \simeq -2.53 \cdot 10^{-4}\, m^2\, c^2 \end{aligned} \tag{5.3.14}$$

5.3.3 Pair Creation $\gamma \to e^+ + e^-$

As in special relativity, also in our model the process of photon decay in an electron-positron pair $\gamma \to e^+ + e^-$ is forbidden by energy-momentum conservation. The conservation equations (for a threshold decay) are:

$$\begin{cases} E_\gamma \; = \; 2\, m_e\, c^2 \\ \\ k \; = \; 4\, |\sigma|\, m_e\, c \end{cases} \tag{5.3.15}$$

and if $|\sigma| \neq 1/2$ the process does not occur.

Chapter 6

Conclusion

6.1 Conclusion

This thesis has dealt with a broad area of research developed in the recent past year, on the subject of violation of Lorentz invariance. There are both theoretical and experimental reasons to doubt exact Lorentz invariance.

From the experimental point of view, although no clear signal of Lorentz violation has been found, we have to observe that special relativity has been tested with great accuracy only at low energy [42, 87, 88, 89, 90, 91]. Since we have observational evidence only in a small range of energies (relative to the expected Planck energy scale of quantum gravity), it is plausible that Lorentz invariance is broken and as yet unobserved. The investigation of such phenomena is a new domain of physics, where astroparticle and cosmology would play a major role.

These two branches of physics will probably be an area of further advances in the coming years. As an example one of the most puzzling current experimental physics paradoxes is the arrival on Earth of ultra-high-energy cosmic rays with energies above the GZK threshold. The recent observation, by HEGRA, of 20 TeV photons from Mk 501 (a BL Lac object at a distance of 150 Mpc) is another somewhat similar paradox [92, 93, 94].

The GZK cutoff question has generated a lot of interest, and is currently the only observational phenomenon thought to indicate a possible violation of Lorentz symmetry. The advent of more precise observations will probably open the way to definitive judgments about the plethora of models proposed for high-energy phenomenae and for the evolution of the very early universe. Present theories make predictions but often we do not have instruments sensitive enough to test them. It is conceivable that this situation will change in the near future [10, 11, 12].

On the theoretical side, while general relativity possesses local Lorentz invariance, both canonical quantum gravity and string theory suggest that Lorentz invariance may be broken at high energies. Broken Lorentz invariance has also been postulated as an explanation for astrophysical anomalies such as the missing GZK cutoff.

About Lorentz violation many ideas was proposed:

- DSR's theories: in these models a Lorentz symmetry violation can be generated at the Planck scale, or at some other fundamental length scale. Lorentz symmetry is preserved as a low-energy limit (deformed Lorentz symmetry) [18, 23, 24, 25, 27, 28];

- anisotropic spacetime: in this model the continuity of spacetime is manteined and the group of lorentz transformation is generalized [13, 14, 15, 17];

- existence of a preferred reference frame: in this models, an absolute local rest frame exists and special relativity is a low-momentum limit [64, 95]. The preferred frame can be a "fixed external structure", for instance the thermal cosmic microwave background radiation. In this case general covariance of the theory is violated [32, 33, 62]. Lorentz symmetry violation by preferred frame effects has been much studied in non-gravitational physics, and is currently receiving attention as a possible window on quantum gravity. If we include gravity in a framework with a preferred frame, we have to remember that general covariance is a deep symmetry of general relativity. If we want to preserve general covariance of the theory we have to consider another possibility, that is, a inherent and unavoidable dynamical unit time-like vector field exist. As a consequence a preferred rest frame at each spacetime point can be defined. Local Lorentz invariance is broken while covariance of the theory is preserved [63].

In DSR's theories a critical distance scale, allow us to consider models, compatible with standard tests of special relativity, where a small violation of Lorentz symmetry leads to a deformed relativistic kinematics producing dramatic effects on the properties of very high-energy cosmic rays. For instance, GZK cut-off does no longer apply and particles which are unstable at low energy (for example some hadronic resonances, possibly several nuclei) become stable at very high energy.

Indeed, in this Ph.D. thesis, our attempt was to generalize special relativity maintaining the relativity principle and the usual description with classical configuration variables such as position and velocity, in contrast with DSR approach in which the central role is played by energy-momentum space.

We descibed a pseudo-Finslerian event spaces with a partially broken local rotational symmetry in 3D space. Since in chapter three we saw that in the $4D$ curved case the locally isotropic Riemannian spacetime is a special case of our pseudo-Finslerian model, corresponding to the vanishing of the parameter (scalar field) σ, we can speak of a joint description of these two geometric models of spacetime. It is important to stress that each of the above-mentioned models possesses different local relativistic invariances: local invariance may take either the form of full Lorentz invariance (3D rotational symmetry not broken) or in the (1+1)-dimensional model the form of generalized Lorentz invariance, that is to say invariance under the transformations (3.2.7) (partial breaking of isotropy).

The general lessons we can extract from this research is:

- spacetime is a pseudo-Finslerian manifold rather than a pesudo-Riemannian one;
- with our approach we can work towards a generalization of the Mansouri-Sexl test theory of special relativity;
- hopefully this research will provide a hint towards a full covariant theory with a preferred frame.

In this thesis we demonstrated that the physical model that envisages a "real" spatial anisotropy, provides equations for the threshold energies of the physical processes that should cause the GZK cutoff that have only a qualitative correct trend. The differences on threshold energies between those calculated on the anisotropic model and those calculated using special relativity are insufficient to explain from a quantitative point of view the lack of GZK cutoff.

Probably this problem can be cured in a a local anisotropic spacetime. Indeed, the generalization towards a local spacetime anisotropy is the natural continuation of this work.

Moreover Finsler's geometry is a mathematic instrument by means of we can treat, in an handled way, the existence of a possible "privileged" dynamic time-like vector field. This field can be used to identify in each spacetime point a privileged reference system. Therefore, using Finsler's geometry, is particularly natural to abandon relativity principle.

These observations suggest us that Finsler's geometry seems to be the most suitable mathematic instrument to further generalize our work towards a theory that describes a local anisotropic spacetime where a preferred frame is determined by the spacetime metric itself.

Bibliography

[1] C.W. Misner, K.S. Thorne, J.A. Wheeler: *Gravitation*. W. H. Freeman and Company San Francisco, 1973.

[2] K. Greisen: “End to the cosmic-ray spectrum?”. Phys. Rev. Lett. **16** (1966) 748-750.

[3] G.T. Zatsepin, V.A. Kuz’min: “Upper Limit Of The Spectrum Of Cosmic Rays”. JETP Lett. **4** (1966) 78-80.

[4] D. De Marco, P. Blasi and A.V. Olinto: “On the statistical significance of the GZK feature in the spectrum of ultra high energy cosmic rays”. [astro-ph/0301497].

[5] F.W. Stecker: “Cosmic physics: The high energy frontier”. J. Phys. G **29** R47 (2003). [astro-ph/0309027].

[6] J. Linsley: “Evidence for a Primary Cosmic-Ray Particle with Energy 10^{20}eV”. Phys. Rev. Lett. **10** (1963) 146-148.

[7] A.A. Penzias, R.W. Wilson: “A Measurement of Excess Antenna Temperature at 4080 mc/s. ”. Astrophys. J. **142** (1965) 419-421.

[8] R. Torretti: *Relativity and Geometry*. Pergamon Press, Oxford, 1983.

[9] W.A. von Ignatowsky: “Das Relativitätsprinzip”. Arch. Math. Phys. **3** (17) (1911) 1-24.

[10] http://www.auger.org/icrc2001/icrc2001.html

[11] A. De Angelis: “GLAST, the Gamma-ray Large Area Space Telescope”. [astro-ph/0009271].

[12] http://wwwmagic.mppmu.mpg.de/

[13] G.Yu. Bogoslovsky, H.F. Goenner: “Finslerian Spaces Possessing Local Relativistic Symmetry”. [gr-qc/9904081].

[14] H.F. Goenner, G.Yu. Bogoslovsky: “A class of anisotropic (Finsler-) space-time geometries”. [gr-qc/9701067].

[15] G.Yu. Bogoslovsky, H.F. Goenner: “On a possibility of phase transictions in the geometric structure of space-time”. [gr-qc/9804082].

[16] G.Yu. Bogoslovsky: “From the Weyl theory to a theory of a locally anisotropic spacetime”. [hep-ph/0009305].

[17] G.S. Asanov: “Can Neutrinos and High-Energy Particles Test Finsler Metric of Space-Time?”. [hep-ph/0009305].

[18] G. Amelino-Camelia: “Relativity in space-times with short-distance structure governed by an observer-indipendent (Planckian) length scale”. Int. J. Mod. Phys. D **11** (2002) 35-59. [gr-qc/0012051].

[19] S. Gao, X. Wu: “Position space of Doubly Special Relativity”. [gr-qc/0311009].

[20] G. Amelino-Camelia, T. Piran: “Planck-scale deformation of Lorentz symmetry as a solution to the UHECR and the TeV-γ paradoxes”. Int. J. Mod. Phys. D **11** (2002) 35-59. [astro-ph/0008107].

[21] J. Magueijo, L. Smolin: “Generalized Lorentz invariance with an invariant energy scale”. [gr-qc/0207085].

[22] L. Gonzalez-Mestres: “Deformed Lorentz Symmetry and High-Energy Astrophysics (III)”. [hep-th/0208064].

[23] G. Amelino-Camelia: “Planck-scale deformation of Lorentz symmetry as a solution to the UHECR and the TeV- paradoxes”. [astro-ph/0008107].

[24] M. Toller: “Lorentz Symmetry and Ultra-high-Energy Cosmic Rays”. [hep-ph/0211094].

[25] M. Toller: “On the Lorentz Transformations of Momentum and Energy”. [hep-th/0301153].

[26] C. Rovelli, S. Speziale: “Reconcile Planck-scale discreteness and the Lorentz-Fitzgerald contraction”. Phys. Rev. D **67** 064019 (2003).

[27] J. Magueijo: “New varying speed of light theories”. [astro-ph/0305457].

[28] J. Magueijo L. Smolin: “Lorentz invariance with an invariant energy scale”. Phys. Rev. Lett. **88** 190403 (2002) [hep-th/0112090].

[29] S. Liberati, S. Sonego, M. Visser: "Interpreting doubly special relativity as a modified theory of measurement". [gr-qc/04100113].

[30] R. Schutzhold, W.G. Unruh: "Problems of doubly special relativity with variable speed of light". [arXiv:gr-qc/0308049]

[31] V.G. Gurzadyan et al.: "Probing the Light Speed ansotropy with respect to the Cosmic Microwave Background Radiation Dipole". [astro-ph/0410742]

[32] R. Mansouri, R.U. Sexl: "A test Theory of Special Relativity: I. Simultaneity and clock Synchronization". Gen. Rel. Grav. **8** (1977) 497-513.

[33] F. Selleri: "The Relativity Principle and the Nature of Time". Found. Phys. **27** (1997) 1527-158.

[34] http://lambda.gsfc.nasa.gov/product/cobe/bibliography.cfm

[35] H. Vucetich: "Testing Lorentz Invariance Violation in Quantum Gravity Theories". [gr-qc/0502093].

[36] D. Sudarsky, L. Urrutia, H. Vucetich: "Observational Bounds on Quantum Gravity Signals using Existing Data". Phys. Rev. Lett. **89** (1997) 231301.

[37] D. Sudarsky, L. Urrutia, H. Vucetich: "Bounds on stringy quantum gravity from low energy existing data". Phys. Rev. D **68** 024010 (1997).

[38] A. Einstein: "Zur Elektrodynamik betwegter Körper". Ann. Phys. **17** (1905) 891-921.

[39] M.V. Lalan: "Sur les postulats qui sont a la base des cinematiques". Bull. Soc. Math. France **65** (1937) 83-99.

[40] D. Bao, S.-S. Chern, Z. Sheng: *An Introduction to Riemann-Finsler Geometry*. Springer-Verlag, 2000.

[41] A. Einstein: Relativity: *The Special and the General Theory*. 15th ed., crown, New York, 1952.

[42] A. Brillet, J.L. Hall: "Improved Laser Test of the Isotropy of Space". Phys. Rev. Lett. **42** (1979) 549-552.

[43] S.K. Lamoreaux et al.: "New limits on spatial anisotropy from optically-pumped ^{201}Hg and ^{199}Hg". Phys. Rev. Lett. **57** (1986) 3125-4128.

[44] R. Anderson, I. Vetharaniam, G.E. Stedman: "Conventionality of Synchronisation, Gauge Dependenc and Test Theories of Relativity". Phys. Rep. **295** (1998) 93-180.

[45] H.C. Ohanian: "The role of dynamics in the synchronization problem". Am. J. Phys. **72** (2004) 141-148.

[46] A. Valentini: "Hidden Variables and the Large-Scale Structure of Spacetime". [quant-ph/0504011].

[47] http://relativity.livingreviews.org/Articles/lrr-2003-1/

[48] H. Reichenbach: *The Philosophy of Space and Time*. Dover, New York, 1957.

[49] M. Friedman: *Foundations of Space-Time Theories, Relativistic Physics and Philosophy of Science*. Princeton University Press, 1983.

[50] J.A. Winnie: "Special Relativity without One-way Velocity Assumption". Philos. Sci. **37** (1970) 81-99, 223-238.

[51] D. Malament, Noûs. **11** (1977) 293-300.

[52] A.A. Robb: *A Theory of Time and Space*. Cambridge University Press, Cambridge, 1914.

[53] J. Norton, in: M.H. Salmon et al. (Ed.) *Philosophy of Space and Time in Introduction to the Philosophy of Science*. Prentice-Hall, Englewood Cliffs,, NJ, 1992.

[54] M.L.G. Redhead, in: J. Earman (Ed.) *Philosophical Problems of the Internal and External World*. Univ. of Pittsburgh Press, Pittsburgh, 1993.

[55] P. Havas: "Simultaneity, Conventionalism, General Covariance, and the Special Theory of Relativity". Gen. Rel. Grav. **19** (1987) 435-453.

[56] A.A. Ungar: "Formalism to Deal with Reichenbach's Special Theory of Relativity". Found. Phys. **21** (1991) 691-725.

[57] F.R. Tangherlini: "An Introduction to the General Theory of Relativity". Suppl. Nuovo Cimento **20** (1961) 1-86.

[58] A. Grümbaum: *Philosophical Problems of Space and Time*. A. Knopf, New York, 1963.

[59] R.D. Klauber: " 'Pseudo'pseudo forces in gauge synchronization theories". [physics/0404037].

[60] S. Liberati, S. Sonego, M. Visser: "Faster-than-c-Signals, Special Relativity, and Causality". Ann. Phys. **298** (2002) 167-185.

[61] T.A. Debs, M.L.G. Redhead: "The twin 'paradox' and the conventionality of simultaneity". Am. J. Phys. **64** (1996) 384-392.

[62] F. Selleri: "Noninvariant One-Way Velocity of Light". Found. Phys. **26** (1996) 641-664.

[63] T. Jacobson, D. Mattingly: "Gravity with a dynamical preferred frame". Phys. Rev. D **64** (2000) 024028.

[64] D.A. Kirzhnits, V.A. Chechin: "Ultra-High-Energy Cosmic Rays and a Possible Generalization of Relativistic Theory". Sov. J. Nucl. Phys. **15** (5) (1972) 585-589.

[65] S. Sonego, M. Pin: "Generalised relativistic dynamics". To be published.

[66] R.M. Wald: *General Relativity.* University of Chicago Press, Chicago, 1984.

[67] W.M. Boothby: *An Introduction to Differentiable Manifolds and riemannian Geometry.* Academic Press, 1986.

[68] P.L. Antonelli, R. S. Ingarden, M. Matsumoto: *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology.* FTPH 58, Kluwer Academic Publishers, 1993.

[69] S.S. Chern: "Finsler Geometry Is Just Riemannian Geometry without the Quadratic Restriction". Notices of the AMS, September 1996.

[70] M. Crampin, F.A.E. Pirani: *Applicable Differential Geometry.* (London Math. Soc. Lect. Notes Series 59) Cambridge University Press, 1986.

[71] Z. Shen: "Geometric Meanings of Curvatures in Finsler Geometry". [math.DG/0011136].

[72] M. Panahi: "Hidden connection between general relativity and Finsler geometry". [gr-qc/0312053].

[73] L.D. Landau, E.M. Lifšits: *Teoria dei campi.* Editori Riuniti, 1994.

[74] C. Møller: *The Theory of Relativity.* Oxford University Press, 1972.

[75] N.D. Mermin: "Relativity without light". Am. J. Phys. **52** (1984) 119-124. Reissued in N.D. Mermin: *Boojums All The Way Through*. Cambridge University Press, 1990.

[76] J.-M. Lévy-Leblond: "One more derivation of the Lorentz transformation". Am. J. Phys. **44** (1976) 271-277.

[77] W. Rindler: *Relativity, special, general, and cosmological*. Oxford University Press, 2001.

[78] W. Rindler: *La Relatività Ristretta*. Edizioni Cremonese, 1971.

[79] Wu-Ki Tung: *Group Theory in Physics*. World Scientific, Philadelphia, 1985.

[80] V. Berzi, V. Gorini: "Reciprocity principle and the Lorentz transformations". J. Math. Phys. **21** (1969) 1518-1524.

[81] J. Lukierski, H. Ruegg, W.J. Zakrzewski: "Classical and Quantum Mechanics of Free k-Relativistic Systems". [hep-th/9312153].

[82] S. Sonego, M. Pin: "Deriving Relativistic Momentum and Energy". Eur. J. Phys. **26** (2005) 33-45.

[83] S. Sonego, M. Pin: "Deriving Relativistic Momentum and Energy II". In press to Eur. J. Phys.

[84] J.D. Bekenstein: "Relation between physical and gravitational geometry". Phys. Rev. D. **48** (1993) 3641-3647.

[85] D.I. Blokhintsev: Usp. Fiz. Nauk. **89** 185 (1966) [Sov. Phys.-Usp. **9** 405 (1966)]

[86] T. Jacobson, S. Liberati, D. Mattingly: "Astrophysical Bounds on Planck Suppressed Lorentz Violation". [hep-th/0407370].

[87] D.W. MacArthur et al.: "Test of the Special-Relativistic Doppler Formula at $\beta = 0.84$". Phys. Rev. Lett. **42** (1986) 282-285.

[88] P.S. Cooper et al.: "Experimental Test of Special Relativity from a High-γ Electron $g-2$ Measurement". Phys. Rev. Lett. **56** (1986) 1386-1389.

[89] D. Newman, G.W. Ford, A. Rich, E. Sweetman: "Precision experimental Verification of special Relativity". Phys. Rev. Lett. **40** (1978) 1355-1358.

[90] D.W. MacArthur: “Special relativity: Understanding experimental tests and formulations”. Phys. Rev. A **33** (1986) 1-5.

[91] A.K. Maciel, J. Tiommo: “Experiments to detect Possible Weak Violations of Special Relativity”. Phys. Rev. A **55** (1985) 143-146.

[92] R.J. Gould, G. Schéder: “Opacity of the Universe to High-Energy Photons”. Phys. Rev. Lett. **16** (1966) 252-254.

[93] R.J. Gould, G. Schéder: “Pair Production in Photon-Photon Collision”. Phys. Rev. **155** (1967) 1404-1407.

[94] R.J. Gould, G. Schéder: “Opacity of the Universe to High-Energy Photons”. Phys. Rev. **155** (1967) 1408-14011.

[95] L. Gonzalez-Mestres: “Physics Opportunities Above the Greisen-Zatsepin-Kuzmin Cutoff: Lorentz Symmetry Violation at the Planck Scale”. [physics/9712047].