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ELASTIC BODIES WITH RESIDUAL STRESS: VARIATIONAL MODELS BY Γ-CONVERGENCE

Relatori:
Prof. LORENZO FREDDI
Prof. ROBERTO PARONI

Dottorando:
LONDERO ALESSANDRO

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Introduction

It is common practice in structural engineering to consider "thin structures", in which one or more dimensions are small compared to others. For instance, we can think of plates, in which thickness is small, of rods, where trasversal section has small diameter, or of thin-walled beams, that is a rod in which the thickness of the trasversal cross-section is smaller than its diameter.

The research for this type of structures goes back to the pioneering works in mechanics of Euler, D. Bernoulli, Navier, Cauchy and Kirchhoff, see [36] and [44]. In particular, Cauchy, in 1829, derived the equations of a elastic material with residual stress.

The classical theories are usually based on some a-priori assumptions, due to the smallness of some directions with respect to the remaining, on the deformation of the body or on the induced stress field.

Many papers has been published in the last two-three decades to rigorously justify the a priori-assumptions on which classical theories are based. In particular, approaches based on the Γ-convergence of energy functionals (introduced by De Giorgi in [21]) have been used to derive one or two dimensional mechanical models for thin structures in linear elasticity starting from three-dimensional problems. For instance, we can recall [4] and [5] for the theory of isotropic plates, [4] and [22] for the theory of isotropic rods and [23] for the theory of isotropic thin-walled beams.

Recently, motivated by the necessity to obtain more accurate models, anisotropic thin structures with residual stress have been studied (see, for instance, Paroni [39] in the case of plates).

The aim of this work is to explain the results found, via Γ-convergence, for thin-walled beams with residual stress and for slender rods with residual stress. Moreover, in the last chapter, we obtain a first step to find a variational model for junction of two plates with residual stress: in fact we derive the kinematics of the body.

The thesis is divided in two parts: in the first two chapters, we recall the notion of Γ-convergence and the theory of linear elasticity with residual stress, while, in the last three chapters, we derive some models for thin structures with residual stress.

In Chapter 1, after recalling the direct method of the Calculus of Variations and the relaxation method, we introduce the concept of Γ-convercenge, following [6] and [17] and we study its main properties, in particular the variational one. Moreover, we extend the notion of Γ-limit to the case in which sequences of functionals and their limits are defined on different domain, following [4], and we recall Korn's inquality.

In Chapter 2, we recall the theory of linear elasticity with residual stress found by Cauchy in [8]. The presence of residual stress introduces in the constitutive equation the dependence of the Piola-Kirchhoff stress from the all displacement gradient in contrast to the case of no residual stress where the dependence is only on the strain. For residually stressed hyperelastic materials, the elastic energy density turns out to be non-convex, and this makes an important mathematical difference from the usual framework of linear elasticity (without residual stress). Moreover, we search for the relation between material symmetries of a body $\Omega \in \mathbb{R}^3$ and the residual stress $\mathring{\mathbf{T}}$ present in its reference configuration, following Hoger [31] and [32].

In Chapter 3, we consider a triclinic, inhomogeneous along the longitudinal axis, residually stressed, rectangular thin-walled beam and we derive, by Γ-convergence, a one-dimensional variational model from the three-dimensional theory of linear elasticity with residual stress. In particular, if we observe the equations of equilibrium (3.35), we note that the equations of longitudinal extension and bending in the (x_2, x_3)-plane are uncoupled, while the equations involving the twist and the displacement along the x_1-axis are coupled. Moreover, the residual stress appears only in the coupled equations. If we consider a monoclinic material symmetry, all equations decouple and the residual stress does not enter into the limit problem, see (3.39).

Chapter 4 is devoted to find a one-dimensional variational model for a triclinic, inhomogeneous along the longitudinal axis, residually stressed slender rod from three-dimensional problem of the theory of linear elasticity with residual stress. Moreover, in Section 4.7, we deduce the equations of equilibrium for a material with monoclinic symmetry. We point out that equations (4.35) depend on the residual stress component $\mathring{T}_{12}$, while the Euler-Lagrange equations for a thin walled beam with monoclinic symmetry are completely independent of $\mathring{\mathbf{T}}$.

Chapter 5 introduces the three-dimensional problem of junction of two triclinic, rectangular plates with residual stress. In particular, in Section 5.5, we derive the junction conditions which characterize the kinematics fields of the cross-section, following the lines traced by [24] and [34].

The contents of the present thesis are based on the papers [19], [20] and

[22].

Notation: for the reader's convenience, we introduce some basic notations that will be used throughout this work.

We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively. We define $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. For a subset M of the topological space X, we denote by $\overline{M}$, ∂M and χ_M the closure, the boundary and the characteristic function of M in X, respectively.

Unless otherwise stated, we use the Einstein summation convention and we index vector and tensor components as follows: Greek indices take values in the set $\{1, 2\}$ and Latin indices in the set $\{1, 2, 3\}$. The component k of a vector $\mathbf{v}$ is denoted either with v_k or $(v)_k$ and an analogous notation is used to denote tensor components. The trace of a matrix or a tensor $\mathbf{A}$ is denoted with $\mathrm{tr}\mathbf{A}$.

The derivative of a function f is denoted with Df or f', the partial derivative of f with respect to the variable x_1 is denoted with $D_{x_1}f$, $D_1 f$, $f_{,x_1}$ or $f_{,1}$, and the divergence of f is denoted with $\mathrm{div} f$. We denote with $L^p(A; B)$, $W^{s,p}(A; B)$ and the usual Lebesgue and Sobolev spaces of functions defined on the domain A and taking values in B, with the usual norm $\| \cdot \|_{L^p(A;B)}$ and $\| \cdot \|_{W^{s,p}(A;B)}$, respectively. The space of all distributions on A is denoted by $\mathcal{D}'(A)$. When $B = \mathbb{R}$ or when the right set B is clear from the context, we write $L^p(A)$ or $W^{s,p}(A)$. Moreover, we denote with $H^s(A; B)$ the Hilbert space $W^{s,2}(A; B)$.

Convergence in the norm, that is the so called strong convergence, is denoted by $\rightarrow$ while weak convergence is denote with $\rightharpoonup$.

With a little abuse of notation, we use to call "sequences" even those families indicized by a continouos parameter ε which is assumed to belong to the interval $(0, 1]$. For $h > 0$, we denote with $o(\varepsilon^h)$ an infinitesimal of ε^h, i.e. $\lim_{\varepsilon \to 0} \dfrac{o(\varepsilon^h)}{\varepsilon^h} = 0$.

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Chapter 1

Calculus of Variations

Let X be a topological space and let F be a functional from X to $\overline{\mathbb{R}}$. One of the main problems of the Calculus of Variations is to find the existence and a characterization of the minimum value and of the set of all minimum points for F in X. The case where both X and F vary can be reduced to the case where X is fixed and only F varies, allowing for functionals which take their values in the extended real line $\overline{\mathbb{R}}$. In fact, the problem can be reformulated in a larger ambient space Y (containing all sets X), by introducing, for every X the functional $F_X : Y \to \overline{\mathbb{R}}$ defined by $F_X(x) = F(x)$ for $x \in X$ and $F_X = +\infty$ for $x \notin X$. It is clear that, if $F \neq +\infty$, the minimum value and the set of minimum points of F are the same of F_X. Therefore, we shall assume that X is fixed.

It is clear that, if we consider a sequence (F_n) of perturbations of F, which converges to F in a very strong way, then, in general, we can prove that the minimum values of the functionals F_n converge to the minimum value of F. For instance, if (F_n) converges to F uniformly on X, then the sequence of minimum values of F_n converges to the minimum value of F. Although this assumption can be useful in some simple situation, it is not suitable for many applications to Physics and Engineering, characterized by perturbations of the minimum problems for integral of the form

$$F(\mathbf{u}) = \int_\Omega f(\mathbf{x}, \mathbf{Du}(\mathbf{x}))\, dx, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a function satisfying suitable conditions and $\mathbf{Du}$ denotes the gradient of the unknown function $\mathbf{u} : \Omega \to \mathbb{R}$.

On the other hand, if we suppose that (F_n) converges only pointwise to F on X, but not uniformly, then, in general, the "variational limit" of (F_n) can be different from the pointwise limit F, i.e. the sequence of minimum values of F_n can not converge to the minimum value of F.

Since its introduction by De Giorgi in [21], Γ-convergence has gained an undiscussed role as the most flexible and natural notion of convergence for variational problems, and is widely used also outside the field of the Calculus of Variations. In fact, good compactness properties are one of the main advantages of Γ-convergence, with respect to other "variational convergences", in particular the compactness of the class of all integral functionals of the form (1.1).

This first chapter is organized as follows. After a review of the direct method of the Calculus of Variations and of the relaxation method, we examine the notion of Γ-convergence for sequences of functionals defined on an arbitrary topological space X and its main properties. We extend the concept of Γ-convergence to the case of a sequence of functionals and of its Γ-limit defined on different topological spaces. At last we recall Korn's inequalities.

1.1 The direct method of the Calculus of Variations

In this section, we briefly recall Tonelli's direct method for the existence of the minimum points for a functional F.

Definition 1.1. *We say that a subset K of a topological space X is* compact *if every sequence in K has at least a cluster point in K. We say that K is* sequentially compact *if every sequence in K has a subsequence which converges to a point of K.*

Definition 1.2. *Let (X, τ) be a topological space. A function $F : X \to \overline{\mathbb{R}}$ is* lower semicontinuous at point $x \in X$ *if the set $F^{-1}(]c, +\infty]) = \{x \in X : F(x) > c\}$ is open in X for every $c \in \mathbb{R}$. We say that F is* lower semicontinuous on X *if F is lower semicontinuous at each point $x \in X$.*

F is sequentially lower semicontinuous in $x \in X$ *if*

$$\liminf_{n\to+\infty} f(x_n) \geq f(x) \quad \textit{for all} \quad x_n \to x$$

We say that F is sequentially lower semicontinuous on X *if F is sequentially lower semicontinuous at each point $x \in X$.*

We note that the definition of lower semicontinuous and sequentially lower semicontinuous are equivalent if X is a metric space.

Theorem 1.3 (Weierstrass). *Let (X, τ) be a topological space and K a closed and compact (respectively sequentially compact) subset of X. If $f : X \to \overline{\mathbb{R}}$ is a lower semicontinuous function (respectively sequentially lower semicontinuous function), then there exists the minimum of f in K.*

To prove Theorem 1.3, we need some auxiliary lemmas (for the proof, see [14]).

Lemma 1.4. *Let K be a closed subset of a compact topological space X. Then K is compact.*

Definition 1.5. *We say that a sequence of sets $(B_\alpha)_{\alpha \in I}$ has* the property of finite intersection *if for every finite subset $J \subseteq I$, the set $\cap_{\alpha \in J} B_\alpha$ is not empty.*

Lemma 1.6. *Let (X, τ) be a topological space. X is compact if and only if the intersection of every sequence of closed subset $(C_\alpha)_{\alpha \in I} \subset X$, that has the property of finite intersection, is not empty.*

Proof. (of Theorem 1.3) First we suppose that X is a metric space, hence we can use the sequential characterization of semicontinuous functions and of compactness of X. Then there exists a sequence (x_n) in K such that

$$\lim_{n \to +\infty} f(x_n) = \inf_K f.$$

By sequential compactness of K, there exists a subsequence (x_{n_h}) of (x_n) such that $x_{n_h} \to x \in K$; moreover f is a sequentially lower semicontinuous function, and hence

$$f(x) \leq \liminf_{h \to +\infty} f(x_{n_h}) = \lim_{n \to +\infty} f(x_n) = \inf_K f,$$

from which it follows that x is a minimum point for f.

We assume that X is a topological space. If f is the constant $+\infty$, we have that every $x \in K$ is a minimum point. Then we suppose

$$\inf\{f(x) : x \in K\} = M < +\infty.$$

From the fact that f is lower semicontinuous and K is a closed set, it follows that, for every $t \in \mathbb{R}$, the set $A := \{x \in K : f(x) \leq M\}$ is a closed subset of K and hence, by Lemma 1.4, A is compact; moreover, for $t > M$, A is not empty. By Lemma 1.6,

$$\{x \in K : f(x) \leq M\} = \bigcap_{t > M} \{x \in K : f(x) \leq t\}$$

is not empty. $\square$

Definition 1.7. *Let (X, τ) be a topological space and $f : X \to \overline{\mathbb{R}}$ a function. f is* coercive *on X if, for every $M \in \mathbb{R}$, there exists a closed and compact set K_M in X such that $\{x \in X : f(x) \leq M\} \subseteq K_M$.*
f is sequentially coercive *on X if, for every $M \in \mathbb{R}$, there exists a closed and sequentially compact set K_M in X such that $\{x \in X : f(x) \leq M\} \subseteq K_M$.*

We give now some example of what conditions are sufficient to obtain the coercivity for integral functionals as

$$F(\mathbf{u}) = \int_\Omega f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{Du}(\mathbf{x}))\, dx,$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\mathbf{u}$ is a function from Ω to $\mathbb{R}^m$ and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \overline{\mathbb{R}}$. We denote with $L^p(\Omega)$, $W^{1,p}(\Omega; \mathbb{R}^m)$ and $W_0^{1,p}(\Omega; \mathbb{R}^m)$ the usual Lebesgue and Sobolev spaces respectively.

- Assume that $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^m)$ with $1 < p < +\infty$. Then F is coercive with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m)$ if

  $$f(\mathbf{x}, \mathbf{z}, \boldsymbol{\xi}) \geq \alpha(|\mathbf{z}|^p + |\boldsymbol{\xi}|^p) + \beta(\mathbf{x}),$$

  where α is a positive constant and $\beta \in L^1(\Omega)$;

- assume that $\mathbf{u} \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ with $1 < p < +\infty$. Then F is coercive with respect to the weak topology of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ if

  $$f(\mathbf{x}, \mathbf{z}, \boldsymbol{\xi}) \geq \alpha|\boldsymbol{\xi}|^p + \beta(\mathbf{x}),$$

  where α is a positive constant and $\beta \in L^1(\Omega)$;

- assume that $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^m)$. Then F is coercive with respect to the weak topology of $W^{1,1}(\Omega; \mathbb{R}^m)$ if

  $$f(\mathbf{x}, \mathbf{z}, \boldsymbol{\xi}) \geq \psi(|\mathbf{z}| + |\boldsymbol{\xi}|) + \beta(\mathbf{x}),$$

  where ψ is a superlinear function and $\beta \in L^1(\Omega)$;

- assume that $\mathbf{u} \in W_0^{1,1}(\Omega; \mathbb{R}^m)$. Then F is coercive with respect to the weak topology of $W_0^{1,1}(\Omega; \mathbb{R}^m)$ if

  $$f(\mathbf{x}, \mathbf{z}, \boldsymbol{\xi}) \geq \psi(|\boldsymbol{\xi}|) + \beta(\mathbf{x}),$$

  where ψ is a superlinear function and $\beta \in L^1(\Omega)$;

- assume that $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. Then F is coercive with respect to the weak* topology of $W^{1,\infty}(\Omega;\mathbb{R}^m)$ if

$$f(\mathbf{x},\mathbf{z},\boldsymbol{\xi}) \geq \chi_{|\mathbf{z}|+|\boldsymbol{\xi}|\geq R} + \beta(\mathbf{x}),$$

  where $R > 0$, $\beta \in L^1(\Omega)$ and χ_X is the characteristic function of the set X.

The direct method in the Calculus of Variation is summerized by the following theorem.

Theorem 1.8 (Tonelli)**.** *Let (X,τ) be a topological space. If $F : X \to \overline{\mathbb{R}}$ is a lower semicontinuous (respectively sequentially lower semicontinuous) and coercive (respectively sequentially coercive) function, then F has a minimum point in X.*

Proof. (Topological case) If F is the constant $+\infty$, then every $x \in X$ is a minimum point. Then we suppose

$$\inf\{F(x) : x \in X\} = m < +\infty.$$

Let $t > m$, then we have

$$\{x \in X \;:\; F(x) \leq t\} \neq \emptyset.$$

By coercivity of F, there exists a closed, compact and no empty set K_t such that $\{x \in X : F(x) \leq t\} \subseteq K_t$. By Theorem 1.3, there exists $\bar{x} \in K_t$ such that $F(\bar{x}) \leq F(y)$ for every $y \in K$. From $\{x \in X : F(x) \leq t\} \neq \emptyset$, then we have $F(\bar{x}) \leq t$. Moreover, for every $y \in X \setminus K$, we have $F(\bar{x}) \leq t < F(y)$. Then $\bar{x}$, that is a minimum point in K_t, is also a minimum point in X.

(Sequential case) If F is identically $+\infty$, the claim of theorem is trivial. If

$$\inf\{F(x) \;:\; x \in X\} < +\infty, \tag{1.2}$$

let (x_n) be a sequence such that

$$\lim_{n\to+\infty} F(x_n) = \inf\{F(x) \;:\; x \in X\} < +\infty.$$

Then the limit of the sequence $F(x_n)$ is convergent or is $-\infty$; then there exists $L \in \mathbb{R}$ such that

$$F(x_n) \leq L \quad \forall\, n \in \mathbb{N}.$$

By sequential coercivity of F, the sequence (x_n) has a subsequence (x_{n_k}) that converges to an element $\bar{x} \in Y$. By lower semicontinuity of F, we have

$$F(\bar{x}) \leq \liminf_{k\to+\infty} F(x_{n_k}) = \inf\{F(x) \;:\; x \in X\},$$

and then

$$F(\bar{x}) \leq \inf\{F(x) \ : \ x \in X\}. \tag{1.3}$$

□

Definition 1.9. *Let X be a topological space. We say that a function $f : X \to \overline{\mathbb{R}}$ is* convex *if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for every $t \in]0,1[$ and $x, y \in X$ such that $f(x) < +\infty$ and $f(y) < +\infty$.
f is strictly convex *if f is not identically $+\infty$ and*

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y),$$

for every $t \in]0,1[$ and $x, y \in X$ such that $x \neq y$, $f(x) < +\infty$ and $f(y) < +\infty$.

Lemma 1.10. *Let $f : X \to \overline{\mathbb{R}}$ be a strictly convex function. Then f has at most one minimum point in X.*

Proof. If we suppose that x and y are two minimum points of f in X, then

$$f(x) = f(y) = \min_{z \in X} f(z) < +\infty.$$

If $x \neq y$, by the fact that$f$ is strictly convex, it follows

$$f(\frac{x}{2} + \frac{y}{2}) < \frac{1}{2}f(x) + \frac{1}{2}f(y) = \min_{z \in X} f(z) < +\infty,$$

that is in contrast with the fact that x and y are minimum points of f. Then we have $x = y$. □

We briefly recall Lax-Milgram lemma (for the proof, see, for instance, [7]).

Lemma 1.11 (Lax-Milgram)**.** *Let H be a Hilbert space, H' its dual space and $C_1, C_2 > 0$ two constant. Assume that $a(\cdot,\cdot)$ is a bilinear form on H such that $|a(x,y)| \leq C_1\|x\|\,\|y\|$ for every $x, y \in H$ and $|a(x,x)| \geq C_2\|x\|^2$. Then, for any $\varphi \in H'$, there is a unique solution $x \in H$ to the equation*

$$a(x,y) = \varphi(y) \qquad \text{for every } y \in H. \tag{1.4}$$

Moreover, if $a(\cdot,\cdot)$ is symmetric, then the solution x of equation (1.4) is the unique solution of minimum problem

$$\min_{y \in H}\{\frac{1}{2}a(y,y) - \varphi(y)\}.$$

1.2 Relaxation

Let (X, τ) be a topologiacal space and $F : X \to \overline{\mathbb{R}}$ a coercive functional, but not lower semicontinuous. Relaxation allows us to describe the minimizing sequences of F in terms of minimum points of semicontinuous functional $\overline{F}$ that checks:

1. $\min_{x \in X} \overline{F}(x) = \inf_{x \in X} F(x)$;
2. every minimum point x of $\overline{F}$ is a limit of a minimizing sequence of F and every minimizing sequence of F has a subsequence converging to a minimum point of $\overline{F}$.

Definition 1.12. *Let (X, τ) be a topological space. For every function $F : X \to \overline{\mathbb{R}}$, the* lower semicontinuous envelope *or* relaxed function *$\overline{F}$ of F is defined for every $x \in X$ by*

$$\overline{F}(x) = \sup_{G \in \mathcal{S}(F)} G(x),$$

where $\mathcal{S}(F)$ is the set of all lower semicontinuous functions G on X such that $G(x) \leq F(x)$ for every $x \in X$.

By definition of lower semicontinuous function, it follows that

$$\overline{F}(x) = \max_{G \in \mathcal{S}(F)} G(x) \tag{1.5}$$

for every $x \in X$.

Proposition 1.13. *For every $x \in X$, let $\mathcal{U}(x)$ the set of all open neighbourhoods of x in X. Then*

$$\overline{F} = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y)$$

Proof. Let

$$\Phi(x) = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y).$$

By definition of Φ and of lower semicontinuous function, we have that $\Phi \leq F$ and Φ is lower semicontinuity, i.e. $\Phi \in \mathcal{S}(F)$. Moreover we have $G \leq \Phi$ for every $G \in \mathcal{S}(F)$. In fact, by lower semicontinuity of G, we find

$$G(x) = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} G(y) \leq \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y) = \Phi(x).$$

Hence we obtain $\overline{F} = \Phi$. $\square$

The following proposition gives a characterization of $\overline{F}$ in terms of sequences.

Proposition 1.14. *Assume that X is a metric space. Then $\overline{F}$ is characterized by the following properties:*

(a) for every sequence (x_k) converging to $x \in X$, we have

$$\overline{F}(x) \leq \liminf_{k\to+\infty} F(x_k);$$

(b) for every $x \in X$, there exists a sequence (x_k) converging to x such that

$$\overline{F}(x) = \lim_{k\to+\infty} F(x_k).$$

Proof. Let prove property *(a)*. If (x_k) converges to x, by $\overline{F}(x) \leq F(x)$ and lower semicontinuity of $\overline{F}$, we have

$$\overline{F}(x) \leq \liminf_{k\to+\infty} \overline{F}(x_k) \leq \liminf_{k\to+\infty} F(x_k).$$

To prove *(b)*, we may suppose $\overline{F} < +\infty$. Let (U_k) be a countable base for the neighbourhood system of x such that $U_{k+1} \subseteq U_k$ for every $k \in \mathbb{N}$. From Proposition 1.13, we find

$$\inf_{y\in U_k} F(y) \leq \overline{F}(x).$$

Hence for every $k \in \mathbb{N}$, there exists a sequence $x_k \in U_k$ such that $F(x_k) \leq \overline{F}(x) + \dfrac{1}{k}$. Then (x_k) converges to x and $\limsup_{k\to\infty} F(x_k) \leq \overline{F}(x)$. □

We consider now the relation between the minimum problem $\min_{x\in X} F(x)$ and the relaxed problem $\min_{x\in X} \overline{F}(x)$. The next theorem describes the behaviour of the minimizing sequences of F in terms of the minimizers of $\overline{F}$.

Theorem 1.15. *Let $F : X \to \overline{\mathbb{R}}$ be a coercive function. Then we have:*

1. *$\overline{F}$ is coercive and it has a minimum point in X;*
2. *$\inf_{x\in X} F(x) = \min_{x\in X} \overline{F}(x)$;*
3. *if (x_n) is a minimizing sequence for F converging to $x \in X$, then x is a minimum point for $\overline{F}$; if X is a metric space, then, for every minimum point x of $\overline{F}$, there exists a minimizing sequence (x_n) for F converging to x.*

Proof. 1. By Proposition 1.13, it follows that for every $t \in \mathbb{R}$

$$\{x \in X : \overline{F}(x) < t\} \subseteq K := \overline{\{x \in X : F(x) < t\}},$$

where $\overline{\{x \in X : F(x) < t\}}$ is the closure of the set $\{x \in X : F(x) < t\}$. By coercivity of F, we have that K is compact. Then it follows that the set $\{x \in X : \overline{F}(x) < t\}$ is also compact and hence $\overline{F}$ is coercive. Moreover $\overline{F}$ has a minimum point in X by its lower semicontinuity.
2. Assume that $c = \inf_{x \in X} F(x)$. Then the constant function $G(x) = c$ is lower semicontinuous and $G(x) \leq F(x)$ for every $x \in X$. Hence, by definition of lower semicontinuous envelope, we have $c \leq \overline{F}(x)$ for every $x \in X$, i.e. $c \leq \min_{x \in X} \overline{F}(x)$. The opposite inequality is obvious, so *2.* is proved.
3. If $x_n \to x$ is a minimizing sequence for F, by lower semicontinuity of $\overline{F}$, it follows

$$\overline{F}(x) \leq \liminf_{n \to +\infty} \overline{F}(x_n) \leq \limsup_{n \to +\infty} F(x_n) = \inf_{y \in X} F(y).$$

Since $\inf_{y \in X} F(y) = \min_{y \in X} \overline{F}(y)$, x is a minimum point for $\overline{F}$. If X is a metric space and x a minimum point for $\overline{F}$, by Proposition 1.14, we have that there exists a sequence (x_n) converging to x in X such that

$$\inf_{y \in X} F(y) = \overline{F}(x) = \lim_{n \to +\infty} F(x_n).$$

Then (x_n) is a minimizing sequence for F. □

1.3 Γ-convergence

In this section we recall the definition of Γ-limit and its main properties. For the proofs omitted, see [6] and [17].

Let X be a topological space. Henceforth, for every $x \in X$, we denote with $\mathcal{U}(x)$ the set of all open neighbourhoods of x in X.

Definition 1.16. *Let (X, τ) be a topological space and F_n a sequence of functions from X into $\overline{\mathbb{R}}$. The* Γ-lower limit *and the* Γ-upper limit *of the sequence (F_n) are respectively the functions from X into $\overline{\mathbb{R}}$ defined by*

$$(\Gamma_\tau - \liminf_{n \to +\infty} F_n)(x) := \sup_{U \in \mathcal{U}(x)} \liminf_{n \to +\infty} \inf_{y \in U} F_n(y),$$

$$(\Gamma_\tau - \limsup_{n \to +\infty} F_n)(x) := \sup_{U \in \mathcal{U}(x)} \limsup_{n \to +\infty} \inf_{y \in U} F_n(y).$$

If there exists a function $F : X \to \overline{\mathbb{R}}$ *such that* $\Gamma_\tau - \liminf_{n\to+\infty} F_n = \Gamma_\tau - \limsup_{n\to+\infty} F_n = F$, *then we write* $F := \Gamma_\tau - \lim_{n\to+\infty} F_n$ *and we say that the sequence* (F_n) Γ_τ-converges *to* F *(in* X*).*

Remark 1.17. A sequence of function $F_\varepsilon : X \to \overline{\mathbb{R}}$ Γ_τ-converges to the functional $F : X \to \overline{\mathbb{R}}$ if for every sequence ε_n of real positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, we have

$$\Gamma_\tau - \lim_{n\to\infty} F_{\varepsilon_n} = F.$$

Henceforth, if it is clear the topology τ of the space X, the explicit dependence on τ of Γ-limits will be suppressed.

Lemma 1.18. *The functions* $\Gamma - \limsup_{n\to+\infty} F_n$ *and* $\Gamma - \liminf_{n\to+\infty} F_n$ *are lower semicontinuous in* X.

To prove Lemma 1.18, we recall the following result.

Proposition 1.19. *Let* X *be a topological space and* $\mathcal{N}$ *the family of all open subset of* X. *Then, if we define a function* $\alpha : \mathcal{N} \to \overline{\mathbb{R}}$, *the function* $F : X \to \overline{\mathbb{R}}$ *defined by*

$$F(x) = \sup_{U\in\mathcal{U}(x)} \alpha(U)$$

is lower semicontinuous in X.

Proof. (of Lemma 1.18) We apply Proposition 1.19 to

$$\alpha(U) = \liminf_{n\to+\infty} \inf_{y\in U} F_n(y)$$

and

$$\beta(U) = \limsup_{n\to+\infty} \inf_{y\in U} F_n(y),$$

defined for every open subset U of X. □

We study now the Γ-limits of the sum of two sequences (F_n) and (G_n) of functions from X to $\overline{R}$.

Proposition 1.20. *Let* (F_n) *and* (G_n) *be two sequences of functions from* X *into* $\overline{\mathbb{R}}$. *If* $(F_n + G_n)$ *is well defined in* X, *i.e.* $(-\infty, +\infty) \neq (F_n(x), G_n(x)) \neq (+\infty, -\infty)$ *for every* $x \in X$ *and* $n \in \mathbb{N}$, *then we find*

$$\Gamma - \liminf_{n\to+\infty}(F_n + G_n) \geq \Gamma - \liminf_{n\to+\infty} F_n + \Gamma - \liminf_{n\to+\infty} G_n, \tag{1.6}$$

$$\Gamma - \limsup_{n\to+\infty}(F_n + G_n) \geq \Gamma - \limsup_{n\to+\infty} F_n + \Gamma - \limsup_{n\to+\infty} G_n. \tag{1.7}$$

In particular, if (F_n) Γ-converges to F, (G_n) Γ-converges to G and $(F_n + G_n)$ Γ-converges to H, then $F + G \leq H$, if we suppose that $(F_n + G_n)$ and $F + G$ are well defined on X.

Inequalities (1.6) and (1.7) can be strict, even if (F_n) and (G_n) Γ-converge (see Dal Maso [17], Example 6.18 and Example 6.19). If one of the two sequences is continuously convergent, then inequalities (1.6) and (1.7) become equalities.

Definition 1.21. *The sequence of functionals (F_n) is* continuously convergent *in X to a functional $F : X \to \overline{\mathbb{R}}$ if for every $x \in X$ and for every neighbourhood V of $F(x)$ in $\overline{\mathbb{R}}$, there exist $k \in \mathbb{N}$ and $U \in \mathcal{U}(x)$ such that $F_n(y) \in V$ for every $n \geq k$ and $y \in U$.*

Remark 1.22. If X is a metric space, (F_n) is continuously convergent in X to F if for every $y \in X$ we have

$$y_n \to y \quad \Rightarrow \quad F_n(y_n) \to F(y).$$

Remark 1.23. (F_n) is continuously convergent in X to F if and only if (F_n) and $(-F_n)$ Γ-converge to F and $-F$ respectively.

Lemma 1.24. *Let (F_n) and (G_n) be two sequence of functions from X into $\overline{\mathbb{R}}$. If $(F_n + G_n)$ is well defined in X, the sequence (G_n) converges with continuity to G and $-\infty < G_n, G < +\infty$ for every $x \in X$ and $n \in \mathbb{N}$, then*

$$\Gamma - \liminf_{n\to+\infty}(F_n + G_n) = \Gamma - \liminf_{n\to+\infty} F_n + G,$$

$$\Gamma - \limsup_{n\to+\infty}(F_n + G_n) = \Gamma - \limsup_{n\to+\infty} F_n + G.$$

In particular, if (F_n) Γ-converges to F then $(F_n + G_n)$ Γ-converges to $F + G$ in X.

Proof. We prove the lemma for $\Gamma - \liminf$, the proof for $\Gamma - \limsup$ is similar. From Remark 1.23, we have that the sequence (G_n) Γ-converges to G in X; from Proposition 1.20, we find

$$\Gamma - \liminf_{n\to+\infty}(F_n + G_n) \geq \Gamma - \liminf_{n\to+\infty} F_n + G.$$

Moreover, by Remark 1.23, we obtain that the sequence $(-G_n)$ Γ-converges to $-G$ in X. By Proposition 1.20, we have

$$\Gamma - \liminf_{n\to+\infty}(F_n) = \Gamma - \liminf_{n\to+\infty}(F_n + G_n - G_n) \geq \Gamma - \liminf_{n\to+\infty}(F_n + G_n) - G,$$

and hence
$$\Gamma - \liminf_{n\to+\infty}(F_n) + G \geq \Gamma - \liminf_{n\to+\infty}(F_n + G_n).$$
□

The following theorem provides a characterization of $\Gamma - \liminf_{n\to+\infty} F_n$ and $\Gamma - \limsup_{n\to+\infty} F_n$ when X is a metric space.

Theorem 1.25. *Let X be a metric space. If (F_n) is a sequence of functions from X to $\overline{\mathbb{R}}$, then the $\Gamma - \liminf_{n\to+\infty} F_n$ is characterized by the following properties:*

(a) for every $x \in X$ and for every sequence (x_n) converging to x in X, we have
$$(\Gamma - \liminf_{n\to+\infty} F_n)(x) \leq \liminf_{n\to+\infty} F_n(x_n);$$

(b) for every $x \in X$, there exists a sequence (x_n) converging to x in X such that
$$(\Gamma - \liminf_{n\to+\infty} F_n)(x) \geq \liminf_{n\to+\infty} F_n(x_n).$$

The $\Gamma - \limsup_{n\to+\infty} F_n$ is characterized by the following properties:

(c) for every $x \in X$ and for every sequence (x_n) converging to x in X, we have
$$(\Gamma - \limsup_{n\to+\infty} F_n)(x) \leq \limsup_{n\to+\infty} F_n(x_n);$$

(d) for every $x \in X$, there exists a sequence (x_n) converging to x in X such that
$$(\Gamma - \limsup_{n\to+\infty} F_n)(x) \geq \limsup_{n\to+\infty} F_n(x_n).$$

Moreover (F_n) Γ-converges to F if and only if:

(e) (liminf inequality) for every $x \in X$ and for every sequence (x_n) converging to x in X, we have
$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n); \tag{1.8}$$

(f) (limsup inequality) for every $x \in X$, there exists a sequence (x_n) converging to x in X such that
$$F(x) \geq \limsup_{n\to+\infty} F_n(x_n). \tag{1.9}$$

Proof. We prove points (a) and (c). Let (x_n) be a sequence converging to x in X and $U \in \mathcal{U}(x)$. Then there exists $k \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq k$; moreover we have $\inf\limits_{y\in U} F_n(y) \leq F_n(x_n)$ for every $h \geq k$. Hence we find

$$\liminf_{n\to+\infty} \inf_{y\in U} F_n(y) \leq \liminf_{n\to+\infty} F_n(x_n),$$

and

$$\limsup_{n\to+\infty} \inf_{y\in U} F_n(y) \leq \limsup_{n\to+\infty} F_n(x_n),$$

for every $U \in \mathcal{U}(x)$ and hence we have

$$(\Gamma - \liminf_{n\to+\infty} F_n)(x) \leq \liminf_{n\to+\infty} F_n(x_n),$$

$$(\Gamma - \limsup_{n\to+\infty} F_n)(x) \leq \limsup_{n\to+\infty} F_n(x_n).$$

We prove point (b). We suppose to fix a point $x \in X$ such that $(\Gamma - \liminf\limits_{n\to+\infty} F_n)(x) \leq +\infty$. Let (U_k) be a numerable base of neighbourhoods' set of x such that $U_{k+1} \subseteq U_k$ for every $k \in \mathbb{N}$ and let (s_k) be a sequence converging to $(\Gamma - \liminf\limits_{n\to+\infty} F_n)(x)$ in $\overline{\mathbb{R}}$ such that $s_k > (\Gamma - \liminf\limits_{n\to+\infty} F_n)(x)$ for every $k \in \mathbb{N}$. By definition of $\Gamma - \liminf\limits_{n\to+\infty} F_n$, we find

$$s_k > \liminf_{n\to+\infty} \inf_{y\in U_k} F_n(y),$$

for every $k \in \mathbb{N}$; hence, there exists a strictly increasing sequence of integer number (n_k) such that

$$s_k > \inf_{y\in U_k} F_{n_k}(y)$$

for every $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$, there exists $y_k \in U_k$ such that $s_k > F_{n_k}(y_k)$. We define a sequence (x_n) such that $x_n = y_k$ if $n = n_k$ for some $k \in \mathbb{N}$ and $x_n = x$ if $n \neq n_k$, for every $k \in \mathbb{N}$. From $x_n \in U_k$ for every $n \geq n_k$, it follows that the sequence (x_n) converges to x in X and, from $x_{n_k} = y_k$, we find

$$(\Gamma - \liminf_{n\to+\infty} F_n)(x) = \lim_{k\to+\infty} s_k \geq \liminf_{n\to+\infty} F_{n_k}(y_k) \geq \liminf_{n\to+\infty} F_n(x_n).$$

We prove point (d). We suppose to fix a point $x \in X$ such that $(\Gamma - \limsup\limits_{n\to+\infty} F_n)(x) \leq +\infty$. Let (U_k) be a sequence as in point (b) of the proof and let t_k be a decreasing sequence that converges to $(\Gamma - \limsup\limits_{n\to+\infty} F_n)(x)$ in $\overline{\mathbb{R}}$ such that $t_k > (\Gamma - \limsup_{n\to+\infty} F_n)(x)$ for every $k \in \mathbb{N}$. By definition of $\Gamma - \limsup_{n\to+\infty} F_n$, we find

$$t_k > \limsup_{n\to+\infty} \inf_{y\in U_k} F_n(y),$$

for every $k \in \mathbb{N}$; hence there exists a strictly increasing sequence of integer number (n_k) such that

$$t_k > \inf_{y\in U_k} F_n(y)$$

for every $n \geq n_k$. Hence, for every $n \geq n_k$, there exists $y_k^n \in U_k$ such that $t_k > F_n(y_k^n)$. Then we define a sequence (x_n) such that $x_n = y_k^n$ if $n_k \leq n < n_{k+1}$ and $x_n = x$ if $n < n_1$. From $x_n \in U_k$ for every $n \geq n_k$, it follows that the sequence (x_n) converges to x in X and, from $t_k > F_n(n_k)$ for every $n > n_k$, we obtain

$$(\Gamma - \limsup_{n\to+\infty} F_n)(x) = \lim_{k\to+\infty} t_k \geq \limsup_{n\to+\infty} F_n(x_n).$$

From points (a), (b), (c) e (d) of theorem, it follows that F_n Γ-converges to F if and only if points (e) and (f) are check. □

Remark 1.26. If a sequence (x_n) checks the limsup inequality condition, then

$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n) \leq \limsup_{n\to+\infty} F_n(x_n) \leq F(x),$$

and hence $F(x) = \lim_{n\to+\infty} F_n(x_n)$; therefore condition (f) of the previous theorem can be replaced by:
$(f)'$ (existence of a recovery sequence) there exists a sequence (x_n) that converges to x such that

$$F(x) = \lim_{n\to+\infty} F_n(x_n).$$

Definition 1.27. *A sequence of functions (F_n) is* equi-coercive *on X if for every $t \in \mathbb{R}$ there exists a closed and compact subset K_t of X such that $\{x \in X : F_n(x) \leq t\} \subseteq K_t$ for every $n \in \mathbb{N}$.*

The following lemma gives a characterization of Γ-convergence if X is endowed with weak topology.

Lemma 1.28. *Assume that X is a reflexive Banach space endowed with its weak topology and F_n a equi-coercive sequence in the weak topology of X. Then the Γ-liminf in the the weak topology of X is characterized by conditions (a) and (b) of Lemma 1.25, where convergence means now weak convergence in X. Moreover if F satisfies conditions (e) and (f) of Lemma 1.25, then F_n Γ-converges to F.*

The following theorem concerns the convergence of the minimum values of an equi-coercive sequence of functions.

Theorem 1.29 (variational property of Γ-convergence)**.** *If (F_n) is equi-coercive in a topological space X and Γ-converges to a function F in X, then F is coercive and*

$$\min_{x\in X} F(x) = \lim_{n\to+\infty} \inf_{x\in X} F_n(x). \tag{1.10}$$

Moreover if x_n is a minimum point of F_n, then:

1. *if $x_n \to x$ and $\min_{x\in X} F(x) < +\infty$ then x is a minimum point of F;*
2. *if $\min_{x\in X} F(x) < +\infty$, then there exist a minimum point x of F and a subsequence (x_{n_k}) such that $x_{n_k} \to x$.*

To prove Theorem 1.29, we need some auxiliary results.

Proposition 1.30. *Let U be an open subset of X. Then*

$$\inf_{x\in U}(\Gamma - \liminf_{n\to+\infty} F_n)(x) \geq \liminf_{n\to+\infty} \inf_{x\in U} F_n(x)$$

and

$$\inf_{x\in U}(\Gamma - \limsup_{n\to+\infty} F_n)(x) \geq \limsup_{n\to+\infty} \inf_{x\in U} F_n(x).$$

Proof. We prove the first inequality, the proof of the second one is similar. For every $x \in U$, we have $U \in \mathcal{U}(x)$. Moreover, by definition of $\Gamma - \liminf$, we find

$$(\Gamma - \liminf_{n\to+\infty} F_n)(x) \geq \liminf_{n\to+\infty} \inf_{x\in U} F_n(x).$$

Then

$$\inf_{x\in U}(\Gamma - \liminf_{n\to+\infty} F_n)(x) \geq \liminf_{n\to+\infty} \inf_{x\in U} F_n(x).$$

□

The proof of next proposition follows from definition of Γ-convergence and from the fact that a functional $G : X \to \overline{\mathbb{R}}$ is lower semicontinuous if and only if

$$G(x) \leq \sup_{U\in\mathcal{U}(x)} \inf_{y\in U} G(y).$$

Proposition 1.31. *Let (F_n) be a sequence of functionals from X to $\overline{\mathbb{R}}$ and $H : X \to \overline{\mathbb{R}}$ a lower semicontinuous functional such that $H \leq F_n$ in X for every $n \in \mathbb{N}$. Then*

$$H \leq \Gamma - \liminf_{n\to+\infty} F_n \leq \Gamma - \limsup_{n\to+\infty} F_n.$$

In particular, if (F_n) Γ-converges to F, then $H \leq F$.

Proposition 1.32. *A sequence of functionals (F_n) from X to $\overline{\mathbb{R}}$ is equi-coercive if and only if there exists a lower semicontinuous and coercive functional $\Psi : X \to \overline{\mathbb{R}}$ such that $F_n \geq \Psi$ in X for every $n \in \mathbb{N}$.*

Proof. We suppose that a lower semicontinuous and coercive functional Ψ exists. Then

$$\{x \in X : F_n(x) \leq t\} \subseteq \{x \in X : \Psi(x) \leq t\}$$

for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Moreover, from coercivity of Ψ, we can define a sequence of closed and compact sets K_t such that $\{x \in X : \Psi(x) \leq t\} \subseteq K_t$ for every $t \in \mathbb{R}$. Hence (F_n) is equi-coercive.

We suppose that the sequence (F_n) is equi-coercive. Then it exists a sequence $(K_t)_{t\in\mathbb{R}}$ of closed and compact subsets of X such that $\{x \in X : F_n(x) \leq t\} \subseteq K_t$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Let $\Psi : X \to \overline{\mathbb{R}}$ be a functional defined by

$$\Psi(x) := \inf\{s \in \mathbb{R} : x \in K_t \text{ for every } t > s\},$$

if $\{s \in \mathbb{R} : x \in K_t \text{ for every } t > s\}$ is not empty and $\Psi(x) := +\infty$ otherwise. If $F_n(x) \leq s$ then $x \in K_t$ for every $t > s$ and hence $\Psi(x) \leq s$. This implies that $\Psi \leq F_n$ in X for every $n \in \mathbb{N}$. Moreover, from

$$\{x \in X : \Psi(x) \leq s\} = \bigcap_{t>s} K_t,$$

we have that $\{x \in X : \Psi(x) \leq s\}$ is a closed and compact set for every $s \in \mathbb{R}$. Then Ψ is coercive and lower semicontinuous in X. □

Proposition 1.33. *Let K be a closed and compact subset of X and (F_n) a sequence of functionals from X to $\overline{\mathbb{R}}$. Then we have*

$$\min_{x\in K}(\Gamma - \liminf_{n\to\infty} F_n)(x) \leq \liminf_{n\to\infty} \inf_{x\in K} F_n(x).$$

Proof. By Proposition 1.18 and Theorem 1.3, there exists the minimum of $\Gamma - \liminf_{n\to\infty} F_n$ in K. Let (F_{n_k}) be a subsequence of (F_n) such that

$$\lim_{k\to\infty} \inf_{x\in K} F_{n_k}(x) = \lim_{n\to\infty} \inf_{x\in K} F_n(x)$$

and (y_k) a sequence in K such that

$$\lim_{k\to\infty} F_{n_k}(y_k) = \lim_{k\to\infty} \inf_{x\in K} F_{n_k}(x).$$

From the fact that K is closed and compact, then it follows that $y_k \to y \in K$ for $k \to +\infty$. For every $U \in \mathcal{U}(y)$ and $m \in \mathbb{N}$, there exists $k \geq m$ such that

$y_k \in U$ and hence $\inf_{x\in U} F_{n_k}(x) \le F_{n_k}(y_k)$. Then we have

$$\begin{aligned}\liminf_{n\to+\infty} \inf_{x\in U} F_n(x) &\le \liminf_{k\to+\infty} \inf_{x\in U} F_{n_k}(x) \\ &\le \lim_{k\to+\infty} F_{n_k}(y_k) \\ &= \lim_{k\to+\infty} \inf_{x\in K} F_{n_k}(x) \\ &= \liminf_{n\to\infty} \inf_{x\in K} F_n(x).\end{aligned}$$

If we consider $\sup_{U\in\mathcal{U}(y)}$ in the inequality above, we find

$$(\Gamma - \liminf_{n\to\infty} F_n)(y) \le \liminf_{n\to\infty} \inf_{x\in K} F_n(x).$$

Since $y \in K$, we obtain $\min_{x\in K}(\Gamma - \liminf_{n\to\infty} F_n)(x) \le (\Gamma - \liminf_{n\to\infty} F_n)(y)$, from which it follows the claim of the proposition. □

Theorem 1.34. *Let K be a closed and compact subset of X such that*

$$\inf_{x\in X} F_n(x) = \inf_{x\in K} F_n(x) \tag{1.11}$$

for every $n \in \mathbb{N}$. Then there exists a minimum point for $\Gamma - \liminf_{n\to\infty} F_n$ in X and

$$\min_{x\in X}(\Gamma - \liminf_{n\to\infty} F_n)(x) = \liminf_{n\to\infty} \inf_{x\in X} F_n(x). \tag{1.12}$$

Moreover, if (F_n) Γ-converges to F in X, then it exists a minimum point for F in X and

$$\min_{x\in X} F(x) = \lim_{n\to\infty} \inf_{x\in X} F_n(x). \tag{1.13}$$

Proof. By Proposition 1.30 applied to $U = X$, we have

$$\inf_{x\in X}(\Gamma - \liminf_{n\to+\infty} F_n)(x) \ge \liminf_{n\to+\infty} \inf_{x\in X} F_n(x).$$

By Proposition 1.33 and equation (1.11), we find

$$\begin{aligned}\inf_{x\in X}(\Gamma - \liminf_{n\to+\infty} F_n)(x) &\le \min_{x\in K}(\Gamma - \liminf_{n\to+\infty} F_n)(x) \\ &\le \liminf_{n\to+\infty} \inf_{x\in K} F_n(x) \\ &\le \liminf_{n\to+\infty} \inf_{x\in X} F_n(x),\end{aligned}$$

and hence

$$\inf_{x\in X}(\Gamma - \liminf_{n\to+\infty} F_n)(x) = \min_{x\in K}(\Gamma - \liminf_{n\to+\infty} F_n)(x) = \liminf_{n\to+\infty} \inf_{x\in X} F_n(x). \tag{1.14}$$

Then there exists a minimum point for $\Gamma - \liminf\limits_{n\to+\infty} F_n$ in X. If (F_n) Γ-converges to F, then, by Proposition 1.30 applied to $U = X$, we obtain

$$\inf_{x\in X} F(x) \geq \limsup_{n\to+\infty} \inf_{x\in X} F_n(x).$$

Hence, by inequality (1.14), we have

$$\liminf_{n\to+\infty} \inf_{x\in X} F_n(x) = \inf_{x\in X} F(x) \geq \limsup_{n\to+\infty} \inf_{x\in X} F_n(x).$$

□

The proof of next proposition follows by definition of Γ-limit.

Proposition 1.35. *Let (F_{n_k}) be a subsequence of functionals (F_n). Then we have*

$$\Gamma - \liminf_{n\to\infty} F_n \leq \Gamma - \liminf_{k\to\infty}(F_{n_k}),$$

$$\Gamma - \limsup_{n\to\infty} F_n \geq \Gamma - \limsup_{k\to\infty}(F_{n_k}).$$

Moreover, if (F_n) Γ-converges to F in X, then (F_{n_k}) Γ-converges to F in X.

Proof. (of Theorem 1.29) By Proposition 1.32, we find that there exists a lower semicontinuous and coercive functional $\Psi : X \to \overline{\mathbb{R}}$ such that $F_n \geq \Psi$ in X for every $n \in \mathbb{N}$. Hence, by Proposition 1.31, we have that $\Gamma - \lim_{n\to\infty} F_n \geq \Psi$. Then F is coercive; moreover, by Proposition 1.18, F is also lower semicontinuous. By Tonelli theorem, there exists a minimum point for F in X.
We prove that

$$\min_{x\in X}(\Gamma - \liminf_{n\to\infty} F_n)(x) = \liminf_{n\to\infty} \inf_{x\in X} F_n(x). \tag{1.15}$$

By a similar argument seen above, we find that $\Gamma - \liminf\limits_{n\to\infty} F_n$ is coercive and lower semicontinuous and hence there exists its minimum in X. By Proposition 1.30 applied to $U = X$, it follows that

$$\min_{x\in X}(\Gamma - \liminf_{n\to\infty} F_n)(x) \geq \liminf_{n\to\infty} \inf_{x\in X} F_n(x).$$

To prove equation (1.15) is sufficient showing that

$$\min_{x\in X}(\Gamma - \liminf_{n\to\infty} F_n)(x) \leq \liminf_{n\to\infty} \inf_{x\in X} F_n(x). \tag{1.16}$$

Without loss of generality, we can assume that $\liminf_{n\to\infty} \inf_{x\in X} F_n(x) \le +\infty$. Hence there exist a constant $t \in \mathbb{R}$ and a subsequence (F_{n_k}) of (F_n) such that

$$\lim_{k\to+\infty} \inf_{x\in X} F_{n_k} = \liminf_{n\to\infty} \inf_{x\in X} F_n(x) < t.$$

Moreover, we can assume that

$$\inf_{x\in X} F_{n_k} < t \tag{1.17}$$

for every $k \in \mathbb{N}$. Since (F_n) is equi-coercive, there exists a closed and compact subset K of X such that $\{x \in X : F_{n_k} \le t\} \subseteq K$ for every $k \in \mathbb{N}$. By inequality (1.17), we find that $\{x \in X : F_{n_k} \le t\}$ is not empty, and hence

$$\inf_{x\in X} F_{n_k} = \inf_{x\in K} F_{n_k}$$

for every $k \in \mathbb{N}$. By Theorem 1.34 applied to (F_{n_k}), we obtain

$$\min_{x\in X}(\Gamma - \liminf_{k\to\infty} F_{n_k})(x) = \lim_{k\to+\infty} \inf_{x\in X} F_{n_k}(x) = \liminf_{n\to\infty} \inf_{x\in X} F_n(x) < t.$$

By Proposition 1.35, we have $\Gamma - \liminf_{n\to\infty} F_n \le \Gamma - \liminf_{n\to\infty} F_{n_k}$, and hence

$$\min_{x\in X}(\Gamma - \liminf_{n\to\infty} F_n)(x) \le \liminf_{n\to\infty} \inf_{x\in X} F_n(x).$$

We suppose that (F_n) Γ-converges to F; then, by Proposition 1.30 applied to $U = X$, we find

$$\min_{x\in X} F(x) \ge \limsup_{n\to\infty} \inf_{x\in X} F_n(x).$$

By equation (1.15) and inequality above, we find that

$$\min_{x\in X} F(x) = \lim_{n\to+\infty} \inf_{x\in X} F_n(x).$$

Henceforth we suppose that X is a metric space (for the topological case, see [17]).

Let x_n be a minimum point for F_n, $x_n \to x$ and $\min_{x\in X} F(x) < +\infty$. By Theorem 1.25 and equation (1.10), we have

$$\begin{aligned} F(x) &\le \liminf_{n\to+\infty} F_n(x_n) \\ &\le \inf_{x\in X} F(x), \end{aligned}$$

and hence x is a minimum point for F.

At last, we prove that, if x_n is a minimum point for F_n and $\min_{x\in X} F(x) < +\infty$, then there exist a minimum point x for F and a subsequence (x_{n_k}) such that $x_{n_k} \to x$. If $\min_{x\in X} F(x) = m$, then we have $m = \lim_{n\to+\infty} \min_{x\in X} F_n(x)$. Hence there exists $t \in \mathbb{R}$ such that $F_n(x_n) = \min_{x\in X} F_n(x) \leq t$ for every $n \in \mathbb{N}$. By equi-coercivity of (F_n), there exists a closed and compact subset K_t of X such that $x_n \in K_t$ for every $n \in \mathbb{N}$. Then we have a subsequence (x_{n_k}) of (x_n) converging to a point $x \in K_t$. By equation (1.10), x is a minimum point for F in X. $\square$

Remark 1.36. Under the same assumptions of Theorem 1.29, if it exists a unique minimum point x for F in X and x_n is a minimum point for F_n in X, we have that $x_n \to x$ and $F_n(x_n) \to F(x)$.

1.4 Γ-limits of functionals defined on a variable domain

In this section we extend the concept of Γ-convergence to the case of a sequence of functionals $F_\varepsilon : X \to \overline{\mathbb{R}}$ and of its Γ-limit F defined on different domains. It was introduced by Anzellotti, Baldo and Percivale in [4].

Let X be a set, (Y, τ) a topological space and $q_\varepsilon : X \to Y$ and $q_n : X \to Y$ two sequences of functions. If $F_n : X \to \overline{\mathbb{R}}$ is a sequence of functionals and $y \in Y$, we define

$$\begin{aligned} \Gamma(q_n, \tau Y) \liminf_{n\to\infty} F_n(y) &:= \inf\{\liminf_{n\to\infty} F_n(x_n) \;:\; q_n(x_n) \xrightarrow{\tau} y\}, \\ \Gamma(q_n, \tau Y) \limsup_{n\to\infty} F_n(y) &:= \inf\{\limsup_{n\to\infty} F_n(x_n) \;:\; q_n(x_n) \xrightarrow{\tau} y\}, \end{aligned} \tag{1.18}$$

that we call, respectively, *sequential* Γ-*liminf* and *sequential* Γ-*limsup* in y.

Definition 1.37. *Let ε_n be a sequence of positive real numbers, we say that the sequence $F_{\varepsilon_n} : X \to \overline{\mathbb{R}}$* sequentially $\Gamma(q_{\varepsilon_n}, \tau Y)$-converges *to $F : Y \to \overline{\mathbb{R}}$ in $y \in Y$ and we write*

$$\Gamma(q_{\varepsilon_n}, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y)$$

if

$$\Gamma(q_{\varepsilon_n}, \tau Y) \liminf_{n\to\infty} F_{\varepsilon_n}(y) = \Gamma(q_{\varepsilon_n}, \tau Y) \limsup_{n\to\infty} F_{\varepsilon_n}(y) = F(y). \tag{1.19}$$

We say that $F_\varepsilon : X \to \overline{\mathbb{R}}$ sequentially $\Gamma(q_\varepsilon, \tau Y)$-converges *to $F : Y \to \overline{\mathbb{R}}$ in $y \in Y$ and we write*

$$\Gamma(q_\varepsilon, \tau Y) \lim_{\varepsilon\to 0} F_\varepsilon(y) = F(y) \tag{1.20}$$

if for every sequence ε_n *of positive real numbers such that* $\lim_{n\to\infty} \varepsilon_n = 0$ *we have*

$$\Gamma(q_{\varepsilon_n}, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y).$$

We say that F_ε $\Gamma(q_\varepsilon, \tau Y)$-converges on Y *if* F_ε $\Gamma(q_\varepsilon, \tau Y)$*-converges for every* $y \in Y$.

Remark 1.38. We note that (1.19) is satisfied if and only if the two following conditions are checked:

1. for every sequence $x_n \in X$ such that $q_{\varepsilon_n}(x_n) \xrightarrow{\tau} y$ we have
$$\liminf_{n\to\infty} F_{\varepsilon_n}(x_n) \geq F(y);$$

2. there exists a sequence $\bar{x}_n \in X$ such that $q_{\varepsilon_n}(\bar{x}_n) \xrightarrow{\tau} y$ and
$$\lim_{n\to\infty} F_{\varepsilon_n}(\bar{x}_n) \geq F(y);$$

If $X = Y$ and q_ε is the identity function for every ε, then the Γ-limit defined in 1.37 coincides with the characterization of Γ-limit of Theorem 1.25. Moreover, if

$$I_\varepsilon(y) = \inf\{F_\varepsilon(x) \,:\, q_\varepsilon(x) = y\},$$

with $\varepsilon > 0$, then (1.20) is satisfied if and only if

$$\Gamma_\tau \lim_{\varepsilon\to 0} I_\varepsilon(y) = F(y).$$

When the topological space (Y, τ) satisfies the first axiom of countability and F_ε is equi-coercive, then the $\Gamma(q_\varepsilon, \tau Y)$-convergence has the variational property that is typical of Γ-convergence, i.e. the Γ-limit is lower semicontinuity and coercive. Moreover we have the convergence of minima and minimizer.

Definition 1.39. *We say that* F_ε *is* $(q_\varepsilon, \tau Y)$-equi-coercive *if for every real number* M *there exists a* τ*-closed and* τ*-compact subset* K_M *of* Y *such that*

$$\{q_\varepsilon(x) \,:\, F_\varepsilon(x) \leq M\} \subseteq K_M$$

for every $\varepsilon > 0$.

Lemma 1.40. *Let* (Y, τ) *be a topological space that satisfies the first axiom of countability (or* (Y, τ) *is a separable Banach space) and* τ *its weak topology. We assume that* $\Gamma(q_\varepsilon, \tau Y) \lim_{\varepsilon\to 0} F_\varepsilon = F$ *on* Y *and* F_ε *is* $(q_\varepsilon, \tau Y)$*-equi-coercive. Then we have:*

1. *F is τ-lower semicontinuous;*

2. *F is τ-coercive;*

3. *if $x_\varepsilon \in X$ satisfies $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf F_\varepsilon$ (i.e. x_e is a minimum point of F_ε) then:*

   (a) if ε_n is a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $q_{\varepsilon_n}(x_{\varepsilon_n}) \xrightarrow{\tau} y$ then y is a minimizer of F on Y and $\lim_{n\to\infty} F_{\varepsilon_n}(x_{\varepsilon_n}) = F(y)$;

   (b) there exists a sequence ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$ and a minimizer y of F on Y such that $q_{\varepsilon_n}(x_{\varepsilon_n}) \xrightarrow{\tau} y$.

1.5 Korn's inequalities

In this section we introduce Korn's inequalities (for the proofs, see [38]).
We denote with $\mathbf{u}, \mathbf{v}$ two vectorial functions $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$ and with $\mathbf{Du}$, $\mathbf{E}(\mathbf{u})$ two matrices defined by

$$(Du)_{ij} = \frac{\partial u_i}{\partial x_j}$$

and

$$E_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

respectively. We have

$$|\mathbf{Du}|^2 = \frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_h},$$

and

$$|\mathbf{E}(\mathbf{u})|^2 = \frac{1}{4}\Big(\frac{\partial u_i}{\partial x_h} + \frac{\partial u_h}{\partial x_i}\Big)\Big(\frac{\partial u_i}{\partial x_h} + \frac{\partial u_h}{\partial x_i}\Big).$$

Moreover we denote with $H^1(\Omega)$ the Sobolev space $W^{1,2}(\Omega)$ and with $H^1_0(\Omega)$ the Sobolev space $W^{1,2}_0(\Omega)$.

Theorem 1.41 (first Korn's inequality)**.** *Let Ω be a bounded subset of $\mathbb{R}^n$. If $\mathbf{u} \in H^1_0(\Omega)$ then $\mathbf{u}$ satisfies*

$$\|\mathbf{Du}\|^2_{L^2(\Omega)} \le 2\|\mathbf{E}(\mathbf{u})\|^2_{L^2(\Omega)}. \tag{1.21}$$

In the following theorem, we introduce second Korn's inequality.

Theorem 1.42 (second Korn's inequality)**.** *Let Ω be a bounded Lipschitz domain. If $\mathbf{u} \in H^1(\Omega)$, then $\mathbf{u}$ satisfies*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C\Big(\|\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{E}(\mathbf{u})\|_{L^2(\Omega)}\Big)$$

where C is a constant which depend only on Ω.

In many applications it is important the following version of the second Korn's inequality:

$$\|\mathbf{v}\|^2_{H^1(\Omega)} \leq C\|\mathbf{E}(\mathbf{v})\|^2_{L^2(\Omega)} \tag{1.22}$$

for every $\mathbf{v} \in V$, where V is an appropriate subspace of $H^1(\Omega)$.
We denote with $\mathcal{R}_2$ the space of rigid displacement on $\mathbb{R}^n$, i.e. the set of all functions $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ such that $\boldsymbol{\eta} = \mathbf{a} + \mathbf{W}\mathbf{x}$, where $\mathbf{a} = (a_1, \dots, a_n)$ is a vector of real components, and $\mathbf{W}$ is a $n \times n$ skew-matrix.

Theorem 1.43. *Let Ω a bounded Lipschitz domain and V a closed subspace of $H^1(\Omega)$ such that $V \cap \mathcal{R}_2 = \{0\}$, where $\mathcal{R}_2$ is the space of rigid displacement. Then $\mathbf{v} \in V$ satisfies*

$$\|\mathbf{v}\|^2_{H^1(\Omega)} \leq C\|\mathbf{E}(\mathbf{v})\|^2_{L^2(\Omega)}$$

Remark 1.44. Some examples of subspace V defined in Theorem 1.43 are:

1. $V = \{\mathbf{v} \in H^1(\Omega) : (\mathbf{v}, \boldsymbol{\eta})_{H^1(\Omega)} = 0 \quad \text{for every } \boldsymbol{\eta} \in \mathcal{R}_2\}$;
2. $V = \{\mathbf{v} \in H^1(\Omega) : (\mathbf{v}, \boldsymbol{\eta})_{L^2(\Omega)} = 0 \quad \text{for every } \boldsymbol{\eta} \in \mathcal{R}_2\}$,

where $(\cdot, \cdot)$ denotes the scalar product.

The next theorem gives a condition on the elements of space $H^1(\Omega)$ to verify inequality (1.22).

Theorem 1.45. *Let Ω a bounded domain with Lipschitz boundary. We suppose that $\gamma \subset \partial\Omega$ has strictly positive measure. Then every $\mathbf{v} \in H^1(\Omega)$ with null trace on γ satisfies*

$$\|\mathbf{v}\|^2_{H^1(\Omega)} \leq C\|\mathbf{E}(\mathbf{v})\|^2_{L^2(\Omega)}.$$

Chapter 2

The linear theory of elasticity with residual stress

The history of linear elasticity with residual stress is quite controversial. According to Truesdell [44], Cauchy in 1829 derived the equations of a pre-stressed elastic material but his results have been misunderstood and reported obscurely or even incorrectly by nineteenth century expositors. The correct theory reappeared much later in a few books, for instance in Truesdell [45] and Gurtin [29], but merely as exercises.
In this chapter we briefly recall the most important results of the theory of infinitesimal deformations and we justify the constitutive equation for hyperelastic materials with residual stress, following Gurtin [29] and [30]. Moreover, we consider some examples of material symmetries and we search for the relationship between them and the residual stress $\mathring{\mathbf{T}}$ present in the reference configuration of a body $\Omega \in \mathbb{R}^3$, following Hoger [31] and [32].

2.1 Infinitesimal deformations

In this section we recall some notions of the theory of infinitesimal deformations (see Gurtin [30]).
Hereafter, we shall identify a body Ω with the region occupied in a fixed reference configuration in $\mathbb{R}^3$.

Definition 2.1. *A* deformation *of Ω is a smooth homeomorphism $\mathbf{f}$ of Ω onto a region $\mathbf{f}(\Omega)$ with $det\mathbf{Df} > 0$. The point $\mathbf{f}(\mathbf{x})$ is the* place occupied *by the material point $\mathbf{x} \in \Omega$ in the deformation $\mathbf{f}$, while*

$$\mathbf{u}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) - \mathbf{x} \tag{2.1}$$

is the displacement *of* $\mathbf{x}$. *The tensor fields*

$$\mathbf{F}(\mathbf{x}) := \mathbf{Df}(\mathbf{x}) \qquad \mathbf{Hu}(\mathbf{x}) := \mathbf{Du}(\mathbf{x}) \tag{2.2}$$

are called, respectively, the deformation gradient *and the* displacement gradient.

From the definitions above, it follows that

$$\mathbf{Hu}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) - \mathbf{I}(\mathbf{x}),$$

where $\mathbf{I}$ is the identity tensor. If we denote with $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ the *right Cauchy-Green tensor*, we define the *strain tensor*

$$\mathbf{Eu}(\mathbf{x}) := \frac{1}{2}(\mathbf{C} - \mathbf{I}).$$

A deformation is *infinitesimal* if $\varepsilon := |\mathbf{F} - \mathbf{I}| = |\mathbf{Hu}|$ is, in some sense, "small". If we define

$$\mathbf{Hu} = \varepsilon\frac{\mathbf{Hu}}{|\mathbf{Hu}|} := \varepsilon\tilde{\mathbf{H}}\mathbf{u},$$

then $|\tilde{\mathbf{H}}\mathbf{u}| = 1$, and, neglecting ε^2 terms. we have

$$\begin{aligned}
\mathbf{C} &= (\mathbf{I} + \varepsilon\tilde{\mathbf{H}}\mathbf{u})^T(\mathbf{I} + \varepsilon\tilde{\mathbf{H}}\mathbf{u}) \\
&= \mathbf{I} + \varepsilon\tilde{\mathbf{H}}\mathbf{u}^T + \varepsilon\tilde{\mathbf{H}}\mathbf{u} + \varepsilon^2\tilde{\mathbf{H}}\mathbf{u}^T\tilde{\mathbf{H}}\mathbf{u} \\
&\cong \mathbf{I} + \mathbf{Hu}^T + \mathbf{Hu},
\end{aligned}$$

from which we have that the *infinitesimal strain tensor* is

$$\mathbf{Eu}(\mathbf{x}) = \frac{1}{2}(\mathbf{Hu}^T(\mathbf{x}) + \mathbf{Hu}(\mathbf{x})).$$

Remark 2.2. If we denote by $\mathbf{W}(\mathbf{x})$ the skew symmetric part of the displacement gradient $\mathbf{Du}(\mathbf{x})$, i.e.

$$\mathbf{W}(\mathbf{x}) = \frac{1}{2}(\mathbf{Du}(\mathbf{x}) - \mathbf{Du}^T(\mathbf{x})),$$

and by $\mathbf{E}(\mathbf{x})$ the symmetric part of $\mathbf{Du}(\mathbf{x})$, i.e.

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2}(\mathbf{Du}(\mathbf{x}) + \mathbf{Du}^T(\mathbf{x})),$$

we find that

$$\mathbf{Du}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + \mathbf{W}(\mathbf{x}). \tag{2.3}$$

Moreover we have

$$\frac{\partial W_{ij}(\mathbf{x})}{\partial x_k} = \frac{\partial E_{ik}(\mathbf{x})}{\partial x_j} - \frac{\partial E_{jk}(\mathbf{x})}{\partial x_i}, \tag{2.4}$$

for $i, j, k = 1, 2, 3$.

Definition 2.3. *An* infinitesimal rigid displacement *is a displacement* $\mathbf{u}$ *of the form*

$$\mathbf{u}(\mathbf{x}) := \mathbf{a} + \mathbf{W}(\mathbf{x} - \mathbf{x}_O),$$

where $\mathbf{x}_O \in \Omega$*,* $\mathbf{a}$ *a vector in* $\mathbb{R}^3$ *and* $\mathbf{W}$ *a skew symmetric tensor.*

We denote by $\mathcal{R}_2$ the set of the infinitesimal rigid displacements on Ω.

Theorem 2.4 (Characterization of rigid displacement)**.** *Let* $\mathbf{u}$ *be a displacement.* $\mathbf{u} \in \mathcal{R}_2$ *if and only if* $\mathbf{E}\mathbf{u} = 0$*.*

Proof. If $\mathbf{u}$ is rigid, then

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}) = \mathbf{W}(\mathbf{x} - \mathbf{y}),$$

with $\mathbf{W}$ a skew symmetric tensor and $\mathbf{x}, \mathbf{y} \in \Omega$. Hence we have

$$(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) = (\mathbf{x} - \mathbf{y}) \cdot \mathbf{W}(\mathbf{x} - \mathbf{y}) = 0. \tag{2.5}$$

Then, differentiating equation (2.5) with respect to $\mathbf{x}$, we find

$$\mathbf{D}\mathbf{u}^T(\mathbf{x})(\mathbf{x} - \mathbf{y}) + \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}) = 0.$$

Differentiating this equation with respect to $\mathbf{y}$ and evaluating the result at $\mathbf{y} = \mathbf{x}$, we have

$$-\mathbf{D}\mathbf{u}^T(\mathbf{x}) - \mathbf{D}\mathbf{u}(\mathbf{x}) = 0.$$

If $\mathbf{E}\mathbf{u} = 0$, then, by equation (2.4), we have $\dfrac{\partial W u_{ij}(\mathbf{x})}{\partial x_k} = 0$, and hence $\mathbf{D}\mathbf{u}(\mathbf{x}) = \mathbf{H}\mathbf{u}(\mathbf{x}) = \mathbf{C}$ for every $\mathbf{x} \in \Omega$, where $\mathbf{C}$ is a constant matrix. The fact that $\mathbf{E}\mathbf{u} = 0$ implies that $\mathbf{C}$ is a skew symmetric tensor. □

2.2 Constitutive equation for hyperelastic materials with residual stress

In this section, we find the first Piola-Kirchhoff tensor for a hyperelastic material with residual stress (see Gurtin [29]).

Let $\Omega = \Omega_1 \cup \Omega_2$ be a partition of Ω. We assume that the time will be fixed and will not appear explicity. We suppose that the common boundary Σ of Ω_1 and Ω_2 is sufficiently regular.

Concerning the action exerted by Ω_2 on Ω_1, we make the following assumptions introduced by Cauchy:

1. the forces exerted by Ω_2 on Ω_1 are contact forces, which means that they can be represented by a vector measure $d\varphi$ concentrated on Σ;

2. the measure $d\varphi$ is absolutely continuous with respect to the surface measure $d\Sigma$, that is
$$d\varphi = \boldsymbol{\sigma} d\Sigma,$$
where $\boldsymbol{\sigma}$ is the surface density of the forces;

3. the function $\boldsymbol{\sigma}$ depends only on the point $\mathbf{x}$ of Σ and on the unit normal $\mathbf{n}$ to Σ at point $\mathbf{x}$,
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, \mathbf{n});$$

4. for a fixed $\mathbf{n}$, the function $\boldsymbol{\sigma}(\cdot, \mathbf{n})$ is continuous.

The vector $\boldsymbol{\sigma}(\mathbf{x}, \mathbf{n})$ is called the *stress vector* at $\mathbf{x}$ for the direction $\mathbf{n}$. It can be proved that the stress vector $\boldsymbol{\sigma}(\mathbf{x}, \mathbf{n})$ is a linear function of the components of $\mathbf{n}$ (see, for instance, [37]). Hence we can introduce an operator $\mathbf{T}(\mathbf{x})$ such that
$$\boldsymbol{\sigma}(\mathbf{x}, \mathbf{n}) = \mathbf{T}(\mathbf{x})\mathbf{n}.$$
$\mathbf{T}(\mathbf{x})$ is called the *Cauchy stress tensor* at $\mathbf{x}$. It measures the conctact force in the deformed configuration.
In many problems it is not convenient to work with $\mathbf{T}$, since the deformed configuration is not known in advance. For this reason we introduce the *Piola-Kirchhoff tensor* which gives the force measured in the reference configuration. The first Piola-Kirchhoff tensor $\mathbf{S}$ is defined by
$$\mathbf{S}(\mathbf{x}) := (\det \mathbf{F})\mathbf{T}(\mathbf{x})\mathbf{F}^{-T},$$
where $\mathbf{T}(\mathbf{x})$ is the Cauchy tensor and $\mathbf{F}$ is the deformation gradient.

Definition 2.5. *A material is said to be* hyperelastic*, with respect to the reference configuration, if there exists a smooth function $W : \Omega \times Lin^+ \to \overline{\mathbb{R}}$ such that*
$$\mathbf{S}(\mathbf{x}) = D_{\mathbf{F}} W(\mathbf{x}, \mathbf{F}),$$
where Lin^+ is the set of linear application $\mathbf{F}$ with $det \mathbf{F} > 0$ and $D_{\mathbf{F}} W(\cdot, \mathbf{F})$ denotes the Frechet derivative of $W(\mathbf{x}, \cdot)$ for a fixed $\mathbf{x} \in \Omega$. The function W is called strain energy density.
If W is independent on $\mathbf{x}$, we say that the material is homogeneous.

Since the strain energy density function is defined up to a constant, hereafter we assume that $W(\cdot, \mathbf{I}) = 0$.

W is required to be independent of the observer, i.e. it satisfies the *principle of material frame indifference*

$$W(\cdot, \mathbf{QF}) = W(\cdot, \mathbf{F}), \tag{2.6}$$

for every rotation $\mathbf{Q}$ and $\mathbf{F} \in Lin^+$. Differentiating equation (2.6), we find that

$$\begin{aligned}(D_{\mathbf{F}} W(\cdot, \mathbf{F}))_{ij} &= \partial_{F_{rs}} W(\cdot, \mathbf{QF}) \frac{\partial Q_{rk} F_{ks}}{\partial F_{ij}} \\ &= \partial_{F_{rs}} W(\cdot, \mathbf{QF}) Q_{ri} \delta_{js},\end{aligned}$$

and hence

$$D_{\mathbf{F}} W(\cdot, \mathbf{F}) = \mathbf{Q}^T D_{\mathbf{F}} W(\cdot, \mathbf{QF}).$$

Denoting by

$$\mathcal{F}(\mathbf{x}, \mathbf{F}) := D_{\mathbf{F}} W(\mathbf{x}, \mathbf{F}),$$

we have

$$\mathbf{S}(\mathbf{x}) = \mathcal{F}(\mathbf{x}, \mathbf{F}) = \mathbf{Q}^T \mathcal{F}(\mathbf{x}, \mathbf{QF})$$

for every rotation $\mathbf{Q}$. From the equation above, it follows that

$$\mathbf{Q}\mathcal{F}(\cdot, \mathbf{F}) = \mathcal{F}(\cdot, \mathbf{QF})$$

for every rotation $\mathbf{Q}$. Since

$$\mathbf{Q}_\varepsilon := \mathbf{I} + \sin\varepsilon \mathbf{W} + (1 - \cos\varepsilon)\mathbf{W}^2$$

is a rotation for every $\varepsilon \in (0, \pi)$ and every skew symmetric tensor $\mathbf{W}$, we find that

$$\mathbf{Q}_\varepsilon \mathcal{F}(\cdot, \mathbf{F}) - \mathcal{F}(\cdot, \mathbf{F}) = \mathcal{F}(\cdot, \mathbf{Q}_\varepsilon \mathbf{F}) - \mathcal{F}(\cdot, \mathbf{F}).$$

From the fact that $\sin\varepsilon = \varepsilon + o(\varepsilon)$ and $1 - \cos\varepsilon = o(\varepsilon)$, where $o(\varepsilon)$ is a function such that $\lim\limits_{\varepsilon \to 0} \dfrac{o(\varepsilon)}{\varepsilon} = 0$, we have

$$\begin{aligned}\varepsilon \mathbf{W} \mathcal{F}(\cdot, \mathbf{F}) + o(\varepsilon) &= \mathbf{Q}_\varepsilon \mathcal{F}(\cdot, \mathbf{F}) - \mathcal{F}(\cdot, \mathbf{F}) \\ &= \mathcal{F}(\cdot, \mathbf{F} - \varepsilon \mathbf{WF} + o(\varepsilon)) - \mathcal{F}(\cdot, \mathbf{F}) \\ &= \varepsilon D_{\mathbf{F}} \mathcal{F}(\cdot, \mathbf{F}) \mathbf{WF} + o(\varepsilon).\end{aligned}$$

Dividing by ε the equation above and passing to the limit for $\varepsilon \to 0$, we obtain

$$\mathbf{W}\mathcal{F}(\cdot, \mathbf{F}) = D_{\mathbf{F}} \mathcal{F}(\cdot, \mathbf{F}) \mathbf{WF}, \tag{2.7}$$

for every skew symmetric tensor $\mathbf{W}$ and $\mathbf{F} \in Lin^+$.

If we consider a body Ω made of a hyperelastic material and we suppose that the body is clamped on $\Gamma_u \subset \partial\Omega$, i.e. $\mathbf{u}(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Gamma_u$, and it is subjected only to body forces $\mathbf{b}$, then the equilibrium equations are (see Gurtin [29] and [30]):

$$\begin{cases} \operatorname{div}\mathbf{S} + \mathbf{b} = 0 & \text{in } \Omega, \\ \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T & \text{in } \Omega, \\ \mathbf{S} = D_{\mathbf{F}} W(\cdot, \mathbf{F}) & \text{in } \Omega, \\ \mathbf{F} = \mathbf{Df} & \text{in } \Omega, \\ \mathbf{Sn} = 0 & \text{on } \partial\Omega \setminus \Gamma_u, \\ \mathbf{u} = 0 & \text{on } \Gamma_u. \end{cases} \tag{2.8}$$

Henceforth, we suppose

$$W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, \mathbf{I} + \varepsilon\tilde{\mathbf{H}}\mathbf{u}),$$

i.e. deformations are infinitesimal. In what follows, the explicit dependence on $\mathbf{x}$ of W will be suppressed except where its appearance is needed for clarity. Hence we appeal to Taylor's formula to find

$$\begin{aligned} W(\mathbf{I} + \varepsilon\tilde{\mathbf{H}}\mathbf{u}) &= W(\mathbf{I}) + \varepsilon D_{\mathbf{F}} W(\mathbf{I}) \cdot \tilde{\mathbf{H}}\mathbf{u} + \frac{1}{2}\varepsilon^2 D_{\mathbf{FF}} W(\mathbf{I})\tilde{\mathbf{H}}\mathbf{u} \cdot \tilde{\mathbf{H}}\mathbf{u} + o(\varepsilon^2) \\ &= \mathcal{F}(\mathbf{I}) \cdot \mathbf{Hu} + \frac{1}{2} D_{\mathbf{F}}\mathcal{F}(\mathbf{I})\mathbf{Hu} \cdot \mathbf{Hu} + o(\varepsilon^2), \end{aligned}$$

where $o(\varepsilon^2)$ is a function such that $\lim\limits_{\varepsilon \to 0} \dfrac{o(\varepsilon^2)}{\varepsilon^2} = 0$.

We define $\mathring{\mathbf{T}} := \mathcal{F}(\mathbf{I})$. $\mathring{\mathbf{T}}$ is called *residual stress*, i.e. $\mathring{\mathbf{T}}$ is the stress field present in the reference configuration. Recalling the definitions of the Cauchy symmetric tensor $\mathbf{T}$ and the first Piola-Kirchhoff tensor $\mathbf{S}$, we find that $\mathring{\mathbf{T}}$ is a symmetric tensor.

By equation (2.7), we obtain

$$D_{\mathbf{F}}\mathcal{F}(\mathbf{I})\mathbf{Wu} = \mathbf{Wu}\mathcal{F}(\mathbf{I}) = \mathbf{Wu}\mathring{\mathbf{T}}, \tag{2.9}$$

and hence, recalling equation (2.3), we have

$$
\begin{aligned}
W(\mathbf{I}+\mathbf{Hu}) &\cong \mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}D_{\mathbf{F}}\mathcal{F}(\mathbf{I})(\mathbf{Eu}+\mathbf{Wu})\cdot\mathbf{Hu}\\
&=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}(\mathbf{Wu}\mathring{\mathbf{T}}+D_{\mathbf{F}}\mathcal{F}(\mathbf{I})\mathbf{Eu})\cdot\mathbf{Hu}\\
&=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Wu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}D_{\mathbf{F}}\mathcal{F}(\mathbf{I})\mathbf{Hu}\cdot\mathbf{Eu}\\
&=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Wu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Wu}\mathring{\mathbf{T}}\cdot\mathbf{Eu}+\\
&\quad+\frac{1}{2}D_{\mathbf{F}}\mathcal{F}(\mathbf{I})\mathbf{Eu}\cdot\mathbf{Eu}\\
&=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Hu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}-\frac{1}{2}\mathbf{Eu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\\
&\quad+\frac{1}{2}\mathbf{Eu}\mathring{\mathbf{T}}\cdot\mathbf{Wu}+\frac{1}{2}D_{\mathbf{FF}}W(\mathbf{I})\mathbf{Eu}\cdot\mathbf{Eu}\\
&=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Hu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}-\frac{1}{2}\mathbf{Eu}\mathring{\mathbf{T}}\cdot\mathbf{Eu}+\\
&\quad+\frac{1}{2}D_{\mathbf{FF}}W(\mathbf{I})\mathbf{Eu}\cdot\mathbf{Eu}.
\end{aligned}
$$

We define

$$\mathbb{L}\mathbf{Eu}\cdot\mathbf{Eu}:=D_{\mathbf{FF}}W(\mathbf{I})\mathbf{Eu}\cdot\mathbf{Eu}-\mathbf{Eu}\mathring{\mathbf{T}}\cdot\mathbf{Eu}$$

the *incremental elasticity tensor*. Then we find

$$W(\mathbf{I}+\mathbf{Hu})=\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbf{Hu}\mathring{\mathbf{T}}\cdot\mathbf{Hu}+\frac{1}{2}\mathbb{L}\mathbf{Eu}\cdot\mathbf{Eu}.$$

Then the first Piola-Kirchhoff tensor $\mathbf{S}$ becomes

$$\mathbf{S}=D_{\mathbf{F}}W(\mathbf{I}+\mathbf{Hu})=\mathring{\mathbf{T}}+\mathbf{H}\mathring{\mathbf{T}}+\mathbb{L}\mathbf{Eu}, \tag{2.10}$$

while the Cauchy tensor $\mathbf{T}$ can be expressed as

$$
\begin{aligned}
\mathbf{T} &= (1+\varepsilon\mathrm{tr}\tilde{\mathbf{H}}\mathbf{u}+o(\varepsilon))^{-1}(\mathring{\mathbf{T}}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}\mathring{\mathbf{T}}+\varepsilon\mathbb{L}\tilde{\mathbf{E}}\mathbf{u})(\mathbf{I}+\varepsilon\tilde{\mathbf{H}}\mathbf{u})^T\\
&=(1-\varepsilon\mathrm{tr}\tilde{\mathbf{H}}\mathbf{u}+o(\varepsilon))(\mathring{\mathbf{T}}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}\mathring{\mathbf{T}}+\varepsilon\mathbb{L}\tilde{\mathbf{E}}\mathbf{u})(\mathbf{I}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}^T)\\
&\cong(\mathring{\mathbf{T}}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}\mathring{\mathbf{T}}+\varepsilon\mathbb{L}\tilde{\mathbf{E}}\mathbf{u}-\varepsilon\mathring{\mathbf{T}}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{u})(\mathbf{I}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}^T)\\
&=\mathring{\mathbf{T}}+\varepsilon\tilde{\mathbf{H}}\mathbf{u}\mathring{\mathbf{T}}+\varepsilon\mathbb{L}\tilde{\mathbf{E}}\mathbf{u}-\varepsilon\mathring{\mathbf{T}}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{u}+\varepsilon\mathring{\mathbf{T}}\tilde{\mathbf{H}}\mathbf{u}^T,
\end{aligned}
$$

where $\mathrm{tr}\tilde{\mathbf{H}}\mathbf{u}$ is the trace of the tensor $\tilde{\mathbf{H}}\mathbf{u}$ and $\tilde{\mathbf{E}}\mathbf{u}=\dfrac{\mathbf{Eu}}{|\mathbf{Eu}|}$. Hence we have

$$\mathbf{T}=\mathring{\mathbf{T}}(1-\mathrm{tr}\mathbf{Hu})+\mathbf{Hu}\mathring{\mathbf{T}}+\mathring{\mathbf{T}}\mathbf{Hu}^T+\mathbb{L}\mathbf{Eu}\neq\mathbf{S}. \tag{2.11}$$

By equation (2.9), we obtain $\mathbb{L}\mathbf{W} = 0$ for every skew symmetric tensor $\mathbf{W}$, and hence

$$\mathbb{L}_{ijpq} = \mathbb{L}_{ijqp},$$

for $i, j, p, q = 1, 2, 3$.

Remark 2.6. If we suppose that $\overset{\circ}{\mathbf{T}} = 0$, we find

$$W(\mathbf{I} + \mathbf{Hu}) = \frac{1}{2}\mathbb{L}\mathbf{Eu} \cdot \mathbf{Eu},$$

where

$$\mathbb{L} = D_{\mathbf{FF}}W(\mathbf{I}) =: \mathbb{C}.$$

$\mathbb{C}$ is called the *elasticity tensor*. Recalling equations (2.10) and (2.11), we find that

$$\mathbf{S} = \mathbf{T} = \mathbb{C}\mathbf{Eu}.$$

We can note that the presence of residual stress $\overset{\circ}{\mathbf{T}}$ introduces the dependence of the Piola-Kirchhoff tensor from the displacement gradient $\mathbf{Hu}$, while in the case of no residual stress the dependence is only on the infinitesimal strain tensor $\mathbf{Eu}$. From the fact that $\mathbf{S}$ is symmetric, it follows that $\mathbb{C}_{ijpq} = \mathbb{C}_{jipq}$, and, by definition of $\mathbb{C}$, we have $\mathbb{C}_{ijpq} = \mathbb{C}_{pqij}$ for $i, j, p, q = 1, 2, 3$. Hence we obtain the major and minor symmetries of $\mathbb{C}$, i.e.

$$\mathbb{C}_{ijpq} = \mathbb{C}_{jipq} = \mathbb{C}_{pqij},$$

for $i, j, p, q = 1, 2, 3$.

In the case of infinitesimal deformations, the equilibrium equations (2.8) can be rewritten as

$$\begin{cases} \operatorname{div}\mathbf{S} + \mathbf{b} = 0 & \text{in } \Omega, \\ \mathbf{S} = \overset{\circ}{\mathbf{T}} + \mathbf{Hu}\overset{\circ}{\mathbf{T}} + \mathbb{L}\mathbf{Eu} & \text{in } \Omega, \\ \mathbf{Sn} = 0 & \text{on } \partial\Omega \setminus \Gamma_u, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_u. \end{cases} \tag{2.12}$$

2.3 Material symmetries

In order to specify the symmetry properties of a material, we introduce the following definition.

Definition 2.7. *The* symmetry group $\mathcal{G}$ *for a material at the point* $\mathbf{x} \in \Omega$ *is the subgroup of all orthogonal tensors* $\mathbf{Q}$ *that checks*

$$\mathbf{Q}\mathcal{F}(\mathbf{F})\mathbf{Q}^T = \mathcal{F}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T), \tag{2.13}$$

for every tensor $\mathbf{F} \in Lin^+$.
We say that the material is isotropic *at* $\mathbf{x}$ *if the symmetry group* $\mathcal{G}$ *is equal to the orthogonal group,* anisotropic *if it is not isotropic.*

Although there are an infinite number of subgroups of the proper orthogonal group, twelve of them seem to exhaust the kinds of symmetries being appropriate to describe the behaviour of real elastic materials. The first eleven of these subgroups correspond to the thirty-two crystal class (see, for example, [15]). The last type of anisotropy, called *transverse isotropy* (with respect to a direction $\mathbf{k}$), is characterized by the group consisting of the rotations $R_{\mathbf{k}}^{\varphi}$, with $0 \leq \varphi < 2\pi$, where $R_{\mathbf{k}}^{\varphi}$ is the orthogonal tensor corresponding to a right-handed rotation through the angle φ around the direction $\mathbf{k}$.
The following lemma is proven by Gurtin in [30].

Lemma 2.8. *Given any two vectors* $\mathbf{a}, \mathbf{b}$, *let*

$$\mathbf{A}(\mathbf{a}, \mathbf{b}) = \frac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}),$$

where $\mathbf{a} \otimes \mathbf{b}$ *is the tensor product of the two vectors* $\mathbf{a}, \mathbf{b}$. *Then*

$$\mathbb{L}\mathbf{A}(\mathbf{m}, \mathbf{n}) \cdot \mathbf{A}(\mathbf{a}, \mathbf{b}) = \mathbb{L}\mathbf{A}(\mathbf{Q}\mathbf{m}, \mathbf{Q}\mathbf{n}) \cdot \mathbf{A}(\mathbf{Q}\mathbf{a}, \mathbf{Q}\mathbf{b}),$$

for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{m}, \mathbf{n}$ *and every element* $\mathbf{Q}$ *of the symmetry group* $\mathcal{G}$.

We give now some examples of material symmetries and their relation with the components of the incremental elasticity tensor $\mathbb{L}$ (for the relation with all symmetries and the components of $\mathbb{L}$, see [30]). Let $\{\mathbf{e}_i\}_{i=1}^3$ be an orthonormal basis of $\mathbb{R}^3$ and let $\mathbb{L}_{ijpq}$ denote the elasticities relative to this basis.
The symmetry group for *triclinic system* is the trivial one generated by the identity tensor; for this material system, there is no restrictions placed on $\mathbb{L}$ by material symmetry.
The symmetry group for a *monoclinic material* is generated by $R_{\mathbf{e}_3}^{\pi}$. By Lemma 2.8, it follows

$$\mathbb{L}_{ijpq} = \mathrm{sym}(\mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j) \cdot \mathbb{L}\,\mathrm{sym}(\mathbf{Q}\mathbf{e}_p \otimes \mathbf{Q}\mathbf{e}_q)$$

for every element $\mathbf{Q}$ of the symmetric group, where $\mathrm{sym}(\mathbf{A})$ is the symmetric part of the tensor $\mathbf{A}$. If we take $\mathbf{Q} = -R_{\mathbf{e}_3}^{\pi}$, then

$$\mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1 \qquad \mathbf{Q}\mathbf{e}_2 = \mathbf{e}_2 \qquad \mathbf{Q}\mathbf{e}_3 = -\mathbf{e}_3,$$

and the above equations imply

$$\mathbb{L}_{\alpha\beta\gamma 3} = 0, \qquad \mathbb{L}_{\alpha 333} = 0,$$

for all $\alpha, \beta, \gamma = 1, 2$.
The symmetry group for a *rhombic material* is generated by $R^{\pi}_{\mathbf{e}_2}$ and $R^{\pi}_{\mathbf{e}_3}$. In a similar way to the monoclinic symmetry, we obtain

$$\mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1 \qquad \mathbf{Q}\mathbf{e}_2 = \mathbf{e}_2 \qquad \mathbf{Q}\mathbf{e}_3 = -\mathbf{e}_3,$$

$$\mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1 \qquad \mathbf{Q}\mathbf{e}_2 = -\mathbf{e}_2 \qquad \mathbf{Q}\mathbf{e}_3 = \mathbf{e}_3,$$

if we take, respectively, $\mathbf{Q} = -R^{\pi}_{\mathbf{e}_3}$ and $\mathbf{Q} = -R^{\pi}_{\mathbf{e}_2}$. Hence we find

$$\mathbb{L}_{\alpha\beta\gamma 3} = \mathbb{L}_{\alpha 333} = \mathbb{L}_{ii12} = \mathbb{L}_{2313} = 0,$$

for all $\alpha, \beta, \gamma = 1, 2$ and $i = 1, 2, 3$.
For an isotropic material, i.e. when the material symmetry group is the orthogonal group, the incremental elasticity tensor $\mathbb{L}$ has the form

$$\mathbb{L}\mathbf{E}\mathbf{u} = 2\mu\mathbf{E}\mathbf{u} + \lambda(\mathrm{tr}\mathbf{E}\mathbf{u})\mathbf{I}.$$

The scalars $\mu > 0, \lambda \geq 0$ are called the Lamé moduli of the material (see, for instance, [30], Theorem 22.2).

2.4 Relation between residual stress and material symmetry

The residual stress $\mathring{\mathbf{T}}$ is defined in Section 2.2 to be the stress present in a body in the reference configuration, i.e.

$$\mathring{\mathbf{T}} = \mathcal{F}(\mathbf{I}).$$

The residual stress $\mathring{\mathbf{T}}$ must satisfy the equilibrium equation

$$\mathrm{div}\mathring{\mathbf{T}} = 0$$

in the body Ω, and the zero traction condition

$$\mathring{\mathbf{T}}\mathbf{n} = 0$$

on the boundary of the body, $\partial\Omega$, where $\mathbf{n}$ denotes the outward unit normal to the boundary of Ω. Recalling that $\mathring{\mathbf{T}}$ is symmetric (see Section 2.2), we find that it must satisfy

$$\begin{cases} \operatorname{div} \mathring{\mathbf{T}} = 0 & \text{in } \Omega, \\ \mathring{\mathbf{T}} = \left(\mathring{\mathbf{T}}\right)^T & \text{in } \Omega, \\ \mathring{\mathbf{T}}\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

By evaluating (2.13) at $\mathbf{F} = \mathbf{I}$, we find that the residual stress must satisfy

$$\mathring{\mathbf{T}}\mathbf{Q} = \mathbf{Q}\mathring{\mathbf{T}} \tag{2.15}$$

for every $\mathbf{Q} \in \mathcal{G}$, where $\mathcal{G}$ is the symmetry group of the material. Therefore a material with a particular symmetry can support only those residual stress tensors that commute with all elements of its symmetry group.

We assume that the body Ω is made entirely of material having a particular symmetry, i.e. the symmetry group is the same for all $\mathbf{x} \in \Omega$, and Ω is a regular body, i.e. $\partial\Omega$ is a union of a finite number of non intersecting closed regular surfaces.

There are no restrictions imposed on $\mathring{\mathbf{T}}$ by (2.15) for triclinic materials, so in this case no additional information can be gained by consideration of equations (2.14), unless the shape of the body is specified. On the contrary, by equation (2.15), we find that an isotropic body and a body of rhombic symmetry with uniform axes of symmetry can not support residual stress (see Hoger [31]).

For a body made of monoclinic material, we find the next result (that was proven in [31]).

Lemma 2.9. *In a body Ω with monoclinic symmetry and uniform axis of symmetry $\mathbf{k}$, the residual stress tensor $\mathring{\mathbf{T}}$ is planar, i.e.*

$$\mathring{\mathbf{T}}(\mathbf{x}) = \mathring{\mathbf{P}}(\mathbf{x})$$

for every $\mathbf{x} \in \Omega$, where $\mathring{\mathbf{P}}$ is a two dimensional state of stress in the plane whose normal is $\mathbf{k}$, and satisfies

$$div \mathring{\mathbf{P}} = 0 \qquad in\ \Omega,$$

and

$$\mathring{\mathbf{P}}\mathbf{n} = 0 \qquad on\ \partial\Omega.$$

Since $\mathbf{k}$ is uniform, an orthogonal basis can be chosen such that $\mathbf{k} = \mathbf{e}_3$, and, with respect to this basis, we obtain

$$\mathring{\mathbf{T}} = \mathring{\mathbf{P}} = \begin{pmatrix} \mathring{P}_{11} & \mathring{P}_{12} & 0 \\ \mathring{P}_{12} & \mathring{P}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Chapter 3

Thin-walled beams with residual stress

In this chapter, we derive a variational model for a thin-walled beam with residual stress. We start by considering a sequence of regions

$$\Omega_\varepsilon = \left(-\frac{a\varepsilon^2}{2}, \frac{a\varepsilon^2}{2}\right) \times \left(-\frac{b\varepsilon}{2}, \frac{b\varepsilon}{2}\right) \times (0, \ell) \subset \mathbb{R}^3,$$

where $\varepsilon \in (0, 1]$ is a smallness parameter and ℓ is the length of the beam. The different scalings in the first and second variable are introduced to model the thin-walled beam. We denote by $S_\varepsilon(x_3)$ a generic cross-section relative to the abscissa x_3. The body is clamped on $S_\varepsilon(0)$ and subject to dead-body forces of density $\mathbf{b}^\varepsilon$ and null contact-forces over their lateral surfaces. Moreover, the material is assumed to be triclinic and non homogeneous along the x_3-axis, so to model beams made of several different thin layers jointed along the longitudinal axis. After rescaling the problem to a fixed domain, we find the right scalings of the displacement components and of the residual stress tensor. By letting ε go to zero we then find, using Γ-convergence, the one-dimensional limit problem.

The chapter is organized as follows. In Section 3.1 we briefly describe the 3d-problem. We first state the strong formulation of the 3d-equilibrium problem, and then its weak form. The problem of existence of solutions is discussed in Section 3.2, where we make a wide use of results obtained by Paroni [39]. In Section 3.3, following a standard procedure, we rewrite the three dimensional variational problem on a fixed domain Ω. Section 3.4 is devoted to the Γ-convergence theoretical result. In our proof we use some compactness theorems obtained by Freddi, Morassi and Paroni [23]. The difficulty in passing to the limit is due to the lack of convexity of the rescaled energy. Subsequently, we study the convergence of minima and minimizers, in Section 3.5, and we deduce the equations of equilibrium. In particular, it is interesting to note that the equations of longitudinal extension and bending in the (x_2, x_3)-plane are uncoupled, while the equations involving the twist and the displacement along the x_1-axis are coupled, see (3.35). Moreover, the residual stress appears only in the coupled equations. For a material with monoclinic symmetry several coefficients of the incremental elasticity tensor and components of the residual stress are equal to zero. For this symmetry all equations decouple and the residual stress does not enter into the limit problem, see (3.39).

3.1 The 3-Dimensional problem

We consider the following sequence of open subsets of $\mathbb{R}^3$

$$\Omega_\varepsilon := \omega_\varepsilon \times (0, \ell) \subset \mathbb{R}^3,$$

where

$$\omega_\varepsilon := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \frac{a\varepsilon^2}{2}, |x_2| < \frac{b\varepsilon}{2}\right\},$$

$\varepsilon \in (0, 1]$ and $\ell > 0$. For any $x_3 \in (0, \ell)$ we further set $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$. Henceforth we shall refer to Ω_ε as the reference configuration of an elastic body.

We assume that the body responds elastically to deformations from the reference configuration. If $\mathbf{u}$ is a smooth displacement field defined on Ω_ε and $\mathbf{Du}$ denotes the gradient of $\mathbf{u}$, the first Piola-Kirchhoff stress field, $\mathbf{S}$, can be expressed as (see Section 2.2)

$$\mathbf{S}(\mathbf{x}) = \mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbf{Du}(\mathbf{x})\mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbb{L}^\varepsilon(\mathbf{x})\mathbf{Eu}(\mathbf{x}), \tag{3.1}$$

where $\mathbb{L}^\varepsilon(\mathbf{x})$ is the incremental elasticity tensor, $\mathring{\mathbf{T}}^\varepsilon$ is the Cauchy stress field present in the reference configuration and

$$\mathbf{Eu}(\mathbf{x}) = \operatorname{sym}(\mathbf{Du}(\mathbf{x}))$$

denotes the strain of $\mathbf{u}$. By using the results of Section 2.2, we have $\mathbb{L}^\varepsilon_{ijkl} = \mathbb{L}^\varepsilon_{ijlk} = \mathbb{L}^\varepsilon_{jikl}$. We further assume that $\mathbb{L}^{\varepsilon^T} = \mathbb{L}^\varepsilon$, i.e. $\mathbb{L}^\varepsilon_{ijkl} = \mathbb{L}^\varepsilon_{klij}$.

In what follows we consider the situation in which the body is clamped on $S_\varepsilon(0)$ and is subject only to dead body forces $\mathbf{b}^\varepsilon$, so that the equilibrium equations can be written as

$$\begin{cases} \operatorname{div}\mathbf{S} + \mathbf{b}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathbf{S} = \mathring{\mathbf{T}}^\varepsilon + \mathbf{Du}\mathring{\mathbf{T}}^\varepsilon + \mathbb{L}^\varepsilon\mathbf{Eu} & \text{in } \Omega_\varepsilon, \\ \mathbf{Sn} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon \setminus S_\varepsilon(0), \\ \mathbf{u} = \mathbf{0} & \text{on } S_\varepsilon(0), \end{cases} \tag{3.2}$$

where $\mathbf{n}$ denotes the outward unit normal to the boundary of Ω_ε.

We assume that $\mathring{\mathbf{T}}^\varepsilon$ is a residual stress, so that it satisfies the following equations:

$$\begin{cases} \operatorname{div} \mathring{\mathbf{T}}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon = \left(\mathring{\mathbf{T}}^\varepsilon\right)^T & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.3}$$

Hereafter, we suppose that $\mathbb{L}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3\times3\times3})$, $\mathring{\mathbf{T}}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3})$ and $\mathbf{b}^\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3)$. The weak form of the problem defined by equations (3.2) and (3.3) can be written as: find $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ such that

$$\int_{\Omega_\varepsilon} \left(\mathbf{D}\mathbf{u}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{D}\mathbf{v} + \mathbb{L}^\varepsilon \mathbf{E}\mathbf{u} \cdot \mathbf{E}\mathbf{v}\right) dx = \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{v}\, dx, \tag{3.4}$$

for every $\mathbf{v} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$, where

$$H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3) := \{\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\}.$$

3.2 Existence of the solution

In this section we study the existence of a solution to the problem defined by equation (3.4), following the same approach of Paroni [39]. A key ingredient in proving the existence of a solution is Korn's inequality. The following version was proven by Freddi, Morassi and Paroni [23], Theorem 4.1.

Theorem 3.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_{\Omega_\varepsilon} \left(|\mathbf{u}|^2 + |\mathbf{D}\mathbf{u}|^2\right) dx \le \frac{C}{\varepsilon^4} \int_{\Omega_\varepsilon} |\mathbf{E}\mathbf{u}|^2 dx, \tag{3.5}$$

for every $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ and for every $\varepsilon \in (0, 1]$.

Let C_K denote the smallest of all constants C for which inequality (3.5) holds. We assume that there exists a constant $C > 0$, independent of ε, such that

$$\mathbb{L}^\varepsilon(\mathbf{x})\mathbf{E} \cdot \mathbf{E} \ge C|\mathbf{E}|^2, \tag{3.6}$$

for every $\mathbf{E} \in \mathbb{R}^{3\times3}_{\text{sym}}$ and for a.e. $\mathbf{x} \in \Omega_\varepsilon$ and we denote by C_L the largest of all such constants C.

We shall prove the existence of a solution provided that the absolute value of the smallest eigenvalue of $\mathring{\mathbf{T}}^\varepsilon$,

$$\mathring{\tau}^\varepsilon_m := \operatorname*{essinf}_{\mathbf{x}\in\Omega_\varepsilon} \min_{\mathbf{a}\in\mathbb{R}^3} \left\{\mathring{\mathbf{T}}^\varepsilon(\mathbf{x})\mathbf{a} \cdot \mathbf{a} \;:\; |\mathbf{a}| = 1\right\}, \tag{3.7}$$

is not too large.

Lemma 3.2. *Let* $\mathbf{S} \in \mathbb{R}^{3\times 3}_{sym}$ *and* λ_m *its smallest eigenvalue. Then, for all* $\mathbf{A} \in \mathbb{R}^{3\times 3}$ *it holds that*

$$\mathbf{AS} \cdot \mathbf{A} \geq \lambda_m |\mathbf{A}|^2.$$

Proof. It is sufficient to write down the components of $\mathbf{S}$ and $\mathbf{A}$ in the orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$ that diagonalizes $\mathbf{S}$. Let λ_i be the eigenvalues of $\mathbf{S}$, and A_{ij} the components of $\mathbf{A}$ on the basis $\{\mathbf{e}_i\}_{i=1}^3$. Then

$$\mathbf{AS} \cdot \mathbf{A} = \sum_{i,l=1}^{3} A_{li}^2 \lambda_i \geq \lambda_m \sum_{i,l=1}^{3} A_{li}^2 = \lambda_m |\mathbf{A}|^2$$

□

Indeed, if there are no compressions due to the residual stress $\mathring{\mathbf{T}}^\varepsilon$, i.e. the smallest eigenvalue of $\mathring{\mathbf{T}}$ is always non- negative, then no assumption will be needed to prove the existence of the solution. The following lemma, though, proves that $\mathring{\tau}_m^\varepsilon$ is either equal to 0 or that it also takes negative values.

Lemma 3.3. *The essential infimum, on* Ω_ε*, of* $\mathring{\tau}_m^\varepsilon$ *is non-positive.*

Proof. By definition of residual stress (3.3), and applying the Divergence theorem, we find

$$0 = \int_{\Omega_\varepsilon} \mathbf{x} \otimes \mathrm{div}\mathring{\mathbf{T}}^\varepsilon \, dx = \int_{\partial\Omega_\varepsilon} \mathbf{x} \otimes \mathring{\mathbf{T}}^\varepsilon \mathbf{n} \, dx - \int_{\Omega_\varepsilon} \mathbf{Dx}\mathring{\mathbf{T}}^\varepsilon \, dx,$$

where $\mathbf{a} \otimes \mathbf{b}$ is the tensor product of two vectors $\mathbf{a}, \mathbf{b}$, and hence we have

$$\int_{\Omega_\varepsilon} \mathring{\mathbf{T}}^\varepsilon \, dx = 0.$$

Hence, for every unit vector $\mathbf{a} \in \mathbb{R}^3$, we obtain

$$\int_{\Omega_\varepsilon} \mathring{\mathbf{T}}^\varepsilon \mathbf{a} \cdot \mathbf{a} \, dx = 0,$$

from which it follows

$$|\Omega_\varepsilon| \mathring{\tau}_m^\varepsilon \leq \int_{\Omega_\varepsilon} \mathring{\mathbf{T}}^\varepsilon \mathbf{a} \cdot \mathbf{a} \, dx = 0.$$

□

By using Lemma 3.2 and Lax-Milgram lemma, we deduce the following existence theorem.

Theorem 3.4. *Assume that*

$$C_L > C_K \frac{|\overset{\circ}{\tau}{}^{\varepsilon}_m|}{\varepsilon^4}. \tag{3.8}$$

Then there exists a unique solution $\mathbf{u}^\varepsilon \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ *of problem* (3.4).

Proof. From equation (3.7) and inequality (3.6) we have

$$\begin{aligned}\int_{\Omega_\varepsilon}\left(\mathbb{L}^\varepsilon\mathbf{E}\mathbf{v}\cdot\mathbf{E}\mathbf{v}+\mathbf{D}\mathbf{v}\overset{\circ}{\mathbf{T}}{}^{\varepsilon}\cdot\mathbf{D}\mathbf{v}\right)dx &\geq C_L\|\mathbf{E}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}-(|\overset{\circ}{\tau}{}^{\varepsilon}_m|)\|\mathbf{D}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}\\ &\geq\left(C_L-C_K\frac{|\overset{\circ}{\tau}{}^{\varepsilon}_m|}{\varepsilon^4}\right)\|\mathbf{E}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)},\end{aligned}$$

where in the last inequality we used Theorem 3.1. Existence and uniqueness of the solution of problem (3.4) follow from an application of the Lax-Milgram lemma. □

Hereafter, we will always assume inequality (3.8) to hold. Since we have supposed $\mathbb{L}^\varepsilon = \mathbb{L}^{\varepsilon^T}$, the energy functionals

$$J_\varepsilon(\mathbf{u}) := \frac{1}{2}\int_{\Omega_\varepsilon}\left(\mathbf{D}\mathbf{u}\overset{\circ}{\mathbf{T}}{}^{\varepsilon}\cdot\mathbf{D}\mathbf{u}+\mathbb{L}^\varepsilon\mathbf{E}\mathbf{u}\cdot\mathbf{E}\mathbf{u}\right)dx-\int_{\Omega_\varepsilon}\mathbf{b}^\varepsilon\cdot\mathbf{u}\,dx,$$

admit, for $\varepsilon > 0$, a unique minimizer among all functions $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$.

We note that the strain energy density, at fixed point $\mathbf{x} \in \Omega_\varepsilon$, that is defined by

$$W(\mathbf{H}) := \frac{1}{2}(\mathbf{H}\overset{\circ}{\mathbf{T}}{}^{\varepsilon}\cdot\mathbf{H}+\mathbb{L}^\varepsilon\mathrm{sym}\mathbf{H}\cdot\mathrm{sym}\mathbf{H}),$$

for any square matrix $\mathbf{H}$, is not convex in general. This is easily seen by taking the zero matrix $\mathbf{0}$ and a skew-symmetric $\mathbf{W}$, then, for $t \in (0,1)$, we have

$$W(t\mathbf{W}+(1-t)\mathbf{0}) = W(t\mathbf{W}) = \frac{1}{2}t^2\mathbf{W}\overset{\circ}{\mathbf{T}}{}^{\varepsilon}\cdot\mathbf{W},$$

and, by Lemma 3.3, the term $\mathbf{W}\overset{\circ}{\mathbf{T}}{}^{\varepsilon}\cdot\mathbf{W}$ can be negative.

3.3 The rescaled problem

To state our results, it is convenient to stretch the domain Ω_ε along the transverse directions x_1 and x_2 in a way that the transformed domain does not depend on ε. Let us therefore set $\Omega := \Omega_1$, $\omega := \omega_1$, $S(x_3) := S_1(x_3)$ and let $p_\varepsilon : \Omega \to \Omega_\varepsilon$ be defined by $p_\varepsilon(\mathbf{y}) = p_\varepsilon(y_1, y_2, y_3) = (\varepsilon^2 y_1, \varepsilon y_2, y_3)$. Let us consider the following 3×3 matrix

$$\mathbf{H}^\varepsilon \mathbf{v} := (\frac{D_1\mathbf{v}}{\varepsilon^2}, \frac{D_2\mathbf{v}}{\varepsilon}, D_3\mathbf{v}),$$

where $D_i\mathbf{v}$ denotes the column vector of the partial derivatives of $\mathbf{v}$ with respect to y_i. We will use moreover the following notation

$$\mathbf{E}^\varepsilon \mathbf{v} := \mathrm{sym}(\mathbf{H}^\varepsilon \mathbf{v}), \quad \mathbf{W}^\varepsilon \mathbf{v} := \mathrm{skw}(\mathbf{H}^\varepsilon \mathbf{v})$$

and also denote by $\mathbf{W}\mathbf{v} := \mathbf{W}^1\mathbf{v}$ the skew symmetric part of the gradient of $\mathbf{v}$. We define

$$\mathbb{L}^\varepsilon = \mathbb{L} \circ p_\varepsilon^{-1} \quad \text{and} \quad \mathring{\mathbf{T}}^\varepsilon = \varepsilon^4 \mathring{\mathbf{T}} \circ p_\varepsilon^{-1},$$

where $\mathbb{L} \in L^\infty(\Omega; \mathbb{R}^{3\times3\times3\times3})$ and $\mathring{\mathbf{T}} \in L^\infty(\Omega; \mathbb{R}^{3\times3})$. In what follows we shall always assume that inequality (3.8) holds and we shall denote by $|\mathring{\tau}_m|$ the absolute value of the smallest eigenvalue of $\mathring{\mathbf{T}}$, so that

$$C_L > C_K |\mathring{\tau}_m|, \tag{3.9}$$

where C_K is Korn's constant for Ω and $\mathbb{L}(\mathbf{y})\mathbf{E}\cdot\mathbf{E} \geq C_L|\mathbf{E}|^2$, for every $\mathbf{E} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and a.e. $\mathbf{y} \in \Omega$.

The rescaled energy $F_\varepsilon : H^1_\flat(\Omega; \mathbb{R}^3) \to \mathbb{R}$ is defined by

$$F_\varepsilon(\mathbf{v}) := \frac{1}{\varepsilon^3} J_\varepsilon(\mathbf{v} \circ p_\varepsilon^{-1}) = I_\varepsilon(\mathbf{v}) - \int_\Omega \mathbf{b}^\varepsilon \circ p_\varepsilon \cdot \mathbf{v}\, dy,$$

where

$$I_\varepsilon(\mathbf{v}) := \frac{1}{2} \int_\Omega \Big(\mathbb{L}\mathbf{E}^\varepsilon\mathbf{v} \cdot \mathbf{E}^\varepsilon\mathbf{v} + \varepsilon^4 \mathbf{H}^\varepsilon\mathbf{v}\mathring{\mathbf{T}} \cdot \mathbf{H}^\varepsilon\mathbf{v} \Big)\, dy.$$

We suppose the loads to have the following form

$$\begin{aligned}
\mathbf{b}_1^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^4 b_1(\mathbf{y}) + \varepsilon^3 \frac{m(y_3)}{I_3} y_2,\\
\mathbf{b}_2^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^3 b_2(\mathbf{y}) + \varepsilon^2 \frac{m(y_3)}{I_3} y_1,\\
\mathbf{b}_3^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^2 b_3(\mathbf{y}),
\end{aligned} \tag{3.10}$$

with $\mathbf{b} = (b_1, b_2, b_3) \in L^2(\Omega; \mathbb{R}^3)$, $m \in L^2(0, \ell)$, while $I_3 := \int_\omega (y_1^2 + y_2^2)\, dy_1\, dy_2$ denotes the polar moment of inertia of the section ω. With the loads given by (3.10), the energy $F_\varepsilon(\mathbf{v})$ can be rewritten as

$$F_\varepsilon(\mathbf{v}) = I_\varepsilon(\mathbf{v}) - \varepsilon^4 \int_\Omega \mathbf{b} \cdot \big(v_1, \frac{v_2}{\varepsilon}, \frac{v_3}{\varepsilon^2}\big)\, dy - \varepsilon^4 \int_0^\ell m\, \vartheta^\varepsilon(\mathbf{v})\, dy_3, \tag{3.11}$$

where we have set

$$\vartheta^\varepsilon(\mathbf{v})(y_3) := \frac{1}{I_3}\int_\omega \Big(\frac{y_1}{\varepsilon^2} v_2(y_1,y_2,y_3) - \frac{y_2}{\varepsilon} v_1(y_1,y_2,y_3)\Big)\, dy_1 dy_2. \tag{3.12}$$

We note that if $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ then $\vartheta^\varepsilon(\mathbf{v}) \in L^2(0,\ell)$. A similar statement holds if we replace L^2 with H^1.

3.4 The limit energy

Hereafter, we assume the material to be triclinic and non homogeneous along the y_3-axis only, i.e. the incremental elasticity tensor does not depend on y_1 and y_2. This assumption allows to consider beams built with thin layers of different mechanical properties. Moreover we assume that $\mathbb{L} \in W^{2,\infty}((0,\ell);\mathbb{R}^{3\times3\times3\times3})$.

Define

$$f_0(\alpha,\beta) := \min\{f(\mathbf{A}) : \mathbf{A} \in \mathrm{Sym},\ A_{23} = \alpha,\ A_{33} = \beta\}, \tag{3.13}$$

where

$$f(\mathbf{A}) := \mathbb{L}(y_3)\mathbf{A}\cdot\mathbf{A}\ .$$

With a simple but long computation, we can check that

$$f_0(\alpha,\beta) = f(\mathbf{\Lambda}(\alpha,\beta)), \tag{3.14}$$

where $\mathbf{\Lambda}(\alpha,\beta)$ is a symmetric matrix whose components are given by

$$\begin{pmatrix} \mathbb{L}_{1111} & \mathbb{L}_{1122} & 2\mathbb{L}_{1131} & 2\mathbb{L}_{1112} & 0 & 0 \\ \mathbb{L}_{1112} & \mathbb{L}_{2212} & 2\mathbb{L}_{3112} & 2\mathbb{L}_{1212} & 0 & 0 \\ \mathbb{L}_{1131} & \mathbb{L}_{2213} & 2\mathbb{L}_{3131} & 2\mathbb{L}_{3112} & 0 & 0 \\ \mathbb{L}_{1122} & \mathbb{L}_{2222} & 2\mathbb{L}_{2231} & 2\mathbb{L}_{2212} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_{11} \\ \Lambda_{22} \\ \Lambda_{31} \\ \Lambda_{21} \\ \Lambda_{33} \\ \Lambda_{23} \end{pmatrix} =$$

$$= \begin{pmatrix} -2\mathbb{L}_{1123}\alpha - \mathbb{L}_{1133}\beta \\ -2\mathbb{L}_{1223}\alpha - \mathbb{L}_{1233}\beta \\ -2\mathbb{L}_{3123}\alpha - \mathbb{L}_{3133}\beta \\ -2\mathbb{L}_{2223}\alpha - \mathbb{L}_{2233}\beta \\ \beta \\ \alpha \end{pmatrix}.$$

We note that the components of $\mathbf{\Lambda}$ can be written as

$$\begin{aligned} &\Lambda(\alpha,\beta)_{11} = C_1(y_3)\alpha + C_2(y_3)\beta, && \Lambda(\alpha,\beta)_{12} = C_3(y_3)\alpha + C_4(y_3)\beta, \\ &\Lambda(\alpha,\beta)_{22} = C_5(y_3)\alpha + C_6(y_3)\beta, && \Lambda(\alpha,\beta)_{13} = C_7(y_3)\alpha + C_8(y_3)\beta, \\ &\Lambda(\alpha,\beta)_{33} = \beta, && \Lambda(\alpha,\beta)_{23} = \alpha, \end{aligned} \tag{3.15}$$

where, for $i = 1, \dots, 8$, $C_i(y_3)$ are a combination of the components of $\mathbb{L}(y_3)$ only. By using (3.14) and (3.15), we find that f_0 can be expressed as

$$f_0(\alpha, \beta) = 4\widetilde{\mu}(y_3)\alpha^2 + \widetilde{E}(y_3)\beta^2 + 2\widetilde{\gamma}(y_3)\alpha\beta\,, \tag{3.16}$$

where the coefficients $\widetilde{\mu}(y_3)$, $\widetilde{E}(y_3)$ and $\widetilde{\gamma}(y_3)$ depend on the components of the incremental elasticity tensor only.

By inequality (3.9), it follows that $\widetilde{\gamma}(y_3)^2 < 4\widetilde{\mu}(y_3)\widetilde{E}(y_3)$, $\widetilde{\mu}(y_3) > 0$ and $\widetilde{E}(y_3) > 0$ almost everywhere in $(0, \ell)$.

We introduce the space of Bernoulli-Navier displacements on Ω,

$$H_{BN}(\Omega; \mathbb{R}^3) := \{\mathbf{v} \in H^1_\flat(\Omega; \mathbb{R}^3) \,:\, (\mathbf{E}\mathbf{v})_{i\alpha} = 0 \text{ for } i = 1, 2, 3,\ \alpha = 1, 2\},$$

which can be characterized also as follows (see Le Dret [34], Section 4.1)

$$\begin{aligned} H_{BN}(\Omega; \mathbb{R}^3) = \{\ \mathbf{v} \in H^1_\flat(\Omega; \mathbb{R}^3) : \exists\, \xi_\alpha \in H^2_\flat(0, \ell), \exists\, \xi_3 \in H^1_\flat(0, \ell) \\ \text{s.t. } v_\alpha(\mathbf{y}) = \xi_\alpha(y_3),\ v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)\,\}. \end{aligned} \tag{3.17}$$

Theorem 3.5. *Let $F : H^1_\flat(\Omega; \mathbb{R}^3) \times H^1_\flat(\Omega; \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$\begin{aligned} F(\mathbf{v}, \vartheta) :=\ & \frac{1}{2} \int_\Omega f_0\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}} \left(D_3 v_3 + y_1 D_{33} v_1\right), D_3 v_3 \right) + \mathbf{H}\mathbf{v}\mathring{\mathbf{T}} \cdot \mathbf{H}\mathbf{v}\, dy \\ & - \int_\Omega \mathbf{b} \cdot \mathbf{v}\, dy - \int_0^\ell m\vartheta\, dy_3 \end{aligned} \tag{3.18}$$

if $\mathbf{v} \in H_{BN}(\Omega; \mathbb{R}^3)$, and $+\infty$ otherwise, where

$$\mathbf{H}\mathbf{v} := \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & 0 \\ -D_3 v_1 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

As $\varepsilon \to 0$, the sequence of functionals $(1/\varepsilon^4)F_\varepsilon$ defined in (3.11) and (3.12) Γ-converges to the functional F, in the following sense:

1. *(liminf inequality) for every sequence of positive numbers ε_k converging to 0 and for every sequence $\{\mathbf{u}^k\} \subset H^1_\flat(\Omega; \mathbb{R}^3)$ such that*

$$(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2}) \rightharpoonup \mathbf{v} \text{ in } H^1(\Omega; \mathbb{R}^3), \qquad (W^{\varepsilon_k} u^k)_{12} \to -\vartheta \text{ in } L^2(\Omega),$$

we have

$$\liminf_{k \to +\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \geq F(\mathbf{v}, \vartheta);$$

2. *(recovery sequence) for every sequence of positive numbers ε_k converging to 0 and for every $(\mathbf{v},\vartheta)\in H^1_\flat(\Omega;\mathbb{R}^3)\times H^1_\flat(\Omega;\mathbb{R})$, there exists a sequence $\{\mathbf{u}^k\}\subset H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$(u_1^k,\frac{u_2^k}{\varepsilon_k},\frac{u_3^k}{\varepsilon_k^2})\rightharpoonup \mathbf{v} \text{ in } H^1(\Omega;\mathbb{R}^3),\qquad (W^{\varepsilon_k}u^k)_{12}\to-\vartheta \text{ in } L^2(\Omega),$$

and

$$\limsup_{k\to+\infty}\frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4}\le F(\mathbf{v},\vartheta).$$

To prove Theorem 3.5 above, we need some auxiliary results.

Theorem 3.6. *There exists a positive constant K, independent of ε, such that*

$$\int_\Omega\Big(\big|(u_1,\frac{u_2}{\varepsilon},\frac{u_3}{\varepsilon^2})\big|^2+|\mathbf{H}^\varepsilon\mathbf{u}|^2\Big)dy\le\frac{K}{\varepsilon^4}\int_\Omega|\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy,$$

for every $\mathbf{u}\in H^1_\flat(\Omega;\mathbb{R}^3)$ and every $\varepsilon\in(0,1]$.

Proof. The inequality $\int_\Omega|\mathbf{H}^\varepsilon\mathbf{u}|^2\,dy\le(K/\varepsilon^4)\int_\Omega|\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy$ follows by a rescaling of Theorem 3.1. If we set $\mathbf{v}:=(u_1,u_2\varepsilon,u_3/\varepsilon^2)$, we notice that $|\mathbf{E}^\varepsilon\mathbf{u}|\ge\varepsilon^2|\mathbf{E}\mathbf{v}|$. Applying Theorem 1.45 to $\mathbf{v}$ on Ω, we find that

$$\int_\Omega|(u_1,u_2/\varepsilon,u_3/\varepsilon^2)|^2\,dy\le(K/\varepsilon^4)\int_\Omega|\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy.$$

□

Let $\wp$ denote the projection of $L^2(\omega;\mathbb{R}^2)$ on the subspace $\mathcal{R}_2$ of the infinitesimal rigid displacements on ω. It is easy to see that $\mathcal{R}_2$ is a closed subspace of $H^1(\omega;\mathbb{R}^2)$ (see also Freddi, Morassi and Paroni [23]). Moreover, if $\mathbf{w}\in L^2(\omega;\mathbb{R}^2)$ we have that

$$(\wp w)_\alpha=\mathcal{E}_{\beta\alpha}y_\beta\Big(\frac{1}{I_O}\int_\omega\mathcal{E}_{\gamma\delta}y_\gamma w_\delta\,dy_1dy_2\Big)+\frac{1}{|\omega|}\int_\omega w_\alpha\,dy_1dy_2,\tag{3.20}$$

where $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol. The two-dimensional Korn's inequality then writes as

$$\|\mathbf{w}-\wp\mathbf{w}\|^2_{H^1(\omega;\mathbb{R}^2)}\le C\|\mathbf{E}\mathbf{w}\|^2_{L^2(\Omega;\mathbb{R}^{2\times2})}\tag{3.21}$$

for all $\mathbf{w}\in H^1(\omega;\mathbb{R}^2)$.

Lemma 3.7. *Let $\mathbf{u}^\varepsilon$ be a sequence of functions in $H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\le C\varepsilon^2,\tag{3.22}$$

for some constant C and for every $\varepsilon\in(0,1]$. Then

1. *for any sequence of positive numbers* ε_n *converging to 0, there exist a subsequence (not relabelled) and a couple of functions* $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ *and* $\vartheta \in L^2(\Omega)$ *such that (as* $n \to +\infty$*)*
$$(u_1^{\varepsilon_n}, \frac{u_2^{\varepsilon_n}}{\varepsilon_n}, \frac{u_3^{\varepsilon_n}}{\varepsilon_n^2}) \rightharpoonup \mathbf{v} \text{ in } H^1(\Omega;\mathbb{R}^3), \tag{3.23}$$
and
$$\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \rightharpoonup \mathbf{H}\mathbf{v} = \begin{pmatrix} 0 & -\vartheta & D_3v_1 \\ \vartheta & 0 & 0 \\ -D_3v_1 & 0 & 0 \end{pmatrix} \text{ in } L^2(\Omega;\mathbb{R}^{3\times3}); \tag{3.24}$$
2. $\vartheta^\varepsilon(\mathbf{u}^\varepsilon) \rightharpoonup \vartheta$ *in* $L^2(\Omega)$*; therefore* ϑ *does not depend on* y_1 *and* y_2*;*
3. $\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)} \le (K/\varepsilon^2)\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})}$ *holds for some constant* $K > 0$*;*
4. $\vartheta \in H^1_\flat(\Omega)$*;*
5. *the following identities hold in* $L^2(\Omega)$
$$E_{33} = D_3v_3, \qquad \text{and} \qquad E_{23} = y_1D_3\vartheta + \eta,$$
where $\eta \in L^2(\Omega)$ *is independent on* y_1 *and, up to subsequences,* E_{33} *and* E_{23} *are the limits of* $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon^2$ *and* $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon^2$ *in the weak convergence of* $L^2(\Omega)$ *respectively.*

Proof. 1. We set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$. Since $|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon| \ge \varepsilon^2|\mathbf{E}\mathbf{v}^\varepsilon|$, by (3.22), $\mathbf{E}\mathbf{v}^\varepsilon$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$ and by Korn's inequality $\mathbf{v}^\varepsilon$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. There then exist a $\mathbf{v} \in H^1_\flat(\Omega;\mathbb{R}^3)$ and a subsequence ε_n such that $\mathbf{v}^{\varepsilon_n} \rightharpoonup \mathbf{v}$ in $H^1(\Omega;\mathbb{R}^3)$. It is easy to check that $|(E^\varepsilon u^\varepsilon)_{i\alpha}| \ge \varepsilon|(Ev^\varepsilon)_{i\alpha}|$, thus, using (3.22) we deduce that $C\varepsilon \ge \|(Ev^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently, as $n \to \infty$, $(Ev)_{i\alpha} = 0$ for $i = 1,2,3$ and $\alpha = 1,2$. Hence $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$.
Using (3.22) and Theorem 3.6 we obtain that the sequence $\mathbf{H}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n}$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$ so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times3})$ to a matrix $\mathbf{H} \in L^2(\Omega;\mathbb{R}^{3\times3})$. Since, from (3.22), $\mathbf{E}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times3})$, we have $\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \rightharpoonup \mathbf{H}$ in $L^2(\Omega;\mathbb{R}^{3\times3})$. In particular, $\mathbf{H}$ is, almost everywhere, a skew-symmetric matrix. Since $(H^\varepsilon u^\varepsilon)_{13} = u^\varepsilon_{1,3} = v^\varepsilon_{1,3}$ and $(H^\varepsilon u^\varepsilon)_{23} = u^\varepsilon_{2,3} = \varepsilon v^\varepsilon_{2,3}$, we deduce that $H_{13} = v_{1,3}$ and $H_{23} = 0$. We conclude the proof of point *1.* by setting $H_{12} := -\vartheta$.

2. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon/\varepsilon, u_2^\varepsilon/\varepsilon^2, u_3^\varepsilon/\varepsilon^3)$. Then for almost $y_3 \in (0,\ell)$ and any $\varepsilon \in (0,1]$ we consider the projection of the first two components of $\mathbf{v}^\varepsilon(\cdot,\cdot,y_3)$. From (3.20) and recalling (3.12) we have
$$(\wp v^\varepsilon)_\alpha = \mathcal{E}_{\beta\alpha}y_\beta\vartheta^\varepsilon(\mathbf{u}^\varepsilon) + \frac{1}{|\omega|}\int_\omega v^\varepsilon_\alpha \, dy_1dy_2. \tag{3.25}$$

Since, furthermore, $(Ev^\varepsilon)_{11} = \varepsilon(E^\varepsilon u^\varepsilon)_{11}$, $(Ev^\varepsilon)_{12} = (E^\varepsilon u^\varepsilon)_{12}$ and $(Ev^\varepsilon)_{22} = (E^\varepsilon u^\varepsilon)_{22}/\varepsilon$, we get

$$\|(Ev^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \frac{1}{\varepsilon}\|(E^\varepsilon u^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \tag{3.26}$$

for $\alpha, \beta = 1, 2$. Then, integrating equation (3.20) on $(0, \ell)$ and taking into account (3.22), we deduce that

$$\int_0^\ell \|\mathbf{v}^\varepsilon - \wp\mathbf{v}^\varepsilon\|_{H^1(\omega;\mathbb{R}^2)} dy_3 \leq C\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\varepsilon$$

and then

$$\|D_\alpha(v^\varepsilon_\beta - \wp v^\varepsilon_\beta)\|_{L^2(\Omega;\mathbb{R})} \to 0 \tag{3.27}$$

for $\alpha, \beta = 1, 2$. Since $(W\wp v^\varepsilon)_{12} = -\vartheta^\varepsilon(\mathbf{u}^\varepsilon)$ and $(Wv^\varepsilon)_{12} = (W^\varepsilon u^\varepsilon)_{12}$, we obtain

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) = -(W\wp v^\varepsilon)_{12} = (W(v^\varepsilon - \wp\mathbf{v}^\varepsilon))_{12} - (W^\varepsilon u^\varepsilon)_{12}, \tag{3.28}$$

Using (3.24), for $\varepsilon \to 0$, we get that $\vartheta^\varepsilon(\mathbf{u}^\varepsilon) \rightharpoonup \vartheta$ in $L^2(\Omega)$. From the fact that $\vartheta^\varepsilon(\mathbf{u}^\varepsilon)$ does not depend on y_1 and y_2, the same holds for ϑ.

3. From inequalities (3.21) and (3.26) and equation (3.28), we find that

$$\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)} \leq \|\mathbf{W}\mathbf{v}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \frac{K}{\varepsilon}\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

where $\mathbf{v}^\varepsilon := (u^\varepsilon_1/\varepsilon, u^\varepsilon_2/\varepsilon^2, u^\varepsilon_3/\varepsilon^3)$, and the claim follows from Theorem 3.6.

4. Let $\mathbf{v}^\varepsilon := (u^\varepsilon_1/\varepsilon, u^\varepsilon_2/\varepsilon^2, u^\varepsilon_3/\varepsilon^3)$ and $\xi \in C^\infty_0(\omega)$ be such that

$$\int_\omega \xi \, dy_1 dy_2 = -\frac{I_0}{2},$$

where I_O is the polar moment of inertia of the section ω. Hence, from equation (3.25), we get

$$\begin{aligned}
I_0\vartheta^\varepsilon(\mathbf{u}^\varepsilon) &= -2\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \xi \, dy_1 dy_2 = -\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \xi D_\alpha y_\alpha \, dy_1 dy_2 \\
&= \vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega (D_\alpha\xi) y_\alpha \, dy_1 dy_2 = \vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \mathcal{E}_{\alpha\gamma}\mathcal{E}_{\beta\gamma}(D_\alpha\xi) y_\beta \, dy_1 dy_2 \\
&= \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(\mathcal{E}_{\beta\gamma} y_\beta \vartheta^\varepsilon(\mathbf{u}^\varepsilon)) \, dy_1 dy_2 \\
&= \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)\left((\wp v^\varepsilon)_\gamma - \frac{1}{|\omega|}\int_\omega v^\varepsilon_\gamma \, dy_1 dy_2\right) dy_1 dy_2 \\
&= \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(\wp v^\varepsilon)_\gamma \, dy_1 dy_2 \\
&= \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi) v^\varepsilon_\gamma \, dy_1 dy_2 - \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(v^\varepsilon - \wp v^\varepsilon)_\gamma \, dy_1 dy_2.
\end{aligned}$$

Then, denoting by

$$\tilde{\vartheta}^\varepsilon = \frac{1}{I_0}\int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)v^\varepsilon_\gamma \, dy_1 dy_2.$$

and taking into account (3.27), we have

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) - \tilde{\vartheta}^\varepsilon \to 0 \quad \text{in} \quad L^2(\Omega). \tag{3.29}$$

Since $\mathcal{E}_{\alpha\gamma}D_\alpha D_\gamma \xi = 0$ in ω and $D_\alpha \xi = 0$ on $\partial\omega$, we find

$$\begin{aligned}
I_0 D_3\tilde{\vartheta}^\varepsilon &= \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(D_3 v^\varepsilon_\gamma)\, dy_1 dy_2 \\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(Ev^\varepsilon)_{\gamma 3}\, dy_1 dy_2 - \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(D_\gamma v^\varepsilon_3)\, dy_1 dy_2 \\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(Ev^\varepsilon)_{\gamma 3}\, dy_1 dy_2 - \int_\omega D_\gamma(\mathcal{E}_{\alpha\gamma}(D_\alpha\xi)v^\varepsilon_3)\, dy_1 dy_2 + \\
&\quad + \int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha D_\gamma\xi)v^\varepsilon_3\, dy_1 dy_2 \\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha\xi)(Ev^\varepsilon)_{\gamma 3}\, dy_1 dy_2,
\end{aligned}$$

but $(Ev^\varepsilon)_{13} = (E^\varepsilon u^\varepsilon)_{13}/\varepsilon$ and $(Ev^\varepsilon)_{23} = (E^\varepsilon u^\varepsilon)_{23}/\varepsilon^2$, and therefore $D_3\tilde{\vartheta}^\varepsilon$ is bounded in $L^2(0,\ell)$. Hence, from (3.29) and 2. of this lemma we have

$$\tilde{\vartheta}^\varepsilon \rightharpoonup \vartheta \quad \text{in} \quad H^1(\Omega). \tag{3.30}$$

Therefore, as $\tilde{\vartheta}^\varepsilon(0) = 0$, we conclude that $\vartheta \in H^1_\flat(\Omega)$.

5. To prove the first equation of (3.7) it suffices to notice that $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon^2 = D_3(u^\varepsilon_3/\varepsilon^2)$ and apply (3.23). Let's prove the second equation of (3.7). From inequality (3.22) we deduce that, up to subsequences, $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon^2 \rightharpoonup E_{23}$ in $L^2(\Omega)$. To characterize $E_{23} \in L^2(\Omega)$ note that

$$D_3(W^\varepsilon u^\varepsilon)_{12} = D_2\Big(\frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon}\Big) - D_1\Big(\frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon^2}\Big),$$

in the sense of distributions. Hence for $\psi \in C^\infty_c(\Omega)$ we obtain

$$\int_\Omega (W^\varepsilon u^\varepsilon)_{12} D_3\psi\, dy = \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon} D_2\psi\, dy - \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon^2} D_1\psi\, dy.$$

Recalling (3.22), we have $(E^\varepsilon u^\varepsilon)_{13}/\varepsilon \rightharpoonup 0$ in $L^2(\Omega)$. Hence, passing to the limit in the previous equality we find

$$\int_\Omega -\vartheta D_3\psi\, dy = -\int_\Omega E_{23} D_1\psi\, dy.$$

Thus $D_3\vartheta = D_1 E_{23}$ in the sense of distributions, hence in $L^2(\Omega)$ since $\vartheta \in H^1_\flat(\Omega)$. Taking into account that ϑ is independent of y_1 we have that $E_{23} = y_1 D_3\vartheta + \eta$, with η like in the statement of the lemma. $\square$

Remark 3.8. The weak convergence in *2.* is, actually, a strong convergence. In fact, from (3.24) and points *3.* and *4.* of Lemma 3.7, it follows that

$$(H^\varepsilon u^\varepsilon)_{12} \to -\vartheta \ \text{ in } L^2(\Omega), \qquad \text{and} \qquad (H^\varepsilon u^\varepsilon)_{21} \to \vartheta \ \text{ in } L^2(\Omega). \tag{3.31}$$

Lemma 3.9. *Let* $\mathbf{u}^\varepsilon$ *be a sequence of functions in the space* $H^1_\flat(\Omega;\mathbb{R}^3)$*. If*

$$\sup_\varepsilon \big(F_\varepsilon(\mathbf{u}^\varepsilon)/\varepsilon^4\big) < +\infty,$$

then

$$\|\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\varepsilon^2, \tag{3.32}$$

holds for some constant $C > 0$ *and for every* $\varepsilon \in (0,1]$.

Proof. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$ and $R := C_L - C_K|\mathring{\tau}_m|$. By inequality (3.9), we have $R > 0$. With this notation and by using (3.6), (3.11), Theorem 3.6, and 3. of Lemma 3.7 we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) &= \frac{1}{2}\int_\Omega \mathbb{L}\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\cdot\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2} + \mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\mathring{\mathbf{T}}\cdot\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\,dy + \\
&\quad -\int_\Omega \mathbf{b}\cdot\mathbf{v}^\varepsilon\,dy - \int_0^\ell m\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\,dy_3 \\
&\ge \frac{R}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(0,\ell)} \\
&\ge \frac{R}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\
&\quad -\frac{1}{2C_2}\|m\|^2_{L^2(0,\ell)} - \frac{C_2}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)},
\end{aligned}$$

where C_1 and C_2 are arbitrary positive constants. Choosing $C_2 = R/2$ we have

$$\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) \ge \frac{R}{4}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} - \frac{1}{R}\|m\|^2_{L^2(0,\ell)}. \tag{3.33}$$

By Theorem 3.6, we deduce that

$$\begin{aligned}
\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) \ge\ & \frac{R}{4K}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \Big(\frac{1}{K} - \frac{C_1}{2}\Big)\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\
& -\frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{1}{R}\|\ m\|^2_{L^2(0,\ell)}.
\end{aligned}$$

By choosing, for instance, $C_1 = 1/K$, we find that there exists a constant $M > 0$ such that

$$M \geq \frac{R}{4K}\|\mathbf{H}^\varepsilon \mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2K}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}$$

from which follows that the sequence $\mathbf{v}^\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^3)$. Using this fact in (3.33) we get the estimate (3.32). □

Lemma 3.9, and *1.* of Lemma 3.7 imply that the family of functionals $(1/\varepsilon^4)F_\varepsilon$ is coercive in the space $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$ with respect to the weak convergence of the sequence $q_\varepsilon(\mathbf{u}^\varepsilon) := \Big(u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2, (W^\varepsilon u^\varepsilon)_{12}\Big)$, uniformly with respect to ε. Hence, for any sequence $\mathbf{u}^\varepsilon$ which is bounded in energy, that is $(1/\varepsilon^4)F_\varepsilon \leq C$ for a suitable constant $C > 0$, and satisfies the boundary conditions $\mathbf{u}^\varepsilon = 0$ on $S(0)$, the corresponding sequence $q_\varepsilon(\mathbf{u}^\varepsilon)$ is weakly relatively compact in $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$.

Lemma 3.10. *Let $V \subset \mathbb{R}^N$ be open and bounded and $\mathbb{G} \in L^\infty(V;\mathbb{R}^{d\times N\times d\times N})$. For $\boldsymbol{\varphi} \in H^1(V;\mathbb{R}^d)$ let*

$$I(\boldsymbol{\varphi}) := \int_V \mathbb{G}\mathbf{D}\boldsymbol{\varphi}\cdot\mathbf{D}\boldsymbol{\varphi}\,dx,$$

be such that $I(\boldsymbol{\varphi}) \geq 0$ for every $\boldsymbol{\varphi} \in H^1(V;\mathbb{R}^d)$. If $\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi}$ in $H^1(V;\mathbb{R}^d)$ then

$$\liminf_{k\to\infty} I(\boldsymbol{\varphi}_k) \geq I(\boldsymbol{\varphi}),$$

i.e. I is lower semicontinuous with respect to the weak topology of $H^1(V;\mathbb{R}^d)$.

Proof. Let $\{\boldsymbol{\varphi}_k\}$ be a sequence in $H^1(V;\mathbb{R}^d)$ and $\boldsymbol{\varphi} \in H^1(V;\mathbb{R}^d)$ such that $\boldsymbol{\varphi} \rightharpoonup \boldsymbol{\varphi}$ in $H^1(V;\mathbb{R}^d)$. Then we have

$$\begin{aligned} I(\boldsymbol{\varphi}_k) &= \int_V \mathbb{G}(\mathbf{D}\boldsymbol{\varphi}_k - \mathbf{D}\boldsymbol{\varphi} + \mathbf{D}\boldsymbol{\varphi})\cdot(\mathbf{D}\boldsymbol{\varphi}_k - \mathbf{D}\boldsymbol{\varphi} + \mathbf{D}\boldsymbol{\varphi})\,dx \\ &\geq I(\boldsymbol{\varphi}) + \int_V \mathbb{G}(\mathbf{D}\boldsymbol{\varphi}_k - \mathbf{D}\boldsymbol{\varphi})\cdot\mathbf{D}\boldsymbol{\varphi}\,dx + \int_V \mathbb{G}\mathbf{D}\boldsymbol{\varphi}\cdot(\mathbf{D}\boldsymbol{\varphi}_k - \mathbf{D}\boldsymbol{\varphi})\,dx. \end{aligned}$$

The aim of the lemma is proven by letting k go to $+\infty$. □

Proof. (of Theorem 3.5) Let us prove the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} = \lim_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} < +\infty.$$

Then Lemma 3.9 applies to the sequence $(1/\varepsilon_k^4)F_{\varepsilon_k}(\mathbf{u}^k)$. Hence (3.32) is fulfilled and the results of Lemma 3.7 hold true.

From (3.11) and (3.12), and with $L_\varepsilon := I_\varepsilon - F_\varepsilon$ the work done by the loads, we see that, by using Lemma 3.7,

$$\frac{L_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} = \int_\Omega \mathbf{b}\cdot(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2})\,dy + \int_0^\ell m\vartheta^{\varepsilon_k}(\mathbf{u}^k)\,dy_3 \to \int_\Omega \mathbf{b}\cdot\mathbf{v}\,dy + \int_0^\ell m\vartheta\,dy_3.$$

Thus we have only to prove that

$$\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \geq \frac{1}{2}\int_\Omega f_0\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(D_3 v_3 + y_1 D_{33} v_1\right), D_3 v_3\right) + \\ +\mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\,dy.$$

From the definition (3.13) of f_0, we find

$$\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \geq \frac{1}{2}\int_\Omega f_0\left(\frac{(E^{\varepsilon_k}u^k)_{23}}{\varepsilon_k^2}, \frac{(E^{\varepsilon_k}u^k)_{33}}{\varepsilon_k^2}\right) + \mathbf{H}^{\varepsilon_k}\mathbf{u}^k\mathring{\mathbf{T}}\cdot\mathbf{H}^{\varepsilon_k}\mathbf{u}^k\,dy.$$

If we define $\mathbf{z}^k := (u_1^k, u_2^k/\varepsilon_k, u_3^k/\varepsilon_k^2)$, we obtain

$$\begin{aligned}
&\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \geq \\
&\geq \frac{1}{2}\int_\Omega f_0\left(\frac{(Ez^k)_{23}}{\varepsilon_k}, (Ez^k)_{33}\right) + \mathring{T}_{11}(z_{3,1}^k)^2 + \\
&\quad +\mathring{T}_{33}(z_{1,3}^k)^2 + 2C_L\sum_{i=1}^3\sum_{\alpha=1}^2 |(Ez^k)_{i\alpha}|^2 + \\
&\quad -2C_L\sum_{i=1}^3\sum_{\alpha=1}^2 |(Ez^k)_{i\alpha}|^2 + \mathring{T}_{11}\left(\frac{(z_{1,1}^k)^2}{\varepsilon_k^4} + \frac{(z_{2,1}^k)^2}{\varepsilon_k^2}\right) + \\
&\quad +\mathring{T}_{33}\left(\varepsilon_k^2(z_{2,3}^k)^2 + \varepsilon_k^4(z_{3,3}^k)^2\right) + \mathring{T}_{22}\left(\frac{(z_{1,2}^k)^2}{\varepsilon_k^2} + (z_{2,2}^k)^2 + \varepsilon_k^2(z_{3,2}^k)^2\right) + \\
&\quad +2\mathring{T}_{12}\left(\frac{z_{1,1}^k}{\varepsilon_k^2}\frac{z_{1,2}^k}{\varepsilon_k} + \frac{z_{2,1}^k}{\varepsilon_k}z_{2,2}^k + \varepsilon_k z_{3,1}^k z_{3,2}^k\right) + \\
&\quad +2\mathring{T}_{13}\left(\frac{z_{1,1}^k}{\varepsilon_k^2}z_{1,3}^k + z_{2,1}^k z_{2,3}^k + \varepsilon_k^2 z_{3,1}^k z_{3,3}^k\right) + \\
&\quad +2\mathring{T}_{23}\left(\frac{z_{1,2}^k}{\varepsilon_k}z_{1,3}^k + \varepsilon_k z_{2,2}^k z_{2,3}^k + \varepsilon_k^3 z_{3,2}^k z_{3,3}^k\right)\,dy.
\end{aligned}$$

By using (3.23), (3.31) and (3.32), the last five lines in the inequality above converge to $(\mathring{T}_{11} + \mathring{T}_{22})\vartheta^2 - 2\mathring{T}_{23}\vartheta D_3 v_1$, while by using 5. of Lemma 3.7, the inequality

$$\int_\Omega f_0\left(\frac{(Ez^k)_{23}}{\varepsilon_k}, (Ez^k)_{33}\right) + 2C_L \sum_{i=1}^{3}\sum_{\alpha=1}^{2} |(Ez^k)_{i\alpha}|^2\, dy \geq C_L \|\mathbf{E}\mathbf{z}^k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

and Lemma 3.10 on the second and third line, we obtain

$$\begin{aligned}\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} &\geq \frac{1}{2}\int_\Omega f_0(y_1 D_3\vartheta + \eta,\ D_3 v_3) + \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\, dy \\ &\geq \frac{1}{2}\min_\eta \int_\Omega f_0(y_1 D_3\vartheta + \eta,\ D_3 v_3) + \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\, dy\ ,\end{aligned}$$

where the minimun is taken over all functions η in $L^2(\Omega)$ independent of y_1. It is easy to see that the minimum is achieved for $\bar{\eta} := -(\widetilde{\gamma}/4\widetilde{\mu})\,(D_3 v_3 + y_1 D_{33} v_1)$, which depends only on y_2 and y_3 since $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$. Hence we have the liminf inequality.

Let us now find a recovery sequence. Let $F(\mathbf{v},\vartheta) < +\infty$, otherwise there is nothing to prove. Then $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega;\mathbb{R})$.

We first assume further that $\mathbf{v}$ and ϑ are smooth and equal to zero near by $y_3 = 0$. By (3.17), there exists $\boldsymbol{\xi}$ smooth and equal to zero near by $y_3 = 0$ such that $v_\alpha(\mathbf{y}) = \xi_\alpha(y_3)$, and $v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)$. Let $\mathbf{u}^{0,\varepsilon}$ be the sequence defined by

$$\mathbf{u}^{0,\varepsilon} := \mathbf{u}^{f,\varepsilon} + \mathbf{u}^{t,\varepsilon}, \tag{3.34}$$

with

$$\begin{aligned}
u_1^{f,\varepsilon} &:= \xi_1 + \varepsilon^2 C_6 \frac{y_2^2}{2}\xi_1'' + \varepsilon^4 C_2\left(y_1\xi_3' - \frac{y_1^2}{2}\xi_1'' - y_1 y_2 \xi_2''\right) + \\
&\quad -\varepsilon^4 C_1 \frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_1\left(\xi_3' - y_2\xi_2''\right), \\
u_2^{f,\varepsilon} &:= \varepsilon\xi_2 + \varepsilon^3\left(C_6\left(y_2\xi_3' - \frac{y_2^2}{2}\xi_2'' - y_1 y_2\xi_1''\right) - C_5\frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_2\left(\xi_3' - \frac{y_2}{2}\xi_2''\right)\right) + \\
&\quad +\varepsilon^4\left(C_4\left(2y_1\xi_3' - y_1^2\xi_1'' - 2y_1y_2\xi_2''\right) - C_3\frac{\widetilde{\gamma}}{2\widetilde{\mu}} y_1\left(\xi_3' - y_2\xi_2''\right)\right), \\
u_3^{f,\varepsilon} &:= \varepsilon^2(\xi_3 - y_1\xi_1' - y_2\xi_2') - \varepsilon^3\frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_2\left(2\xi_3' - y_2\xi_2''\right) + \\
&\quad +\varepsilon^4\left(C_8\left(2y_1\xi_3' - y_1^2\xi_1'' - 2y_1y_2\xi_2''\right) - C_7\frac{\widetilde{\gamma}}{2\widetilde{\mu}} y_1\left(\xi_3' - y_2\xi_2''\right)\right) + \\
&\quad +\varepsilon^4\left(-C_6\frac{y_2^2}{2} y_1\xi_1''' - D_3(C_6)\frac{y_2^2}{2} y_1\xi_1''\right),
\end{aligned}$$

and

$$
\begin{aligned}
u_1^{t,\varepsilon} &:= -\varepsilon y_2 \vartheta - \varepsilon^2 C_5 \frac{y_2^2}{2} D_3\vartheta + \varepsilon^4 C_1 \frac{y_1^2}{2} D_3\vartheta, \\
u_2^{t,\varepsilon} &:= \varepsilon^2 y_1 \vartheta + \varepsilon^3 C_5 y_1 y_2 D_3\vartheta + \varepsilon^4 C_3 y_1^2 D_3\vartheta, \\
u_3^{t,\varepsilon} &:= \varepsilon^3 y_1 y_2 D_3\vartheta + \varepsilon^4 C_7 y_1^2 D_3\vartheta + \varepsilon^4 \left(C_5 \frac{y_2^2}{2} y_1 D_{33}\vartheta + D_3(C_5)\frac{y_2^2}{2} y_1 D_3\vartheta \right),
\end{aligned}
$$

where, for $i = 1, \ldots, 8$, $C_i = C_i(y_3)$ are defined by equation (3.15). We have that $\mathbf{u}^{0,\varepsilon}$ is equal to zero in $y_3 = 0$ and satisfies the following estimates

$$
\begin{aligned}
&\left\| \frac{\mathbf{E}^\varepsilon \mathbf{u}^{0,\varepsilon}}{\varepsilon^2} - \mathbf{\Lambda}\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}(D_3 v_3 + y_1 D_{33} v_1), D_3 v_3 \right) \right\|_{L^2(\Omega)} \leq \varepsilon C(\mathbf{v}, \vartheta), \\
&\|\mathbf{H}^\varepsilon \mathbf{u}^{0,\varepsilon} - \mathbf{H}\mathbf{v}\|_{L^2(\Omega)} \leq \varepsilon C(\mathbf{v}, \vartheta), \\
&\|(W^\varepsilon u^{0,\varepsilon})_{12} + \vartheta\|_{L^2(\Omega)} \leq \varepsilon C(\mathbf{v}, \vartheta), \\
&\left\| \left(u_1^{0,\varepsilon}, \frac{u_2^{0,\varepsilon}}{\varepsilon}, \frac{u_3^{0,\varepsilon}}{\varepsilon^2} \right) - \mathbf{v} \right\|_{H^1(\Omega)} \leq \varepsilon C(\mathbf{v}, \vartheta),
\end{aligned}
$$

where $C(\mathbf{v}, \vartheta)$ depends only on $\mathbf{v}$ and ϑ and $\mathbf{\Lambda}$ is defined in equation (3.14). Hence, in this case, $(\mathbf{u}^{0,\varepsilon_k})$ is a recovery sequence.

In the general case, i.e. $\mathbf{v} \in H_{BN}(\Omega; \mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega)$, for any $\delta > 0$, we can find, by density, functions $\mathbf{v}^\delta \in C^\infty(\Omega; \mathbb{R}^3)$ and $\vartheta^\delta \in C^\infty(\Omega)$ which are equal to zero near by $y_3 = 0$ and such that

$$
\begin{aligned}
&\|\mathbf{v}^\delta - \mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} < \delta, \\
&\|\vartheta^\delta - \vartheta\|_{L^2(\Omega)} < \delta,
\end{aligned}
$$

Denoting by $\mathbf{u}^{\delta,\varepsilon}$ the sequence defined as $\mathbf{u}^{0,\varepsilon}$ in (3.34) with $\mathbf{v}$ and ϑ replaced with $\mathbf{v}^\delta$ and ϑ^δ, we obtain

$$
\begin{aligned}
&\lim_{\delta\to 0} \lim_{k\to+\infty} \left(u_1^{\delta,\varepsilon_k}, \frac{u_2^{\delta,\varepsilon_k}}{\varepsilon_k}, \frac{u_3^{\delta,\varepsilon_k}}{\varepsilon_k^2}\right) = \mathbf{v} \quad \text{in } H^1(\Omega; \mathbb{R}^3), \\
&\lim_{\delta\to 0} \lim_{k\to+\infty} \mathbf{H}^{\varepsilon_k} \mathbf{u}^{\delta,\varepsilon_k} = \mathbf{H}\mathbf{v} \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}), \\
&\lim_{\delta\to 0} \lim_{k\to+\infty} (W^{\varepsilon_k} u^{\delta,\varepsilon_k})_{12} = \vartheta \quad \text{in } L^2(\Omega), \\
&\lim_{\delta\to 0} \lim_{k\to+\infty} \frac{1}{\varepsilon_k^4} F_{\varepsilon_k}(\mathbf{u}^{\delta,\varepsilon_k}) = F(\mathbf{v}, \vartheta),
\end{aligned}
$$

and, hence, by a standard diagonal argument, we can find a sequence δ_k converging to zero such that

$$\begin{aligned}
&\|(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2}) - \mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} < \delta_k, \\
&\|\mathbf{H}^{\varepsilon_k}\mathbf{u}^k - \mathbf{H}\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} < \delta_k, \\
&\|(W^{\varepsilon_k}u^k)_{12} + \vartheta\|_{L^2(\Omega)} < \delta_k, \\
&\lim_{k\to+\infty} \frac{1}{\varepsilon_k^4} F_{\varepsilon_k}(\mathbf{u}^k) = F(\mathbf{v}, \vartheta),
\end{aligned}$$

where $\mathbf{u}^k := \mathbf{u}^{\delta_k,\varepsilon_k}$. Hence, the sequence $\mathbf{u}^k$ satisfies the recovering sequence condition. □

3.5 Convergence of minima and minimizers

For every $\varepsilon \in (0,1]$ let us denote by $\tilde{\mathbf{u}}^\varepsilon$ the solution of the following minimization problem

$$\min\left\{F_\varepsilon(\mathbf{u}) : \mathbf{u} \in H^1(\Omega;\mathbb{R}^3),\ \mathbf{u} = 0 \text{ on } S(0)\right\}.$$

Corollary 3.11. *The following minimization problem for the* Γ*-limit functional* F *defined in* (3.18)

$$\min\left\{F(\mathbf{v},\vartheta) : \mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3),\ \vartheta \in H^1(0,\ell),\ \mathbf{v} = \mathbf{0} \text{ on } S(0),\ \vartheta(0) = 0\right\}$$

admits a unique solution $(\tilde{v}, \tilde{\vartheta})$*. Moreover, as* $\varepsilon \to 0$*,*

[1. $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon/\varepsilon, \tilde{u}_3^\varepsilon/\varepsilon^2) \rightharpoonup \tilde{\mathbf{v}}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*

2. $(W^\varepsilon \tilde{u}^\varepsilon)_{12} \to -\tilde{\vartheta}$ *in* $L^2(\Omega)$*;*

3. $(1/\varepsilon^4)F_\varepsilon(\tilde{\mathbf{u}}^\varepsilon)$ *converges to* $F(\tilde{\mathbf{v}}, \tilde{\vartheta})$*.*

Proof. Property *3.* and the weak convergence in *1.* and *2.* follow from the Γ-convergence Theorem 3.5, the uniform coercivity of the sequence $(1/\varepsilon^4)F_\varepsilon$ and the variational property of Γ-convergence (Lemma 1.40). The strong convergence in *2.* follows from Remark 3.8. □

3.6 The equations of equilibrium

The limit energy functional $F(\mathbf{v},\vartheta)$ defined in (3.18) can be written in a more explicit form by using (3.17), the fact that ϑ depends only on y_3 and the expressions (3.16) of f_0 and (3.19) of $\mathbf{Hv}$. By using

$$\int_\Omega y_\alpha \xi_3' \xi_\alpha''\, dy = 0, \qquad \int_\Omega y_\alpha \xi_3'\, dy = 0,$$

for $\alpha = 1, 2$ and

$$\int_\Omega y_1 y_2 \xi_1'' \xi_2''\, dy = 0, \qquad \int_\Omega y_1 y_2 \xi_2''\, dy = 0\ ,$$

the limit strain energy can be rewritten as

$$\begin{aligned}
I(\mathbf{v},\vartheta) &= \frac{1}{2}\int_\Omega f_0\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(D_3 v_3 + y_1 D_{33} v_1\right),\ D_3 v_3\right) + \mathbf{Hv}\mathring{\mathbf{T}}\cdot\mathbf{Hv}\, dy \\
&= \frac{1}{2}\int_\Omega f_0\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(\xi_3' - y_2\xi_2''\right),\ \xi_3' - y_1\xi_1'' - y_2\xi_2''\right) dy + \\
&\quad + \frac{1}{2}\int_\Omega \left(\mathring{T}_{11} + \mathring{T}_{33}\right)\xi_1'^2 + \mathring{T}_{\alpha\alpha}\vartheta^2 - 2\mathring{T}_{23}\vartheta\xi_1'\, dy \\
&= \int_0^\ell \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)A\xi_3'^2 + \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)J_1\xi_2''^2 + \frac{1}{2}\widetilde{E}J_2\xi_1''^2\, dy_3 + \\
&\quad + \int_0^\ell \frac{1}{2}\langle\mathring{T}_{11} + \mathring{T}_{33}\rangle\xi_1'^2 - \widetilde{\gamma}J_2\vartheta'\xi_1'' - \langle\mathring{T}_{23}\rangle\vartheta\xi_1'\, dy_3 + \\
&\quad + \int_0^\ell \frac{1}{2}\widetilde{\mu}J\vartheta'^2 + \frac{1}{2}\langle\mathring{T}_{\alpha\alpha}\rangle\vartheta^2\, dy_3,
\end{aligned}$$

where

$$\begin{aligned}
A &:= \int_\omega dy_1 dy_2 = ab, & J_1 &:= \int_\omega y_2^2\, dy_1 dy_2 = \frac{1}{12}ab^3, \\
J_2 &:= \int_\omega y_1^2\, dy_1 dy_2 = \frac{1}{12}a^3 b, & J &:= 4\int_\omega y_1^2\, dy_1 dy_2 = \frac{1}{3}a^3 b,
\end{aligned}$$

and $\langle\cdot\rangle = \int_\omega \cdot\, dy_1 dy_2$ denotes integration over the cross section ω. The work done by the external forces rewrites as

$$\int_\Omega \mathbf{b}\cdot\mathbf{v}\, dy = \int_0^\ell \langle b_i\rangle\xi_i - \langle y_\alpha b_3\rangle\xi_\alpha'\, dy_3.$$

From (3.3), we obtain

$$\langle\mathring{T}_{1i,1} + \mathring{T}_{2i,2} + \mathring{T}_{3i,3}\rangle = 0,$$

for $i = 1, 2, 3$. Using the Divergence theorem on the first two terms of the equation above and recalling that $\mathring{\mathbf{T}}\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, see (3.3), we find $\langle \mathring{T}_{3i} \rangle = 0$, for $i = 1, 2, 3$.

The energy of the beam $F(\mathbf{v},\ \vartheta)$ can be rewritten, with a small abuse of notation, as

$$\begin{aligned} F(\boldsymbol{\xi}, \vartheta) \ &= \int_0^\ell \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) A\xi_3'^2 + \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) J_1 \xi_2''^2 + \frac{1}{2}\widetilde{E} J_2 \xi_1''^2 \, dy_3 + \\ &+ \int_0^\ell \frac{1}{2}\langle \mathring{T}_{11} \rangle \xi_1'^2 - \widetilde{\gamma} J_2 \vartheta' \xi_1'' + \frac{1}{2}\widetilde{\mu} J \vartheta'^2 + \frac{1}{2}\langle \mathring{T}_{\alpha\alpha} \rangle \vartheta^2 \, dy_3 + \\ &- \int_0^\ell \langle b_i \rangle \xi_i - \langle y_\alpha b_3 \rangle \xi_\alpha' + m\vartheta \, dy_3, \end{aligned}$$

which has to be minimized over all functions $(\boldsymbol{\xi}, \vartheta)$ with $\xi_\alpha \in H^2_\flat(0, \ell)$, $\xi_3 \in H^1_\flat(0, \ell)$ and $\vartheta \in H^1_\flat(0, \ell)$. The Euler-Lagrange equations can be written as

$$\begin{cases} J_2 \left(\widetilde{E} \xi_1''\right)'' - \left(\langle \mathring{T}_{11} \rangle \xi_1'\right)' - J_2 (\widetilde{\gamma} \vartheta')'' - \langle b_1 \rangle - \langle y_1 b_3 \rangle' = 0, \\ J_1 \left(\left(\widetilde{E} - \dfrac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) \xi_2''\right)'' - \langle b_2 \rangle - \langle y_2 b_3 \rangle' = 0, \\ A \left(\left(\widetilde{E} - \dfrac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) \xi_3'\right)' + \langle b_3 \rangle = 0, \\ J (\widetilde{\mu} \vartheta')' - \langle \mathring{T}_{11} + \mathring{T}_{22} \rangle \vartheta - J_2 (\widetilde{\gamma} \xi_1'')' + m = 0. \end{cases} \tag{3.35}$$

Remark 3.12. If we suppose that each (y_1, y_2) plane is a plane of symmetry, i.e. the material is monoclinic with uniform axis of symmetry identified with the y_3-axis, the term $\widetilde{\gamma}$ vanishes and, hence, the set of equilibrium equations (3.35) becomes uncoupled. Moreover, see Section 2.3 and 2.4,

$$\mathbf{Q}\left(\mathbb{L}(\cdot)\,\mathbf{E}\right)\mathbf{Q}^T = \mathbb{L}(\cdot)\left(\mathbf{Q}\mathbf{E}\mathbf{Q}^T\right), \tag{3.36}$$

and

$$\mathring{\mathbf{T}} = \mathbf{Q}\mathring{\mathbf{T}}\mathbf{Q}^T, \tag{3.37}$$

for all symmetric tensor $\mathbf{E}$ and for all $\mathbf{Q}$ in the symmetric group. From equation (3.36), for a monoclinic material as specified above, it follows that (see Section 2.3)

$$\mathbb{L}_{\alpha\beta\gamma 3} = 0, \qquad \mathbb{L}_{\alpha 333} = 0,$$

for all $\alpha, \beta, \gamma = 1, 2$, while from equation (3.37) we deduce (see Section 2.4)

$$\mathring{T}_{i3} = 0, \tag{3.38}$$

for all $i = 1, 2, 3$.

Hence, for this symmetry, we only have $\mathring{T}_{\alpha\beta} \neq 0$. But, using equations (3.3) and (3.38), we find

$$\langle \mathring{T}^{\varepsilon}_{\alpha\alpha} \rangle = \langle x_\alpha (\mathring{T}^{\varepsilon}_{\alpha 3,3}) \rangle = 0$$

for $\alpha = 1, 2$. Hence, recalling equations (3.35), we find that the equilibrium equations of the limit problem do not depend on the components of the residual stress. Moreover it can be easily shown that, substituting the condition on $\mathbb{L}$ and $\mathring{\mathbf{T}}$ above in (3.35), the Euler-Lagrange equations become

$$\begin{cases} J_2 \left(\widetilde{E}_m \xi_1'' \right)'' - \langle b_1 \rangle - \langle y_1 b_3 \rangle' = 0, \\ J_1 \left(\widetilde{E}_m \xi_2'' \right)'' - \langle b_2 \rangle - \langle y_2 b_3 \rangle' = 0, \\ A \left(\widetilde{E}_m \xi_3' \right)' + \langle b_3 \rangle = 0, \\ J \left(\widetilde{\mu}_m \vartheta' \right)' + m = 0, \end{cases} \tag{3.39}$$

where $\widetilde{E}_m(y_3)$ and $\widetilde{\mu}_m(y_3)$ are obtained by the definition (3.13) of f_0 and by the assumptions of monoclinic material on $\mathbb{L}$.

Chapter 4

Slender rods with residual stress

Slender rods are widely used in structural engineering since they provide high resistance in front of a reduced structural weight. In this chapter, we derive, by means of Γ-convergence, a variational model for slender rods with residual stress.

The presence of residual stress introduces in the constitutive equation the dependence of the Piola-Kirchhoff stress from the displacement gradient in contrast to the case of no residual stress where the dependence is only on the strain. For residually stressed hyperelastic materials the elastic energy density turns out to be non-convex, and this makes an important mathematical difference from the usual framework of linear elasticity (without residual stress).

In our analysis we shall make no assumption on the symmetries of the material, i.e., full anisotropy will be considered, and we allow the material to be non homogeneous along the x_3-axis, so to model rods made of several different thin layers jointed along the longitudinal axis. Assuming the rod to be clamped to one of its bases, we derive by means of Γ-convergence the one-dimensional limit problem, stated in Theorem 4.9.

We show that in the particular case of a slender rod made of isotropic and homogeneous material with a stress-free reference configuration the deduced one-dimensional limit problem leads to the classical model obtained by Anzellotti, Baldo and Percivale in [4], Percivale in [40] and Freddi, Londero and Paroni in [22].

Looking at the equilibrium equations, written in Section 4.7 for a material with monoclinic symmetry, it is interesting to note that the equations involving the twist and the longitudinal displacement are uncoupled, while the equations of displacements along x_1 and x_2-axes are coupled, see (4.35). Moreover, the residual stress appears only in the coupled equations.

The chapter is organized as follows. In Section 4.1 we briefly describe the three-dimensional problem. The problem of existence of solutions is discussed in Section 4.2, where we follow the same approach of Paroni [39]. In Section 4.3 we re-define the three-dimensional problem as a variational problem for a rescaled energy in a fixed domain Ω. In Section 4.4 we recall some results that can be found in [22] that are essential in Section 4.5 to state and prove the Γ-convergence theoretical result and the convergences of minima and minimizers. In Section 4.6 we re-write the Γ-limit in a simple way for some material symmetries. In Section 4.7 we deduce the equations of equilibrium for a material with monoclinic symmetry. We point out that equations (4.35) depend on the residual stress component $\mathring{T}_{12}$, while the Euler-Lagrange equations for a thin walled beam with monoclinic symmetry are completely independent of $\mathring{\mathbf{T}}$ (see equations (3.39)).

4.1 Setting of the problem

For all $\varepsilon \in (0, 1]$ and $\ell > 0$ let

$$\Omega_\varepsilon := \{\mathbf{x} = (x_\alpha, x_3) \in \mathbb{R}^2 \times \mathbb{R} : x_\alpha \in \omega_\varepsilon,\ x_3 \in (0, \ell)\}$$

where $\omega_\varepsilon := \varepsilon\omega$, and ω is a simply connected, bounded, open subset of $\mathbb{R}^2$ with a Lipschitz boundary. We denote $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$ for any $x_3 \in (0, \ell)$. We can think of Ω_ε as the reference configuration of the body subject to a residual stress $\mathring{\mathbf{T}}^\varepsilon$. In this configuration the body is in equilibrium in the absence of external loads and the Cauchy stress field $\mathring{\mathbf{T}}^\varepsilon$ satisfies the following boundary problem (see Section 2.4)

$$\begin{cases} \operatorname{div} \mathring{\mathbf{T}}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon = (\mathring{\mathbf{T}}^\varepsilon)^T & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.1}$$

where $\mathbf{n}$ is the outward unit normal to the boundary of Ω_ε.

In what follows we consider a material triclinic and inhomogeneous along the x_3-axis, so that the first Piola-Kirchhoff stress field $\mathbf{S}^\varepsilon$ can be expressed as (see Section 2.2)

$$\mathbf{S}^\varepsilon(\mathbf{x}) = \mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbf{Du}(\mathbf{x})\mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbb{L}(x_3)\mathbf{Eu}(\mathbf{x})$$

where $\mathbf{Du}$ denotes the gradient of the displacement $\mathbf{u}$, $\mathbf{Eu}$ its strain,

$$\mathbf{Eu}(\mathbf{x}) = \frac{\mathbf{Du} + (\mathbf{Du})^T}{2},$$

and $\mathbb{L}(x_3)$ is the incremental elasticity tensor.

We assume $\mathbb{L}$ to be essentially bounded,

$$\mathbb{L} \in L^\infty((0,\ell);\mathbb{R}^{3\times3\times3\times3}),$$

to have the major and minor symmetries,

$$\mathbb{L}_{ijkl} = \mathbb{L}_{jikl} = \mathbb{L}_{klji},$$

to be positive definite,

$$\exists\, C > 0, \text{ s.t. } \mathbb{L}(x_3)\mathbf{A}\cdot\mathbf{A} \geq C|\mathbf{A}|^2, \tag{4.2}$$

for all $\mathbf{A} \in \mathbb{R}^{3\times3}_{\mathrm{sym}} := \{\mathbf{A} \in \mathbb{R}^{3\times3} : \mathbf{A} = \mathbf{A}^T\}$ and for a.e. $x_3 \in (0,\ell)$; we denote by C_L the largest of all such constants C. Furthermore we assume $\mathring{\mathbf{T}}^\varepsilon \in L^\infty(\Omega_\varepsilon;\mathbb{R}^{3\times3})$.

We consider the body clamped on $S_\varepsilon(0)$ subject to dead body forces $\mathbf{b}^\varepsilon$ and we assume $\mathbf{b}^\varepsilon \in L^2(\Omega_\varepsilon;\mathbb{R}^3)$. In the domain Ω_ε we study the elasticity problem: find $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ such that

$$\int_{\Omega_\varepsilon} (\mathbf{D}\mathbf{u}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{D}\mathbf{v} + \mathbb{L}\mathbf{E}\mathbf{u}\cdot\mathbf{E}\mathbf{v})\,dx = \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{v}\,dx, \tag{4.3}$$

for all $\mathbf{v} \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$, where

$$H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3) := \{\mathbf{u} \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\}.$$

4.2 Existence of the solution

In this section we discuss the existence of a solution $\mathbf{u}$ of (4.3) following the lines traced by Paroni [39]. A crucial role in the proof of the existence of a solution is played by Korn's inequality (see Anzellotti, Baldo and Percivale [4], Theorem A.1).

Theorem 4.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_{\Omega_\varepsilon} (|\mathbf{u}|^2 + |\mathbf{D}\mathbf{u}|^2)dx \leq \frac{C}{\varepsilon^2}\int_{\Omega_\varepsilon} |\mathbf{E}\mathbf{u}|^2 dx, \tag{4.4}$$

for every $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ and for every $\varepsilon \in (0,1]$.

We denote by C_K the smallest of all constants C for which inequality (4.4) is valid.

Let

$$\mathring{\tau}_m^\varepsilon := \operatorname*{essinf}_{\mathbf{x}\in\Omega_\varepsilon} \min_{\mathbf{a}\in\mathbb{R}^3} \{\mathring{\mathbf{T}}^\varepsilon(\mathbf{x})\mathbf{a}\cdot\mathbf{a} \ : \ |\mathbf{a}| = 1\}, \tag{4.5}$$

denote the smallest eigenvalue of $\mathring{\mathbf{T}}^\varepsilon$. As in Section 3.2, it can be shown that $\mathring{\tau}_m^\varepsilon$ is either identically equal to 0 or that it also take negative values. Hence, for a generic residual stress tensor $\mathring{\mathbf{T}}^\varepsilon$, the bilinear form in the first member of (4.3) is not a priori H^1-coercive. Therefore, to prove existence and uniqueness of the solution of problem (4.3), we shall suppose that the absolute value of $\mathring{\tau}_m^\varepsilon$ is small enough, that is the compressions due to the residual stress are not too large.

Theorem 4.2. *Assume that*

$$C_L > C_K \frac{|\mathring{\tau}_m^\varepsilon|}{\varepsilon^2}. \tag{4.6}$$

Then there exists a unique solution $\mathbf{u}^\varepsilon \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ *of problem* (4.3).

Proof. From (4.2), (4.5), Theorem 4.1 and Lemma 3.2 we have

$$\begin{aligned} \int_{\Omega_\varepsilon} (\mathbf{D}\mathbf{v}\mathring{\mathbf{T}}^\varepsilon\cdot\mathbf{D}\mathbf{v} + \mathbb{L}\mathbf{E}\mathbf{v}\cdot\mathbf{E}\mathbf{v})\,dx \ &\geq \mathring{\tau}_m^\varepsilon \|\mathbf{D}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)} + C_L\|\mathbf{E}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)} \\ &\geq \left(C_L - C_K\frac{|\mathring{\tau}_m^\varepsilon|}{\varepsilon^2}\right)\|\mathbf{E}\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Using Theorem 4.1 in the last term of the previous inequality, existence and uniqueness of the solution of problem (4.3) follow from an application of Lax-Milgram lemma. □

Hereafter, we will always assume inequality (4.6) to hold. Moreover, by Theorem 4.2, we state that the energy functionals

$$J_\varepsilon(\mathbf{u}) = \frac{1}{2}\int_{\Omega_\varepsilon} (\mathbf{D}\mathbf{u}\mathring{\mathbf{T}}^\varepsilon\cdot\mathbf{D}\mathbf{u} + \mathbb{L}\mathbf{E}\mathbf{u}\cdot\mathbf{E}\mathbf{u})\,dx - \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon\cdot\mathbf{u}\,dx \tag{4.7}$$

admit for $\varepsilon > 0$ a unique minimizer among all functions $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$.

4.3 The rescaled problem

In order to study the behaviour of the energy functionals (4.7), as $\varepsilon \to 0$, following the idea of Ciarlet and Destuynder [13], we rescale the problem. We consider the map $p_\varepsilon : \Omega \to \Omega_\varepsilon$, where $\Omega := \Omega_1$, $\omega := \omega_1$ and $S(x_3) := S_1(x_3)$, defined as

$$p_\varepsilon(\mathbf{y}) = (\varepsilon y_1, \varepsilon y_2, y_3)\,.$$

We denote by $\mathbf{E}^\varepsilon\mathbf{v}^\varepsilon := \mathrm{sym}(\mathbf{H}^\varepsilon\mathbf{v}^\varepsilon)$ the rescaled strain, where

$$\mathbf{H}^\varepsilon\mathbf{v}^\varepsilon := \left(\frac{D_1\mathbf{v}^\varepsilon}{\varepsilon}, \frac{D_2\mathbf{v}^\varepsilon}{\varepsilon}, D_3\mathbf{v}^\varepsilon\right),$$

and $D_i\mathbf{v}^\varepsilon$ denotes the column vector of the partial derivatives of $\mathbf{v}^\varepsilon$ with respect to y_i, $i = 1, 2, 3$. Furthermore we also denote $\mathbf{W}^\varepsilon\mathbf{v}^\varepsilon := \mathrm{skw}(\mathbf{H}^\varepsilon\mathbf{v}^\varepsilon)$ the skew-symmetric part of $\mathbf{H}^\varepsilon\mathbf{v}^\varepsilon$. We define

$$\mathring{\mathbf{T}}^\varepsilon = \varepsilon^2\mathring{\mathbf{T}} \circ p_\varepsilon^{-1}, \tag{4.8}$$

where $\mathring{\mathbf{T}} \in L^\infty(\Omega, \mathbb{R}^{3\times3})$ and denote by $\mathring{\tau}_m$ the smallest eigenvalue of $\mathring{\mathbf{T}}$. We note that, in the fixed domain Ω, the inequality (4.6) becomes

$$C_L > C_K|\mathring{\tau}_m|, \tag{4.9}$$

where C_K is the smallest Korn's constant for Ω.
Hence, the rescaled energy $F_\varepsilon : H^1_\flat(\Omega; \mathbb{R}^3) \to \mathbb{R}$ is defined by

$$F_\varepsilon(\mathbf{v}) := \frac{1}{\varepsilon^2}J_\varepsilon(\mathbf{v} \circ p_\varepsilon^{-1}) = I_\varepsilon(\mathbf{v}) - \int_\Omega \mathbf{b}^\varepsilon \circ p_\varepsilon \cdot \mathbf{v}\, dy,$$

where

$$I_\varepsilon(\mathbf{v}) := \frac{1}{2}\int_\Omega (\mathbb{L}\mathbf{E}^\varepsilon\mathbf{v} \cdot \mathbf{E}^\varepsilon\mathbf{v} + \varepsilon^2\mathbf{H}^\varepsilon\mathbf{v}\mathring{\mathbf{T}} \cdot \mathbf{H}^\varepsilon\mathbf{v})dy. \tag{4.10}$$

We suppose the loads to have the following form

$$\begin{gathered} b_1^\varepsilon \circ p_\varepsilon(\mathbf{y}) = \varepsilon^2 b_1(\mathbf{y}) - \varepsilon\frac{m(y_3)}{I_O}y_2, \quad b_2^\varepsilon \circ p_\varepsilon(\mathbf{y}) = \varepsilon^2 b_2(\mathbf{y}) + \varepsilon\frac{m(y_3)}{I_O}y_1, \\ b_3^\varepsilon \circ p_\varepsilon(\mathbf{y}) = \varepsilon b_3(\mathbf{y}), \end{gathered} \tag{4.11}$$

with $\mathbf{b} = (b_1, b_2, b_3) \in L^2(\Omega; \mathbb{R}^3)$, $m \in L^2(0, \ell)$ and $I_O := \int_\omega (y_1^2 + y_2^2)\, dy_1\, dy_2$ the polar moment of inertia of the section ω. With the loads given by (4.11), the energy $F_\varepsilon(\mathbf{v})$ can be rewritten as

$$F_\varepsilon(\mathbf{v}) = I_\varepsilon(\mathbf{v}) - \varepsilon^2\int_\Omega \mathbf{b} \cdot (v_1, v_2, \frac{v_3}{\varepsilon})\, dy - \varepsilon^2\int_0^\ell m\, \vartheta^\varepsilon(\mathbf{v})\, dy_3, \tag{4.12}$$

where we have set

$$\vartheta^\varepsilon(\mathbf{v})(y_3) := \frac{1}{I_O}\int_\omega (\frac{y_1}{\varepsilon}v_2(y_1, y_2, y_3) - \frac{y_2}{\varepsilon}v_1(y_1, y_2, y_3))\, dy_1 dy_2. \tag{4.13}$$

We note that if $\mathbf{v} \in L^2(\Omega; \mathbb{R}^3)$ then $\vartheta^\varepsilon(\mathbf{v}) \in L^2(0, \ell)$. A similar statement holds if we replace L^2 with H^1.

4.4 Compactness lemmata

The following theorem recalls a scaled Korn inequality obtained by Freddi, Londero and Paroni in [22].

Theorem 4.3. *There exists a positive constant K, independent of ε, such that*

$$\int_\Omega \Big(|(u_1, u_2, \frac{u_3}{\varepsilon})|^2 + |\mathbf{H}^\varepsilon \mathbf{u}|^2\Big)\, dy \le \frac{K}{\varepsilon^2} \int_\Omega |\mathbf{E}^\varepsilon \mathbf{u}|^2\, dy,$$

for every $\mathbf{u} \in H^1_\flat(\Omega;\mathbb{R}^3)$ and every $\varepsilon \in (0,1]$.

Proof. The inequality $\int_\Omega |\mathbf{H}^\varepsilon \mathbf{u}|^2\, dy \le (K/\varepsilon^2)\int_\Omega |\mathbf{E}^\varepsilon \mathbf{u}|^2\, dy$ simply follows by rescaling the Korn's inequality of Anzellotti, Baldo and Percivale [4], Theorem A.1. To show that $\int_\Omega |(u_1,u_2,u_3/\varepsilon)|^2\, dy \le (K/\varepsilon^2)\int_\Omega |\mathbf{E}^\varepsilon \mathbf{u}|^2\, dy$, it suffices to set $\mathbf{v} := (u_1, u_2, u_3/\varepsilon)$, notice that $|\mathbf{E}^\varepsilon \mathbf{u}| \ge \varepsilon |\mathbf{E}\mathbf{v}|$ and apply Theorem 1.45 to $\mathbf{v}$ on Ω. □

Let $H_{BN}(\Omega;\mathbb{R}^3)$ be the space of Bernoulli-Navier displacements on Ω, defined as in section 3.4, i.e.

$$\begin{aligned} H_{BN}(\Omega;\mathbb{R}^3) = \{\ \mathbf{v} \in H^1_\flat(\Omega;\mathbb{R}^3) : \exists\, \xi_\alpha \in H^2_\flat(0,\ell), \exists\, \xi_3 \in H^1_\flat(0,\ell) \\ \text{s. t. } v_\alpha(\mathbf{y}) = \xi_\alpha(y_3),\ v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)\ \}. \end{aligned} \tag{4.14}$$

In the remaining part of this section we assume $\mathbf{u}^\varepsilon$ to be a sequence of functions in $H^1_\flat(\Omega;\mathbb{R}^3)$ such that

$$\|\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\varepsilon, \tag{4.15}$$

for some constant C and for every $\varepsilon \in (0,1]$.

As in Section 3.4, let $\wp$ denote the projection of $L^2(\omega;\mathbb{R}^2)$ on the subspace $\mathcal{R}_2$ of the infinitesimal rigid displacements on ω. Hence $\mathcal{R}_2$ is a closed subspace of $H^1(\omega;\mathbb{R}^2)$; if $\mathbf{w} \in L^2(\omega;\mathbb{R}^2)$, we have that

$$(\wp w)_\alpha = \mathcal{E}_{\beta\alpha} y_\beta \Big(\frac{1}{I_O}\int_\omega \mathcal{E}_{\gamma\delta} y_\gamma w_\delta\, dy_1 dy_2\Big) + \frac{1}{|\omega|}\int_\omega w_\alpha\, dy_1 dy_2, \tag{4.16}$$

where $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol, and the two-dimensional Korn's inequality writes as

$$\|\mathbf{w} - \wp\mathbf{w}\|^2_{H^1(\omega;\mathbb{R}^2)} \le C\|\mathbf{E}\mathbf{w}\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \tag{4.17}$$

for all $\mathbf{w} \in H^1(\omega;\mathbb{R}^2)$.

Lemma 4.4. *Let* (4.15) *hold. Then*

1. *for any sequence of positive numbers* ε_n *converging to 0 there exist a subsequence (not relabelled) and a couple of functions* $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ *and* $\vartheta \in L^2(\Omega)$ *such that (as* $n \to +\infty$*)*

$$\left(u_1^{\varepsilon_n}, u_2^{\varepsilon_n}, \frac{u_3^{\varepsilon_n}}{\varepsilon_n}\right) \rightharpoonup \mathbf{v} \ \text{in } H^1(\Omega;\mathbb{R}^3), \tag{4.18}$$

$$\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \rightharpoonup \mathbf{H}\mathbf{v} := \begin{pmatrix} 0 & -\vartheta & D_3v_1 \\ \vartheta & 0 & D_3v_2 \\ -D_3v_1 & -D_3v_2 & 0 \end{pmatrix} \ \text{in } L^2(\Omega;\mathbb{R}^{3\times3}); \tag{4.19}$$

2. $\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)} \le \dfrac{K}{\varepsilon}\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})}$ *holds for some constant* $K > 0$*;*

3. $\vartheta^\varepsilon(\mathbf{u}^\varepsilon) \to \vartheta$ *in* $L^2(\Omega)$*; therefore* ϑ *does not depend on* y_1 *and* y_2*;*

4. $\vartheta \in H^1_\flat(\Omega)$*;*

5. *the following identities hold in* $L^2(\Omega)$

$$E_{33} = D_3v_3, \qquad -D_2E_{13} + D_1E_{23} = D_3\vartheta, \tag{4.20}$$

where, up to subsequences, E_{33}*,* E_{13} *and* E_{23} *are, respectively, the limits of* $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon$*,* $(E^\varepsilon u^\varepsilon)_{13}/\varepsilon$ *and* $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon$ *in the weak convergence of* $L^2(\Omega)$*.*

Proof. 1. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon)$. Since $|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon| \ge \varepsilon|\mathbf{E}\mathbf{v}^\varepsilon|$, by (4.15), $\mathbf{E}\mathbf{v}^\varepsilon$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$ and by Korn's inequality $\mathbf{v}^\varepsilon$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. There then exist a $\mathbf{v} \in H^1_\flat(\Omega;\mathbb{R}^3)$ and a subsequence of ε_n such that $\mathbf{v}^{\varepsilon_n} \rightharpoonup \mathbf{v}$ in $H^1(\Omega;\mathbb{R}^3)$. Again, it is easy to check that $|(E^\varepsilon u^\varepsilon)_{i\alpha}| \ge |(Ev^\varepsilon)_{i\alpha}|$, thus, using (4.15) we deduce that $C\varepsilon \ge \|(Ev^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently, as $n \to \infty$, $(Ev)_{i\alpha} = 0$ for $i = 1,2,3$ and $\alpha = 1,2$. Hence $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$.
Using (4.15) and Theorem 4.3 we obtain that the sequence $\mathbf{H}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n}$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$ so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times3})$ to a matrix $\mathbf{H} \in L^2(\Omega;\mathbb{R}^{3\times3})$. Since, from (4.15), $\mathbf{E}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times3})$, we have $\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \rightharpoonup \mathbf{H}$ in $L^2(\Omega;\mathbb{R}^{3\times3})$. In particular, $\mathbf{H}$ is, almost everywhere, a skew-symmetric matrix. Since $(H^\varepsilon u^\varepsilon)_{13} = u^\varepsilon_{1,3} = v^\varepsilon_{1,3}$ and $(H^\varepsilon u^\varepsilon)_{23} = u^\varepsilon_{2,3} = v^\varepsilon_{2,3}$, we deduce that $(H)_{13} = v_{1,3}$ and $(H)_{23} = v_{2,3}$. We conclude the first claim of the lemma by denoting $(H)_{12} := -\vartheta$.

2. We define $\mathbf{w}^\varepsilon := (u_1^\varepsilon/\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$. Hence, for almost $y_3 \in (0,\ell)$ and any $\varepsilon \in (0,1]$, we consider the projection of the first two components of $\mathbf{w}^\varepsilon(\cdot, y_3)$. From (4.16) and (4.13), we find

$$(\wp w^\varepsilon)_\alpha = \mathcal{E}_{\beta\alpha} y_\beta \vartheta^\varepsilon(u^\varepsilon) + \frac{1}{|\omega|}\int_\omega w^\varepsilon_\alpha \, dy_1 dy_2. \tag{4.21}$$

Since $(Ew^\varepsilon)_{11} = (E^\varepsilon u^\varepsilon)_{11}$, $(Ew^\varepsilon)_{12} = (E^\varepsilon u^\varepsilon)_{12}$ and $(Ew^\varepsilon)_{22} = (E^\varepsilon u^\varepsilon)_{22}$, we have

$$\|(Ew^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} = \|(E^\varepsilon u^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \tag{4.22}$$

for $\alpha, \beta = 1, 2$. Then, integrating inequality (4.17) on $(0, \ell)$ and recalling (4.15), we get

$$\int_0^\ell \|\mathbf{w}^\varepsilon - \wp\mathbf{w}^\varepsilon\|_{H^1(\omega;\mathbb{R}^2)} dy_3 \leq C\|\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\varepsilon$$

and hence

$$\|D_\alpha(w^\varepsilon_\beta - \wp w^\varepsilon_\beta)\|_{L^2(\Omega;\mathbb{R})} \to 0 \tag{4.23}$$

for $\alpha, \beta = 1, 2$. Since $(W\wp w^\varepsilon)_{12} = -\vartheta^\varepsilon(u^\varepsilon)$ and $(Ww^\varepsilon)_{12} = (W^\varepsilon u^\varepsilon)_{12}$, we obtain, from the identity

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) = -(W\wp w^\varepsilon)_{12} = (W(w^\varepsilon - \wp w^\varepsilon))_{12} - (W^\varepsilon u^\varepsilon)_{12}, \tag{4.24}$$

the proof of point *2.* of the lemma.

3. Using equation (4.19), for $\varepsilon \to 0$, we have that $\vartheta^\varepsilon(\mathbf{u}^\varepsilon) \rightharpoonup \vartheta$ in $L^2(\Omega)$. Since $\vartheta^\varepsilon(\mathbf{u}^\varepsilon)$ does not depend on y_1 and y_2, the same holds for ϑ.

4. The proof proceeds along the same lines of that of point *4.* of Lemma 3.7.

5. To prove the first equation of (4.20) we note that $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon = D_3(u^\varepsilon_3/\varepsilon)$ and we apply (4.18). Let's prove the second equation of (4.20). From inequality (4.15) we deduce that, up to subsequences, $(E^\varepsilon u^\varepsilon)_{13}/\varepsilon \rightharpoonup E_{13}$ and $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon \rightharpoonup E_{23}$ in $L^2(\Omega)$. To characterize $E_{13}, E_{23} \in L^2(\Omega)$ note that

$$D_3(W^\varepsilon u^\varepsilon)_{12} = D_2\big(\frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon}\big) - D_1\big(\frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon}\big),$$

in the sense of distributions. Hence for $\psi \in C^\infty_c(\Omega)$ we obtain

$$\int_\Omega (W^\varepsilon u^\varepsilon)_{12} D_3\psi\, dy = \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon} D_2\psi\, dy - \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon} D_1\psi\, dy.$$

Passing to the limit in the previous equality we find

$$\int_\Omega -\vartheta D_3\psi\, dy = \int_\Omega E_{13} D_2\psi\, dy - \int_\Omega E_{23} D_1\psi\, dy.$$

Thus $D_3\vartheta = -D_2E_{13} + D_1E_{23}$ in the sense of distributions, hence in $L^2(\Omega)$ since $\vartheta \in H^1_\flat(\Omega)$. $\square$

Remark 4.5. The strong convergence in *3.* of the previous lemma was shown in [22], even if not explicity stated. Therefore, from *1.* of Lemma 4.4, we find

$$(H^\varepsilon u^\varepsilon)_{12} \to -\vartheta \;\text{ in } L^2(\Omega), \qquad \text{and} \qquad (H^\varepsilon u^\varepsilon)_{21} \to \vartheta \;\text{ in } L^2(\Omega). \tag{4.25}$$

Remark 4.6. From the second equality of (4.20), i.e. $-D_2E_{13}+D_1E_{23} = D_3\vartheta$, which we can rewrite as

$$D_2(E_{13} + \frac{y_2}{2}D_3\vartheta) = D_1(E_{23} - \frac{y_1}{2}D_3\vartheta) \;\text{ in } \mathcal{D}'(\Omega),$$

and the weak version of Poincaré's lemma (see Girault and Raviart [28], Theorem 2.9), there exists a function $\varphi \in Q_1 := L^2((0,\ell); H^1_m(\omega))$ such that

$$\begin{cases} E_{13} = D_1\varphi - \dfrac{y_2}{2}D_3\vartheta \\ E_{23} = D_2\varphi + \dfrac{y_1}{2}D_3\vartheta, \end{cases} \tag{4.26}$$

where $H^1_m(\omega) := \{v \in H^1(\omega) : \fint_\omega v = 0\}$.

To characterize the other components of the limit strain $\mathbf{E}$, we need the two-dimensional Korn's inequality (4.17).

Lemma 4.7. *Let* (4.15) *hold. Then there exists* $\mathbf{w} = (w_1, w_2) \in Q_2$*, where* $Q_2 := H^1_\flat(\omega; L^2((0,\ell);\mathbb{R}^2))$*, such that*

$$E_{11} = E_{11}(\mathbf{w}), \quad E_{22} = E_{22}(\mathbf{w}), \quad E_{12} = E_{12}(\mathbf{w}), \tag{4.27}$$

where, up to subsequences, E_{11}*,* E_{22} *and* E_{12} *are the limits of* $(E^\varepsilon u^\varepsilon)_{11}/\varepsilon$*,* $(E^\varepsilon u^\varepsilon)_{22}/\varepsilon$ *and* $(E^\varepsilon u^\varepsilon)_{12}/\varepsilon$ *in the weak convergence of* $L^2(\Omega)$*.*

Proof. Let $\bar{\mathbf{u}}^\varepsilon$ be the vector whose components are the first two ones of $\mathbf{u}^\varepsilon$, i.e. $\bar{\mathbf{u}}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon)$. We have $(E\bar{\mathbf{u}})_{\alpha\beta}/\varepsilon = (E^\varepsilon\bar{\mathbf{u}})_{\alpha\beta}$, for $\alpha,\beta = 1,2$. Using (4.15) and integrating inequality (4.17) on $(0,\ell)$, we find that

$$\|\frac{\bar{\mathbf{u}}^\varepsilon - \wp\bar{\mathbf{u}}^\varepsilon}{\varepsilon^2}\|_{L^2(\Omega;\mathbb{R}^2)} \le C.$$

Hence, up to subsequences, $(\bar{\mathbf{u}}^\varepsilon - \wp\bar{\mathbf{u}}^\varepsilon)/\varepsilon^2 \rightharpoonup \mathbf{w}$ in $L^2(\Omega;\mathbb{R}^2)$. Moreover

$$\frac{(E^\varepsilon u^\varepsilon)_{\alpha\beta}}{\varepsilon} = \frac{E(\bar{\mathbf{u}}^\varepsilon - \wp\bar{\mathbf{u}}^\varepsilon)_{\alpha\beta}}{\varepsilon^2} \rightharpoonup E(\mathbf{w})_{\alpha\beta} \text{ in } L^2(\Omega),$$

for $\alpha,\beta = 1,2$. $\square$

From (4.20), (4.26) and (4.27) we have that the limit strain can be written as

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w}) \\
&= \begin{pmatrix} E_{11}(\mathbf{w}) & E_{12}(\mathbf{w}) & D_1\varphi - \dfrac{y_2}{2}D_3\vartheta \\ E_{12}(\mathbf{w}) & E_{22}(\mathbf{w}) & D_2\varphi + \dfrac{y_1}{2}D_3\vartheta \\ D_1\varphi - \dfrac{y_2}{2}D_3\vartheta & D_2\varphi + \dfrac{y_1}{2}D_3\vartheta & D_3v_3 \end{pmatrix}.
\end{aligned}
\tag{4.28}
$$

4.5 The main result

Lemma 4.8. *Let* $\mathbf{u}^\varepsilon$ *be a sequence of functions in the space* $H^1_\flat(\Omega;\mathbb{R}^3)$. *If*

$$
\sup_\varepsilon (F_\varepsilon(\mathbf{u}^\varepsilon)/\varepsilon^2) < +\infty,
$$

then (4.15) *holds for some constant* $C > 0$ *and for every* $\varepsilon \in (0,1]$.

Proof. We define $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon)$ and $R := C_L - C_K|\mathring{\tau}_m|$. By inequality (4.9), we have $R > 0$. With this notation and by using (4.2), (4.12) and Lemma 3.2, we find

$$
\begin{aligned}
\frac{1}{\varepsilon^2}F_\varepsilon(\mathbf{u}^\varepsilon) &= \frac{1}{2}\int_\Omega \left(\mathbb{L}\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\cdot\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon} + \mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\mathring{\mathbf{T}}\cdot\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon \right) dy + \\
&\quad - \int_\Omega \mathbf{b}\cdot\mathbf{v}^\varepsilon\,dy - \int_0^\ell m\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\,dy_3 \\
&\geq \frac{C_L}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} + \frac{\mathring{\tau}_m}{2}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \\
&\quad -\|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(0,\ell)} \\
&\geq \frac{R}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(0,\ell)},
\end{aligned}
$$

where in the last inequality we used Theorem 4.3. From *3.* of Lemma 4.4, we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2}F_\varepsilon(\mathbf{u}^\varepsilon) \geq \frac{R}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\
-\frac{1}{2C_2}\|m\|^2_{L^2(0,\ell)} - \frac{C_2}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)}
\end{aligned}
$$

where C_1 and C_2 are arbitrary positive constants. Choosing $C_2 = R/2$ we have

$$\frac{1}{\varepsilon^2}F_\varepsilon(\mathbf{u}^\varepsilon) \geq \frac{R}{4}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} - \frac{1}{R}\|m\|^2_{L^2(0,\ell)}. \quad (4.29)$$

By Theorem 4.3, we deduce that

$$\begin{aligned}\frac{1}{\varepsilon^2}F_\varepsilon(\mathbf{u}^\varepsilon) \geq \frac{R}{4K}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + (\frac{1}{K} - \frac{C_1}{2})\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\ -\frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{1}{R}\| \ m\|^2_{L^2(0,\ell)}.\end{aligned}$$

By choosing, for instance, $C_1 = 1/K$, we find that there exists a constant $M > 0$ such that

$$M \geq \frac{R}{4K}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2K}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}$$

from which follows that the sequence $\mathbf{v}^\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^3)$. Using this fact in (4.29) we get the estimate (4.15). □

From Lemma 4.8 and *1.* of Lemma 4.4, it follows that the family of functionals $(1/\varepsilon^2)F_\varepsilon$ is coercive in the space $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$ with respect to the weak convergence of the sequence $q_\varepsilon(\mathbf{u}^\varepsilon) := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon, (W^\varepsilon u^\varepsilon)_{12})$, uniformly with respect to ε. Then, for any sequence $\mathbf{u}^\varepsilon$ which is bounded in energy, that is $(1/\varepsilon^2)F_\varepsilon \leq C$ for a suitable constant $C > 0$, and satisfies the boundary conditions $\mathbf{u}^\varepsilon = 0$ on $S(0)$, the corresponding sequence $q_\varepsilon(\mathbf{u}^\varepsilon)$ is weakly relatively compact in $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$.

Theorem 4.9 (Γ-convergence)**.** *Let $F : H^1_\flat(\Omega;\mathbb{R}^3) \times H^1_\flat(\Omega;\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$\begin{aligned}F(\mathbf{v},\vartheta) = \ & \frac{1}{2}\min_{\varphi\in Q_1,\mathbf{w}\in Q_2}\left\{\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w}) \cdot \mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\,dy\right\} + \\ & +\frac{1}{2}\int_\Omega \mathbf{H}\mathbf{v}\mathring{\mathbf{T}} \cdot \mathbf{H}\mathbf{v}\,dy - \int_\Omega \mathbf{b}\cdot\mathbf{v}\,dy - \int_0^\ell m\vartheta\,dy_3\end{aligned} \quad (4.30)$$

if $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$, and $+\infty$ otherwise, where Q_1 is defined in Remark 4.6, Q_2 in Lemma 4.7, $\mathbf{H}\mathbf{v}$ and $\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})$ in (4.19) *and* (4.28) *respectively. As $\varepsilon \to 0$, the sequence of functionals $(1/\varepsilon^2)F_\varepsilon$ Γ-converges to the functional F, in the following sense:*

1. *(liminf inequality) for every sequence of positive numbers ε_k converging to 0 and for every sequence $\{\mathbf{u}^k\} \subset H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup \mathbf{v} \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12} \to -\vartheta \text{ in } L^2(\Omega),$$

we have

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} \geq F(\mathbf{v},\vartheta);$$

2. *(recovery sequence) for every sequence of positive numbers* ε_k *converging to* 0 *and for every* $(\mathbf{v},\vartheta) \in H^1_\flat(\Omega;\mathbb{R}^3) \times H^1_\flat(\Omega;\mathbb{R})$ *there exists a sequence* $\{\mathbf{u}^k\} \subset H^1_\flat(\Omega;\mathbb{R}^3)$ *such that*

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup \mathbf{v} \ \textit{in}\ H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12} \to -\vartheta \ \textit{in}\ L^2(\Omega),$$

and

$$\limsup_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} \leq F(\mathbf{v},\vartheta).$$

Proof. Let us prove the liminf inequality. If $\liminf\limits_{k\to+\infty} F_{\varepsilon_k}(\mathbf{u}^k)/\varepsilon_k^2 = +\infty$ the claim follows trivially. Hence let us suppose, by passing to a subsequence, if necessary, that

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} = \lim_{k\to+\infty} \frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} < +\infty.$$

Then Lemma 4.8 applies to the sequence $\sup_k(F_{\varepsilon_k}(\mathbf{u}^k)/\varepsilon_k^2) < +\infty$ and hence the results of Lemma 4.4 hold true. Let $L_{\varepsilon_k} = I_{\varepsilon_k} - F_{\varepsilon_k}$ denote the work done by the loads, where I_{ε_k} is defined in (4.10); recalling (4.12) and (4.13), we deduce that

$$\frac{L_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} = \int_\Omega \mathbf{b}\cdot(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k})\,dy + \int_0^\ell m\vartheta^{\varepsilon_k}(\mathbf{u}^k)\,dy_3 \to \int_\Omega \mathbf{b}\cdot\mathbf{v}\,dy + \int_0^\ell m\vartheta\,dy_3.$$

Thus we have only to prove that

$$\begin{aligned}\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} \geq\ & \frac{1}{2}\min_{\varphi\in Q_1,\mathbf{w}\in Q_2}\left\{\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\cdot\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\,dy\right\} + \\ & +\frac{1}{2}\int_\Omega \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\,dy.\end{aligned}$$

If we define $\mathbf{z}^k := (u_1^k, u_2^k, u_3^k/\varepsilon_k)$ and by using assumption (4.2), we obtain

$$\begin{aligned}\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} \geq\ & \frac{1}{2}\int_\Omega C_L|(Ez^k)_{33}|^2 + \mathring{T}_{11}(z^k_{3,1})^2 + \mathring{T}_{33}((z^k_{1,3})^2 + (z^k_{2,3})^2) + \\ & + \mathring{T}_{22}(z^k_{3,2})^2 + 2\mathring{T}_{12}(z^k_{3,1})(z^k_{3,2}) + 2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2 + \\ & -2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2 + \mathring{T}_{11}\left(\frac{(z^k_{1,1})^2}{\varepsilon_k^2} + \frac{(z^k_{2,1})^2}{\varepsilon_k^2}\right) + \\ & + \mathring{T}_{33}\varepsilon_k^2(z^k_{3,3})^2 + \mathring{T}_{22}\left(\frac{(z^k_{1,2})^2}{\varepsilon_k^2} + \frac{(z^k_{2,2})^2}{\varepsilon_k^2}\right) + \\ & + 2\mathring{T}_{12}\left(\frac{z^k_{1,1}}{\varepsilon_k}\frac{z^k_{1,2}}{\varepsilon_k} + \frac{z^k_{2,1}}{\varepsilon_k}\frac{z^k_{2,2}}{\varepsilon_k}\right) + \\ & + 2\mathring{T}_{13}\left(\frac{z^k_{1,1}}{\varepsilon_k}z^k_{1,3} + \frac{z^k_{2,1}}{\varepsilon_k}z^k_{2,3} + \varepsilon_k z^k_{3,1}z^k_{3,3}\right) + \\ & + 2\mathring{T}_{23}\left(\frac{z^k_{1,2}}{\varepsilon_k}z^k_{1,3} + \frac{z^k_{2,2}}{\varepsilon_k}z^k_{2,3} + \varepsilon_k z^k_{3,2}z^k_{3,3}\right)\, dy.\end{aligned} \tag{4.31}$$

By using (4.18), (3.31) and the weak convergence of the scaled strain $(\mathbf{E}^{\varepsilon_k}\mathbf{u}^k)/\varepsilon$ to $\mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w})$ (see *5.* of Lemma 4.4 and Lemma 4.7), the last five lines in the inequality above converge to $(\mathring{T}_{11} + \mathring{T}_{22})\vartheta^2 + 2\mathring{T}_{13}\vartheta v_{2,3} - 2\mathring{T}_{23}\vartheta v_{1,3}$. From the standard Korn inequality, Lemma 3.2 and recalling the definition of $\mathring{\tau}_m$, we have

$$\begin{aligned}&\int_\Omega C_L|(Ez^k)_{33}|^2 + \mathring{T}_{11}(z^k_{3,1})^2 + \mathring{T}_{33}((z^k_{1,3})^2 + (z^k_{2,3})^2) + \mathring{T}_{22}(z^k_{3,2})^2 + \\ &\quad + 2\mathring{T}_{12}(z^k_{3,1})(z^k_{3,2}) + 2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2\, dy \geq (C_L - C_K|\mathring{\tau}_m|)\|\mathbf{E}\mathbf{z}^k\|^2_{L^2(\Omega)},\end{aligned}$$

and then, from inequality (4.9), it follows that the first two lines of inequality (4.31) check the assumptions of Lemma 3.10. Hence, passing to the limit and using (4.18) and (4.28), we obtain

$$
\begin{aligned}
\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^2} \geq\ & \frac{1}{2}\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\cdot\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w}) + \mathring{T}_{11}v_{3,1}^2 + \\
& +\mathring{T}_{33}(v_{1,3}^2+v_{2,3}^2) + \mathring{T}_{22}v_{3,2}^2 + 2\mathring{T}_{12}v_{3,1}v_{3,2} + \\
& +(\mathring{T}_{11}+\mathring{T}_{22})\vartheta^2 + 2\mathring{T}_{13}\vartheta v_{2,3} - 2\mathring{T}_{23}\vartheta v_{1,3}\,dy \\
=\ & \frac{1}{2}\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\cdot\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w}) + \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\,dy \\
\geq\ & \frac{1}{2}\inf_{\varphi\in Q_1,\mathbf{w}\in Q_2}\left\{\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\cdot\mathbf{E}(\mathbf{v},\vartheta,\varphi,\mathbf{w})\,dy\right\} + \\
& +\frac{1}{2}\int_\Omega \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\,dy.
\end{aligned}
$$

The existence of the minimum in the previous inequality follows from a standard application of the direct method of the Calculus of Variations. Hence we have the liminf inequality.

Let us now find a recovery sequence. Let $F(\mathbf{v},\vartheta) < +\infty$, otherwise there is nothing to prove. Then $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega;\mathbb{R})$.

We first assume further that $\mathbf{v}$ and ϑ are smooth and equal to zero near by $y_3 = 0$. By (4.14), there exists $\boldsymbol{\xi}$ smooth and equal to zero near by $y_3 = 0$ such that $v_\alpha(\mathbf{y}) = \xi_\alpha(y_3)$, and $v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha\xi'_\alpha(y_3)$ for $\alpha = 1, 2$. Let $\hat{\mathbf{w}}$ and $\hat{\varphi}$ be the minimizers in the definition (4.30) of $F(\mathbf{v},\vartheta)$. Let $\mathbf{u}^{0,\varepsilon}$ be the sequence defined by

$$
\begin{aligned}
u_1^{0,\varepsilon} &:= \xi_1 - \varepsilon y_2\vartheta + \varepsilon^2\hat{w}_1, \\
u_2^{0,\varepsilon} &:= \xi_2 + \varepsilon y_1\vartheta + \varepsilon^2\hat{w}_2, \\
u_3^{0,\varepsilon} &:= \varepsilon(\xi_3 - y_1\xi'_1 - y_2\xi'_2) + 2\varepsilon^2\hat{\varphi}.
\end{aligned}
\tag{4.32}
$$

We have that $\mathbf{u}^{0,\varepsilon}$ is equal to zero in $y_3 = 0$ and it is easily checked that, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\frac{(E^\varepsilon u^{0,\varepsilon})_{\alpha\beta}}{\varepsilon} \to E_{\alpha\beta}(\hat{\mathbf{w}}), && \frac{(E^\varepsilon u^{0,\varepsilon})_{33}}{\varepsilon} \to D_3v_3, \\
&\frac{(E^\varepsilon u^{0,\varepsilon})_{13}}{\varepsilon} \to D_1\hat{\varphi} - \frac{y_2}{2}D_3\vartheta, && \frac{(E^\varepsilon u^{0,\varepsilon})_{23}}{\varepsilon} \to D_2\hat{\varphi} + \frac{y_1}{2}D_3\vartheta,
\end{aligned}
$$

and $(W^\varepsilon u^{0,\varepsilon})_{12} \to -\vartheta$ in $L^2(\Omega)$.

It is also easy to check that the following estimates are satisfied

$$|\frac{1}{\varepsilon^2}F_\varepsilon(\mathbf{u}^{0,\varepsilon}) - F(\mathbf{v},\vartheta)| \le \varepsilon C(\mathbf{v},\vartheta),$$
$$\|\mathbf{H}^\varepsilon \mathbf{u}^{0,\varepsilon} - \mathbf{H}\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le \varepsilon C(\mathbf{v},\vartheta),$$
$$\|(W^\varepsilon u^{0,\varepsilon})_{12} + \vartheta\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta),$$
$$\|(u_1^{0,\varepsilon}, u_2^{0,\varepsilon}, \frac{u_3^{0,\varepsilon}}{\varepsilon}) - \mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} \le \varepsilon C(\mathbf{v},\vartheta),$$

where $C(\mathbf{v},\vartheta)$ depends only on $\mathbf{v}$ and ϑ. Hence, in this case, $\mathbf{u}^{0,\varepsilon}$ is a recovery sequence.

In the general case, i.e. $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega)$, for any $\delta > 0$, we can find, by density, functions $\mathbf{v}^\delta \in C^\infty(\Omega;\mathbb{R}^3)$ and $\vartheta^\delta \in C^\infty(\Omega)$ which are equal to zero near by $y_3 = 0$ and such that

$$\|\mathbf{v}^\delta - \mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} < \delta,$$
$$\|\vartheta^\delta - \vartheta\|_{L^2(\Omega)} < \delta.$$

Therefore, we obtain

$$\begin{aligned}\limsup_{\delta\to 0}\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v}^\delta,\vartheta^\delta,\hat{\varphi}^\delta,\hat{\mathbf{w}}^\delta)\cdot\mathbf{E}(\mathbf{v}^\delta,\vartheta^\delta,\hat{\varphi}^\delta,\hat{\mathbf{w}}^\delta)\,dy \le&\\ \le \limsup_{\delta\to 0}\int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v}^\delta,\vartheta^\delta,\hat{\varphi},\hat{\mathbf{w}})\cdot\mathbf{E}(\mathbf{v}^\delta,\vartheta^\delta,\hat{\varphi},\hat{\mathbf{w}})\,dy&\\ = \int_\Omega \mathbb{L}\mathbf{E}(\mathbf{v},\vartheta,\hat{\varphi},\hat{\mathbf{w}})\cdot\mathbf{E}(\mathbf{v},\vartheta,\hat{\varphi},\hat{\mathbf{w}})\,dy,&\end{aligned}$$

where $(\hat{\varphi}^\delta,\hat{\mathbf{w}}^\delta)$ and $(\hat{\varphi},\hat{\mathbf{w}})$ are, respectively, the minimizers for $(\mathbf{v}^\delta,\vartheta^\delta)$ and $(\mathbf{v},\vartheta)$ in the definition (4.30). Hence, we have

$$\limsup_{\delta\to 0} F(\mathbf{v}^\delta,\vartheta^\delta) \le F(\mathbf{v},\vartheta).$$

Denoting by $\mathbf{u}^{\delta,\varepsilon}$ the sequence defined as $\mathbf{u}^{0,\varepsilon}$ in (4.32) with $\mathbf{v}$ and ϑ replaced with $\mathbf{v}^\delta$ and ϑ^δ, we obtain

$$\lim_{\delta\to 0}\lim_{k\to+\infty}(u_1^{\delta,\varepsilon_k}, u_2^{\delta,\varepsilon_k}, \frac{u_3^{\delta,\varepsilon_k}}{\varepsilon_k}) = \mathbf{v} \quad \text{in } H^1(\Omega;\mathbb{R}^3),$$
$$\lim_{\delta\to 0}\lim_{k\to+\infty}\mathbf{H}^{\varepsilon_k}\mathbf{u}^{\delta,\varepsilon_k} = \mathbf{H}\mathbf{v} \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}),$$
$$\lim_{\delta\to 0}\lim_{k\to+\infty}(W^{\varepsilon_k}u^{\delta,\varepsilon_k})_{12} = \vartheta \quad \text{in } L^2(\Omega),$$
$$\limsup_{\delta\to 0}\lim_{k\to+\infty}\frac{1}{\varepsilon_k^2}F_{\varepsilon_k}(\mathbf{u}^{\delta,\varepsilon_k}) \le F(\mathbf{v},\vartheta),$$

and, hence, by a standard diagonal argument, we can find a sequence δ_k converging to zero such that

$$\begin{aligned}
&\|(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) - \mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} < \delta_k,\\
&\|\mathbf{H}^{\varepsilon_k}\mathbf{u}^k - \mathbf{H}\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} < \delta_k,\\
&\|(W^{\varepsilon_k}u^k)_{12} + \vartheta\|_{L^2(\Omega)} < \delta_k,\\
&\limsup_{k\to+\infty} \frac{1}{\varepsilon_k^2} F_{\varepsilon_k}(\mathbf{u}^k) \le F(\mathbf{v}, \vartheta),
\end{aligned}$$

where $\mathbf{u}^k := \mathbf{u}^{\delta_k,\varepsilon_k}$. Hence, the sequence $\mathbf{u}^k$ satisfies the recovery sequence condition. □

For every $\varepsilon \in (0, 1]$, let us denote by $\tilde{\mathbf{u}}^\varepsilon$ the solution of the following minimization problem

$$\min\{F_\varepsilon(\mathbf{u}) : \mathbf{u} \in H^1(\Omega;\mathbb{R}^3),\ \mathbf{u} = \mathbf{0} \text{ on } S(0)\}.$$

The next lemma follows from the Γ-convergence Theorem 4.9, the uniform coercivity of the sequence of the functionals $(1/\varepsilon^2)F_\varepsilon$ and Lemma 1.40.

Lemma 4.10. *The minimization problem for the* Γ*-limit functional* F *defined in* (4.30)

$$\min\{F(\mathbf{v}, \vartheta) : \mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3),\ \vartheta \in H^1(0, \ell),\ \mathbf{v} = \mathbf{0} \textit{ on } S(0),\ \vartheta(0) = 0\}$$

admits a unique solution $(\tilde{\mathbf{v}}, \tilde{\vartheta})$*. Moreover, as* $\varepsilon \to 0$*,*

1. $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon/, \tilde{u}_3^\varepsilon/\varepsilon) \rightharpoonup \tilde{\mathbf{v}}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*

2. $(W^\varepsilon \tilde{u}^\varepsilon)_{12} \to -\tilde{\vartheta}$ *in* $L^2(\Omega)$*;*

3. $(1/\varepsilon^2)F_\varepsilon(\tilde{\mathbf{u}}^\varepsilon)$ *converges to* $F(\tilde{\mathbf{v}}, \tilde{\vartheta})$*.*

4.6 Characterization of Γ-limit for some material symmetries

For $(\mathbf{v},\vartheta) \in H^1_\flat(\Omega;\mathbb{R}^3) \times H^1_\flat(\Omega;\mathbb{R})$, the minimizers $\hat{\mathbf{w}}$ and $\hat{\varphi}$ of the functional F defined in (4.30) satisfy the systems

$$\begin{cases} \mathbb{L}_{1\alpha\beta\gamma}\hat{w}_{\beta,\alpha\gamma} + \mathbb{L}_{331\alpha}v_{3,3\alpha} + 2\mathbb{L}_{3\alpha 1\beta}\hat{\varphi}_{,\alpha\beta} + \mathbb{L}_{1223}\vartheta' - \mathbb{L}_{2231}\vartheta' = 0 \text{ in } \omega \\ \mathbb{L}_{2\alpha\beta\gamma}\hat{w}_{\beta,\alpha\gamma} + \mathbb{L}_{332\alpha}v_{3,3\alpha} + 2\mathbb{L}_{3\alpha 2\beta}\hat{\varphi}_{,\alpha\beta} + \mathbb{L}_{1123}\vartheta' - \mathbb{L}_{1321}\vartheta' = 0 \text{ in } \omega \\ \mathbb{L}_{1\alpha\beta\gamma}\hat{w}_{\beta,\gamma}n_\alpha + \mathbb{L}_{331\alpha}v_{3,3}n_\alpha + 2\mathbb{L}_{3\alpha 1\beta}\hat{\varphi}_{,\alpha}n_\beta = \\ \qquad = \vartheta'(\mathbb{L}_{1113}y_2n_1 + \mathbb{L}_{1321}y_2n_2 - \mathbb{L}_{1123}y_1n_1 - \mathbb{L}_{2321}y_1n_2) \quad \text{on } \partial\omega \\ \mathbb{L}_{2\alpha\beta\gamma}\hat{w}_{\beta,\alpha}n_\alpha + \mathbb{L}_{332\alpha}v_{3,3}n_\alpha + 2\mathbb{L}_{3\alpha 2\beta}\hat{\varphi}_{,\alpha}n_\beta = \\ \qquad = \vartheta'(\mathbb{L}_{2213}y_2n_2 + \mathbb{L}_{1321}y_2n_1 - \mathbb{L}_{2223}y_1n_2 - \mathbb{L}_{2321}y_1n_1) \quad \text{on } \partial\omega, \end{cases} \tag{4.33}$$

and

$$\begin{cases} \mathbb{L}_{\alpha\beta\gamma 3}\hat{w}_{\alpha,\beta\gamma} + \mathbb{L}_{333\alpha}v_{3,3\alpha} + 2\mathbb{L}_{3\alpha 3\beta}\hat{\varphi}_{,\alpha\beta} = 0 \quad \text{in } \omega \\ \mathbb{L}_{\alpha\beta\gamma 3}\hat{w}_{\alpha,\beta}n_\gamma + \mathbb{L}_{333\alpha}v_{3,3}n_\alpha + 2\mathbb{L}_{3\alpha 3\beta}\hat{\varphi}_{,\alpha}n_\beta = \\ \qquad = \vartheta'(\mathbb{L}_{1313}y_2n_1 + \mathbb{L}_{2331}y_2n_2 - \mathbb{L}_{2323}y_1n_2 - \mathbb{L}_{2331}y_1n_1) \quad \text{on } \partial\omega, \end{cases} \tag{4.34}$$

where $\mathbf{n} = (n_1, n_2)$ is the normal unit vector to $\partial\omega$.

If we suppose that each (y_1, y_2) plane is a plane of symmetry, i.e. the material is monoclinic with uniform axis of symmetry identified with the y_3-axis, it follows that (see Section 2.3)

$$\mathbb{L}_{\alpha\beta\gamma 3} = 0, \qquad \mathbb{L}_{\alpha 333} = 0,$$

for all $\alpha, \beta, \gamma = 1, 2$. Hence the two previous systems become decoupled. Recalling that $\mathbf{v}$ belongs to the space of Bernoulli-Navier displacements on Ω, a solution $(\hat{\mathbf{w}}, \hat{\varphi})$ of the two systems is given by

$$\hat{\varphi} = \frac{1}{2}\varphi_1\vartheta',$$

where φ_1 solves

$$\begin{cases} \mathbb{L}_{3\alpha 3\beta}\varphi_{1,\alpha\beta} = 0 & \text{in } \omega \\ \mathbb{L}_{3\alpha 3\beta}\varphi_{1,\alpha}n_\beta = \mathbb{L}_{313\alpha}y_2n_\alpha - \mathbb{L}_{323\alpha}y_1n_\alpha & \text{on } \partial\omega, \end{cases}$$

and

$$\hat{w}_{1,1} = C_1v_{3,3}, \qquad \hat{w}_{2,2} = C_3v_{3,3}, \qquad \hat{w}_{1,2} + \hat{w}_{2,1} = 2C_2v_{3,3},$$

where C_1, C_2, C_3 are given by

$$\begin{pmatrix} \mathbb{L}_{1111} & \mathbb{L}_{1112} & \mathbb{L}_{1122} \\ \mathbb{L}_{1112} & \mathbb{L}_{1212} & \mathbb{L}_{2212} \\ \mathbb{L}_{1122} & \mathbb{L}_{2212} & \mathbb{L}_{2222} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} -\mathbb{L}_{1133} \\ -\mathbb{L}_{1233} \\ -\mathbb{L}_{2233} \end{pmatrix}.$$

Since (4.2), the matrix in the previous equation is definite positive.

In the case of a slender rods made of rhombic material, it follows that (see Section 2.3)

$$\mathbb{L}_{\alpha\beta\gamma 3} = \mathbb{L}_{\alpha 333} = \mathbb{L}_{ii12} = \mathbb{L}_{2313} = 0,$$

for all $\alpha, \beta, \gamma = 1, 2$ and $i = 1, 2, 3$. For $(\mathbf{v}, \vartheta) \in H^1_\flat(\Omega; \mathbb{R}^3) \times H^1_\flat(\Omega; \mathbb{R})$, the minimizers $\hat{\mathbf{w}}$ and $\hat{\varphi}$ of the functional F defined in (4.30) satisfy the systems

$$\begin{cases} \mathbb{L}_{11\alpha\alpha}\hat{w}_{\alpha,\alpha 1} + \mathbb{L}_{1133}v_{3,31} + \mathbb{L}_{1212}(\hat{w}_{1,22} + \hat{w}_{2,12}) = 0 & \text{in } \omega \\ \mathbb{L}_{22\alpha\alpha}\hat{w}_{\alpha,\alpha 2} + \mathbb{L}_{2233}v_{3,32} + \mathbb{L}_{1212}(\hat{w}_{2,11} + \hat{w}_{1,21}) = 0 & \text{in } \omega, \end{cases}$$

and

$$\begin{cases} \mathbb{L}_{\alpha 3\alpha 3}\hat{\varphi}_{,\alpha\alpha} = 0 & \text{in } \omega \\ 2\mathbb{L}_{\alpha 3\alpha 3}\hat{\varphi}_{,\alpha}n_\alpha = \vartheta'(\mathbb{L}_{1313}y_2n_1 - \mathbb{L}_{2323}y_1n_2) & \text{on } \partial\omega. \end{cases}$$

A solution $(\hat{\mathbf{w}}, \hat{\varphi})$ of the two systems is given by

$$\hat{\varphi} = \frac{1}{2}\varphi_2\vartheta',$$

where φ_2 solves

$$\begin{cases} \mathbb{L}_{\alpha 3\alpha 3}\varphi_{2,\alpha\alpha} = 0 & \text{in } \omega \\ 2\mathbb{L}_{\alpha 3\alpha 3}\varphi_{2,\alpha}n_\alpha = \mathbb{L}_{1313}y_2n_1 - \mathbb{L}_{2323}y_1n_2 & \text{on } \partial\omega. \end{cases}$$

and

$$\hat{w}_{1,1} = C_1v_{3,3}, \qquad \hat{w}_{2,2} = C_2v_{3,3}, \qquad \hat{w}_{1,2} + \hat{w}_{2,1} = 0,$$

where C_1, C_2 are given by

$$C_1 = \frac{\mathbb{L}_{1122}\mathbb{L}_{2233} - \mathbb{L}_{3311}\mathbb{L}_{2222}}{\mathbb{L}_{2222}\mathbb{L}_{1111} - \mathbb{L}^2_{1122}}, \qquad C_2 = \frac{\mathbb{L}_{1122}\mathbb{L}_{1133} - \mathbb{L}_{3322}\mathbb{L}_{1111}}{\mathbb{L}_{2222}\mathbb{L}_{1111} - \mathbb{L}^2_{1122}}.$$

In the case of a slender rod made of isotropic, homogeneous material with a stress-free reference configuration, the Γ-limit defined in (4.30) is the same obtained by Anzellotti, Baldo and Percivale [4] (in the case of circular cross-section), Percivale [40] and Freddi, Londero and Paroni [22]. By Section 2.4, we have $\mathring{\mathbf{T}} = \mathbf{0}$. Setting (see Section 2.3)

$$\begin{aligned} \mathbb{L}\mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w}) \cdot \mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w}) =& (2\mu\mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w}) + \lambda \mathrm{tr}(\mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w}))\mathbf{I}) \cdot \\ & \cdot \mathbf{E}(\mathbf{v}, \vartheta, \varphi, \mathbf{w}), \end{aligned}$$

where $\mu > 0$ and $\lambda \geq 0$ are the Lamé's moduli of the material and $\mathbf{I}$ denotes the identity tensor, we compute $F(\mathbf{v}, \vartheta)$. In this case a solution of the systems (4.33) and (4.34) is given by

$$\hat{\varphi} = \frac{1}{2}\varphi_3\vartheta',$$

where φ_3 is the *torsion function*, i.e. φ_3 solves

$$\begin{cases} \triangle\varphi_3 = 0 \quad \text{in } \omega \\ \varphi_{3,\alpha} \cdot n_\alpha = -y_1 n_2 + y_2 n_1 \quad \text{on } \partial\omega, \end{cases}$$

and $\hat{w}_{1,1} = -\nu v_{3,3}$, $\hat{w}_{1,2} + \hat{w}_{2,1} = 0$, $\hat{w}_{2,2} = -\nu v_{3,3}$, where $\nu := \lambda/(2\lambda + 2\mu)$ denotes the Poisson's coefficient of the material.

Hence, we find

$$F(\mathbf{v}, \vartheta) = \int_\Omega (\frac{E}{2}|D_3 v_3|^2 + \frac{\mu}{2}|\mathbf{D}\psi|^2|D_3\vartheta|^2)\, dy - \int_\Omega \mathbf{b} \cdot \mathbf{v}\, dy - \int_0^\ell m\vartheta\, dy_3,$$

that is the Γ-limit obtained in [22], where ψ is the so-called *Prandtl stress function* and $E := (2\mu^2 + 3\lambda\mu)/(\mu + \lambda)$ denotes the Young modulus of the material.

4.7 The equations of equilibrium

In this section, we find the equation of equilibrium of a slender rod with residual stress made of a monoclinic material with uniform axis of symmetry identified with the y_3-axis and non homogeneous only along y_3-axis.
From (4.1) and (4.8), we obtain

$$\langle \mathring{T}_{1i,1} + \mathring{T}_{2i,2} + \mathring{T}_{3i,3} \rangle = 0,$$

for $i = 1, 2, 3$, where $\langle \cdot \rangle = \int_\omega \cdot\, dy_1 dy_2$ denotes integration over the cross section ω. Using the Divergence theorem on the first two terms of the equation above and recalling that $\mathring{\mathbf{T}}\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, see (4.1), we find $\langle \mathring{T}_{3i} \rangle = 0$, for $i = 1, 2, 3$. Besides, a monoclinic material checks

$$\mathring{T}_{\alpha 3} = 0,$$

for all $\alpha = 1, 2$, and the geometry of the domain and the equations of equilibrium (4.1) imply that (see Section 2.4)

$$\mathring{T}_{33} = 0.$$

Hence we find

$$\langle \mathring{T}_{11} \rangle = \langle y_1(\mathring{T}_{13,3}) \rangle = 0 \quad \text{and} \quad \langle \mathring{T}_{22} \rangle = \langle y_2(\mathring{T}_{23,3}) \rangle = 0.$$

By recalling that $\mathbf{v}$ belongs to the space of Bernoulli-Navier displacements, the term concerning the residual stress in the definition of the Γ-limit (4.30) becomes

$$\int_\Omega \mathbf{Hv}\mathring{\mathbf{T}} \cdot \mathbf{Hv}\, dy = \int_\Omega 2\mathring{T}_{12} v_{3,1} v_{3,2}\, dy.$$

By using (4.14) and the fact that ϑ depends only on y_3, the limit energy functional $F(\mathbf{v}, \vartheta)$ defined in (4.30) can be written in a more explicit way. If we suppose that

$$\int_\Omega y_\alpha\, dy = 0,$$

for $\alpha = 1, 2$, we obtain

$$\int_\Omega y_\alpha \xi_3' \xi_\alpha''\, dy = 0, \qquad \int_\Omega y_\alpha \xi_3'\, dy = 0,$$

for $\alpha = 1, 2$ and

$$\int_\Omega y_1 y_2 \xi_1'' \xi_2''\, dy = 0, \qquad \int_\Omega y_1 y_2 \xi_2''\, dy = 0.$$

Hence, the strain limit energy can be rewritten as

$$\begin{aligned} I(\mathbf{v}, \vartheta) &= \frac{1}{2} \int_\Omega \widetilde{E} v_{3,3}^2 + \widetilde{\mu} \vartheta'^2 + 2\mathring{T}_{12} v_{3,1} v_{3,2}\, dy \\ &= \frac{1}{2} \int_\Omega \widetilde{E} \left(\xi_3'^2 + y_\alpha \xi_\alpha''^2 \right) + \widetilde{\mu} \vartheta'^2 + 2\mathring{T}_{12} v_{3,1} v_{3,2}\, dy \\ &= \int_0^\ell \frac{1}{2} A \widetilde{E} \xi_3'^2 + \frac{1}{2} J_2 \widetilde{E} \xi_1''^2 + \frac{1}{2} J_1 \widetilde{E} \xi_2''^2 + \langle \mathring{T}_{12} \rangle \xi_1' \xi_2' + \frac{1}{2} \langle \widetilde{\mu} \rangle \vartheta'^2\, dy_3, \end{aligned}$$

where

$$A := \int_\omega dy_1 dy_2, \qquad J_1 := \int_\omega y_2^2\, dy_1 dy_2,$$
$$J_2 := \int_\omega y_1^2\, dy_1 dy_2,$$

the coefficient $\widetilde{E} := \widetilde{E}(y_3)$ depends on the components of the incremental elasticity tensor and $\widetilde{\mu} := \widetilde{\mu}(y_1, y_2, y_3)$ depends on $\mathbb{L}$ and on the derivatives of $\hat{\varphi}$. The work done by the external forces rewrites as

$$\int_\Omega \mathbf{b} \cdot \mathbf{v}\, dy = \int_0^\ell \langle b_i \rangle \xi_i - \langle y_\alpha b_3 \rangle \xi_\alpha'\, dy_3.$$

The energy of the beam $F(\mathbf{v},\ \vartheta)$ can be rewritten, with a small abuse of notation, as

$$\begin{aligned} F(\boldsymbol{\xi},\vartheta) = & \int_0^\ell \frac{1}{2} A\widetilde{E}\xi_3'^2 + \frac{1}{2} J_2\widetilde{E}\xi_1''^2 + \frac{1}{2} J_1\widetilde{E}\xi_2''^2 - \langle b_i\rangle\xi_i + \langle y_\alpha b_3\rangle\xi_\alpha'\,dy_3 + \\ & + \int_0^\ell \langle \mathring{T}_{12}\rangle\xi_1'\xi_2' + \frac{1}{2}\langle\widetilde{\mu}\rangle\vartheta'^2 - m\vartheta\,dy_3, \end{aligned}$$

which has to be minimized over all functions $(\boldsymbol{\xi},\vartheta)$ with $\xi_\alpha \in H^2_\flat(0,\ell)$, $\xi_3 \in H^1_\flat(0,\ell)$ and $\vartheta \in H^1_\flat(0,\ell)$. The Euler-Lagrange equations can be written as

$$\begin{cases} J_2\left(\widetilde{E}\xi_1''\right)'' - \left(\langle\mathring{T}_{12}\rangle\xi_1'\right)' - \langle b_1\rangle - \langle y_1 b_3\rangle' = 0, \\ J_1\left(\widetilde{E}\xi_2''\right)'' - \left(\langle\mathring{T}_{12}\rangle\xi_2'\right)' - \langle b_2\rangle - \langle y_2 b_3\rangle' = 0, \\ A\left(\widetilde{E}\xi_3'\right)' + \langle b_3\rangle = 0, \\ J\left(\langle\widetilde{\mu}\rangle\vartheta'\right)' + m = 0. \end{cases} \tag{4.35}$$

We can notice that the previous equations depend on the residual stress component $\mathring{T}_{12}$, while the Euler-Lagrange equations for a thin walled beam with monoclinic symmetry are completely independent of $\mathring{\mathbf{T}}$ (see (3.39)). This fact is due to the D_3v_2-term that appears in equation (4.19). Hence, for monoclinic slender rods, the shear stress due to $\mathring{T}_{12}$ can not be disregarded.

Chapter 5

Junction of plates with residual stress

The first attempt to deduce the equations of a linearly elastic plate (without residual stress) was done by Euler (1766). Sophie Germain in 1815 deduced the equations in the form now generally admitted, even if later investigations have shown that her derivation was not completely correct. Finally Kirchhoff settled the problem in 1850. More recently, a rigorous justification to the theory of isotropic plates, by using Γ-convergence, was given by Anzelotti, Baldo and Percivale in [4] and by Bourquin, Ciarlet, Geymonat and Raoult in [5]. Paroni, in [39], derived the theory of a plate for a linearly elastic monoclinic material with residual stress.

In this chapter, we continue the line of research intiated in [39], and we consider the case of junction of two plates with residual stress. This chapter is a preliminary step for the resolution of the Γ-convergence problem: in fact we study the compactness properties of displacements with equi-bounded energy or, in mechanical terms, the kinematics of the model.

The chapter is organized as follows. In Section 5.1 we briefly describe the three-dimensional problem. Section 5.2 is devoted to discuss the problem of existence of solutions, where we follow the same approach of Paroni [39]. In Section 5.3 we re-define the three-dimensional problem as a variational problem for a rescaled energy in a fixed domain. In Section 5.4 we recall some compactness results for a scaled displacement field that can be found in [5]. Section 5.5 is devoted to the establishment of the junction conditions to characterize the kinematics fields of the cross-section, following the lines traced by Freddi, Morassi and Paroni in [24] and Le Dret in [34].

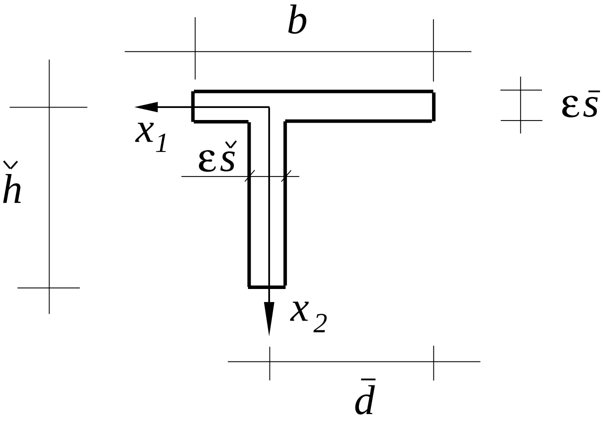
5.1 Setting of the problem

We consider a three-dimensional body $\Omega_\varepsilon \subset \mathbb{R}^3$, with $\Omega_\varepsilon := \bar{\Omega}_\varepsilon \cup \check{\Omega}_\varepsilon$, where

$$\bar{\Omega}_\varepsilon := (-\bar{d}, \bar{b} - \bar{d}) \times (-\varepsilon\frac{\bar{s}}{2}, \varepsilon\frac{\bar{s}}{2}) \times (0, \ell),$$

$$\check{\Omega}_\varepsilon := (-\varepsilon\frac{\check{s}}{2}, \varepsilon\frac{\check{s}}{2}) \times (0, \check{h}) \times (0, \ell),$$

are two no-empty rectangular parallelepipeds and $\varepsilon \in (0, 1]$.



For later convenience we also set

$$\bar{\Omega}_\varepsilon = \bar{\omega}_\varepsilon \times (0, \ell), \quad \check{\Omega}_\varepsilon = \check{\omega}_\varepsilon \times (0, \ell),$$

where

$$\bar{\omega}_\varepsilon := (-\bar{d}, \bar{b} - \bar{d}) \times (-\varepsilon\frac{\bar{s}}{2}, \varepsilon\frac{\bar{s}}{2}), \quad \check{\omega}_\varepsilon := (-\varepsilon\frac{\check{s}}{2}, \varepsilon\frac{\check{s}}{2}) \times (0, \check{h}),$$

and $\omega_\varepsilon := \bar{\omega}_\varepsilon \cup \check{\omega}_\varepsilon$.

We define $S_\varepsilon(x_3) = \omega_\varepsilon \times \{x_3\}$ for any $x_3 \in (0, \ell)$. We shall consider the space

$$H^1_\flat\left(\Omega_\varepsilon; \mathbb{R}^3\right) = \left\{\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\right\},$$

and the spaces $H^1_\flat(\bar{\Omega}_\varepsilon; \mathbb{R}^3)$ and $H^1_\flat(\check{\Omega}_\varepsilon; \mathbb{R}^3)$ defined in a similar way. We can think of Ω_ε as the reference configuration of the body subject to a residual stress $\mathring{\mathbf{T}}^\varepsilon$. In this configuration the body is in equilibrium in the absence of external loads and the Cauchy stress field $\mathring{\mathbf{T}}^\varepsilon$ satisfies the following boundary problem (see Section 2.4)

$$\begin{cases} \operatorname{div} \mathring{\mathbf{T}}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon = (\mathring{\mathbf{T}}^\varepsilon)^T & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.1}$$

where $\mathbf{n}$ is the outward unit normal to the boundary of Ω_ε.

In what follows we consider an inhomogeneous, triclinic material, so that the first Piola-Kirchhoff stress field $\mathbf{S}^\varepsilon$ can be expressed as (see Section 2.2)

$$\mathbf{S}^\varepsilon(\mathbf{x}) = \mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbf{Du}(\mathbf{x})\mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbb{L}^\varepsilon(\mathbf{x})\mathbf{Eu}(\mathbf{x}),$$

where $\mathbf{Eu}$ denotes the strain of the displacement $\mathbf{u}$, $\mathbf{Du}$ the gradient of $\mathbf{u}$ and $\mathbb{L}^\varepsilon(\mathbf{x})$ is the incremental elasticity tensor.

We assume $\mathbb{L}^\varepsilon$ to be essentialy bounded,

$$\mathbb{L}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3\times3\times3}),$$

to have the major and minor symmetries,

$$\mathbb{L}^\varepsilon_{ijkl} = \mathbb{L}^\varepsilon_{jikl} = \mathbb{L}^\varepsilon_{klji},$$

to be positive definite,

$$\exists\, C > 0, \text{ s.t. } \mathbb{L}^\varepsilon(\mathbf{x})\mathbf{A} \cdot \mathbf{A} \geq C|\mathbf{A}|^2, \tag{5.2}$$

for all $\mathbf{A} \in \mathbb{R}^{3\times3}_{\text{sym}} = \{\mathbf{A} \in \mathbb{R}^{3\times3} : \mathbf{A} = \mathbf{A}^T\}$ and for almost every $\mathbf{x} \in \Omega_\varepsilon$ and we denote by $\bar{C}_L$ and $\check{C}_L$ the largest of all such constants C for almost every $\mathbf{x} \in \bar{\Omega}_\varepsilon$ and $\mathbf{x} \in \check{\Omega}_\varepsilon$, respectively. Furthermore we assume $\mathring{\mathbf{T}}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3})$.

We consider the body clamped on $S_\varepsilon(0)$ subject to dead body forces $\mathbf{b}^\varepsilon$ and we assume $\mathbf{b}^\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3)$. In the domain Ω_ε we study the elasticity problem: find $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ such that

$$\int_{\Omega_\varepsilon} \Big(\mathbf{Du}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon \mathbf{Eu} \cdot \mathbf{Ev} \Big)\, dx = \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{v}\, dx, \tag{5.3}$$

for all $\mathbf{v} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$.

5.2 Existence of the solution

In this section we discuss the existence of a solution $\mathbf{u}$ of problem (5.3) following the lines traced by Paroni in his work [39]. We recall a Korn's inequality for plates found by Anzellotti, Baldo and Percivale in [4].

Theorem 5.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_{\bar{\Omega}_\varepsilon} \left(|\mathbf{u}|^2 + |\mathbf{D}\mathbf{u}|^2\right) dx \leq \frac{C}{\varepsilon^2} \int_{\bar{\Omega}_\varepsilon} |\mathbf{E}\mathbf{u}|^2 dx, \tag{5.4}$$

for every $\mathbf{u} \in H^1_\flat(\bar{\Omega}_\varepsilon; \mathbb{R}^3)$ and for every $\varepsilon \in (0, 1]$.

An analogous result can be obtained for $\check{\Omega}_\varepsilon$. We denote by $\bar{C}_K$ and $\check{C}_K$ the smallest among all constants C for which inequality (5.4) is valid on $\bar{\Omega}_\varepsilon$ and $\check{\Omega}_\varepsilon$ respectively.

We shall prove existence and uniquess of the solution of problem (5.3) under the assumption that the absolute values of the smallest eigenvalues of $\mathring{\mathbf{T}}^\varepsilon$ on $\bar{\Omega}$ and $\check{\Omega}$,

$$\bar{\tau}^\varepsilon_m := \operatorname*{essinf}_{\mathbf{x} \in \bar{\Omega}_\varepsilon} \min_{\mathbf{a} \in \mathbb{R}^3} \left\{ \mathring{\mathbf{T}}^\varepsilon(\mathbf{x})\mathbf{a} \cdot \mathbf{a} \ : \ |\mathbf{a}| = 1 \right\},$$

and

$$\check{\tau}^\varepsilon_m := \operatorname*{essinf}_{\mathbf{x} \in \check{\Omega}_\varepsilon} \min_{\mathbf{a} \in \mathbb{R}^3} \left\{ \mathring{\mathbf{T}}^\varepsilon(\mathbf{x})\mathbf{a} \cdot \mathbf{a} \ : \ |\mathbf{a}| = 1 \right\},$$

are not too large, i.e. the compressions due to the residual stress are not too large.

Using the same arguments of Lemma 3.3, it can be proved that the smallest eigenvalues of $\mathring{\mathbf{T}}^\varepsilon$ on $\bar{\Omega}$ and $\check{\Omega}$, are either identically equal to 0 or that they also take negative values.

By using Lemma 3.2 and Lax-Milgram lemma, we deduce the following existence theorem.

Theorem 5.2. *Assume that*

$$\bar{C}_L \geq 2\bar{C}_K \frac{|\bar{\tau}^\varepsilon_m|}{\varepsilon^2} \quad \textit{and} \quad \check{C}_L \geq 2\check{C}_K \frac{|\check{\tau}^\varepsilon_m|}{\varepsilon^2}. \tag{5.5}$$

Then there exists a unique solution $\mathbf{u}^\varepsilon \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ of problem (5.3).

Proof. If $\chi_\varepsilon : \Omega_\varepsilon \to \{1/2, 1\}$ is defined by

$$\chi_\varepsilon(\mathbf{x}) := \begin{cases} \dfrac{1}{2} & \text{if } \mathbf{x} \in \bar{\Omega}_\varepsilon \cap \check{\Omega}_\varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

we have

$$\int_{\Omega_\varepsilon} \left(\mathbf{Dv}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon \mathbf{Ev} \cdot \mathbf{Ev}\right) dx =$$

$$= \int_{\bar{\Omega}_\varepsilon} \chi_\varepsilon(\mathbf{x}) \left(\mathbf{Dv}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon \mathbf{Ev} \cdot \mathbf{Ev}\right) dx +$$
$$+ \int_{\check{\Omega}_\varepsilon} \chi_\varepsilon(\mathbf{x}) \left(\mathbf{Dv}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon \mathbf{Ev} \cdot \mathbf{Ev}\right) dx.$$

Since $\bar{\tau}_m^\varepsilon$ and $\check{\tau}_m^\varepsilon$ are either identically equal to 0 or that they also take negative values and from assumption (5.2) and Theorem 5.1, we obtain

$$\int_{\Omega_\varepsilon} \left(\mathbf{Dv}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon \mathbf{Ev} \cdot \mathbf{Ev}\right) dx \geq$$

$$\geq \bar{\tau}_m^\varepsilon \|\mathbf{Dv}\|^2_{L^2(\bar{\Omega}_\varepsilon)} + \frac{1}{2}\bar{C}_L \|\mathbf{Ev}\|^2_{L^2(\bar{\Omega}_\varepsilon)} +$$
$$+ \check{\tau}_m^\varepsilon \|\mathbf{Dv}\|^2_{L^2(\check{\Omega}_\varepsilon)} + \frac{1}{2}\check{C}_L \|\mathbf{Ev}\|^2_{L^2(\check{\Omega}_\varepsilon)}$$
$$\geq \left(\frac{1}{2}\bar{C}_L - \bar{C}_K \frac{|\bar{\tau}_m^\varepsilon|}{\varepsilon^2}\right) \|\mathbf{Ev}\|^2_{L^2(\bar{\Omega}_\varepsilon)} +$$
$$+ \left(\frac{1}{2}\check{C}_L - \check{C}_K \frac{|\check{\tau}_m^\varepsilon|}{\varepsilon^2}\right) \|\mathbf{Ev}\|^2_{L^2(\check{\Omega}_\varepsilon)}.$$

Applying Theorem 5.1 on $\bar{\Omega}_\varepsilon$ and $\check{\Omega}_\varepsilon$, the existence and uniqueness follow from an application of the Lax-Milgram lemma. □

Hereafter, we will always assume inequalities (5.5) to hold. Moreover, by Theorem 5.2 we state that the energy functionals

$$\mathcal{F}_\varepsilon(\mathbf{u}) := J_\varepsilon(\mathbf{u}) - \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx \tag{5.6}$$

admit for $\varepsilon > 0$ an unique minimizer among all functions $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$, where

$$J_\varepsilon(\mathbf{u}) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(\mathbf{Du}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Du} + \mathbb{L}^\varepsilon \mathbf{Eu} \cdot \mathbf{Eu}\right) dx. \tag{5.7}$$

5.3 The rescaled problem

To state our results it is convenient to stretch the domain $\bar{\Omega}_\varepsilon$ along the direction x_2 and $\check{\Omega}_\varepsilon$ along the direction x_1 in a way that the transformed domains do not depend on ε. Let us therefore set $\bar{\Omega} := \bar{\Omega}_1$ and $\bar{\omega} := \bar{\omega}_1$ and in a similar way $\check{\Omega}$, $\check{\omega}$, Ω and ω. Let

$$\bar{p}_\varepsilon : \bar{\Omega} \to \bar{\Omega}_\varepsilon, \quad \check{p}_\varepsilon : \check{\Omega} \to \check{\Omega}_\varepsilon,$$

be defined by

$$\bar{p}_\varepsilon(\mathbf{y}) = \bar{p}_\varepsilon(y_1, y_2, y_3) = (y_1, \varepsilon y_2, y_3),$$

$$\check{p}_\varepsilon(\mathbf{y}) = \check{p}_\varepsilon(y_1, y_2, y_3) = (\varepsilon y_1, y_2, y_3).$$

Given $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ we define two functions $\bar{\mathbf{u}} \in H^1_\flat(\bar{\Omega}; \mathbb{R}^3)$ and $\check{\mathbf{u}} \in H^1_\flat(\check{\Omega}; \mathbb{R}^3)$ such that

$$\bar{\mathbf{u}} := \mathbf{u} \circ \bar{p}_\varepsilon, \quad \check{\mathbf{u}} := \mathbf{u} \circ \check{p}_\varepsilon.$$

Of course, in the region where the domains overlap, the following "junction condition" must be satisfied

$$\bar{\mathbf{u}} \circ \bar{p}_\varepsilon^{-1} = \check{\mathbf{u}} \circ \check{p}_\varepsilon^{-1} \quad \text{in } \Gamma_\varepsilon, \tag{5.8}$$

where

$$\Gamma_\varepsilon := (-\varepsilon\frac{\check{s}}{2}, \varepsilon\frac{\check{s}}{2}) \times (0, -\varepsilon\frac{\bar{s}}{2}) \times (0, \ell).$$

Let us consider the following 3×3 matrix valued differential operators

$$\bar{\mathbf{H}}^\varepsilon \bar{\mathbf{u}} := (D_1\bar{\mathbf{u}}, \frac{D_2\bar{\mathbf{u}}}{\varepsilon}, D_3\bar{\mathbf{u}}),$$

$$\check{\mathbf{H}}^\varepsilon \check{\mathbf{u}} := (\frac{D_1\check{\mathbf{u}}}{\varepsilon}, D_2\check{\mathbf{u}}, D_3\check{\mathbf{u}}),$$

where $D_i\mathbf{u}$ denotes the column vector of the partial derivatives of $\mathbf{u}$ with respect to y_i. We also set

$$\bar{\mathbf{E}}^\varepsilon \bar{\mathbf{u}} := \operatorname{sym}(\bar{\mathbf{H}}^\varepsilon \bar{\mathbf{u}}), \quad \bar{\mathbf{W}}^\varepsilon \bar{\mathbf{u}} := \operatorname{skw}(\bar{\mathbf{H}}^\varepsilon \bar{\mathbf{u}}),$$

$$\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}} := \operatorname{sym}(\check{\mathbf{H}}^\varepsilon \check{\mathbf{u}}), \quad \check{\mathbf{W}}^\varepsilon \check{\mathbf{u}} := \operatorname{skw}(\check{\mathbf{H}}^\varepsilon \check{\mathbf{u}}),$$

5.4 Compactness lemmata

In this section we establish the compactness of appropriately rescaled sequences of displacements and we prove that the limit functions are displacements of Kirchhoff-Love type.

Theorem 5.3. *There exist two constants $\bar{K}, \check{K} > 0$ such that*

$$\int_{\bar{\Omega}} \big(\big|(\frac{\bar{u}_1}{\varepsilon}, \bar{u}_2, \frac{\bar{u}_3}{\varepsilon})\big|^2 + |\bar{\mathbf{H}}^\varepsilon \bar{\mathbf{u}}|^2\big)\, dy \leq \frac{\bar{K}}{\varepsilon^2} \int_{\bar{\Omega}} |\bar{\mathbf{E}}^\varepsilon \bar{\mathbf{u}}|^2\, dy, \tag{5.9}$$

$$\int_{\check{\Omega}} \big(\big|(\check{u}_1, \frac{\check{u}_2}{\varepsilon}, \frac{\check{u}_3}{\varepsilon})\big|^2 + |\check{\mathbf{H}}^\varepsilon \check{\mathbf{u}}|^2\big)\, dy \leq \frac{\check{K}}{\varepsilon^2} \int_{\check{\Omega}} |\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}}|^2\, dy, \tag{5.10}$$

for every $(\bar{\mathbf{u}}, \check{\mathbf{u}}) \in H^1_\flat(\bar{\Omega}; \mathbb{R}^3) \times H^1_\flat(\check{\Omega}; \mathbb{R}^3)$ and every $\varepsilon \in (0,1]$.

Proof. We prove the theorem for $\check{\Omega}$, the case with $\bar{\Omega}$ is similar. The inequality

$$\int_{\check{\Omega}} |\check{\mathbf{H}}^\varepsilon \check{\mathbf{u}}|^2\, dy \leq \frac{K}{\varepsilon^2} \int_{\check{\Omega}} |\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}}|^2\, dy$$

follows by rescaling the Korn's inequality in Theorem 5.1. To show that

$$\int_{\bar{\Omega}} |(\check{u}_1, \frac{\check{u}_2}{\varepsilon}, \frac{\check{u}_3}{\varepsilon})|^2\, dy \leq \frac{K}{\varepsilon^2} \int_{\check{\Omega}} |\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}}|^2\, dy,$$

it suffices to set $\mathbf{w}^\varepsilon := (\check{u}_1, \check{u}_2/\varepsilon, \check{u}_3/\varepsilon)$, notice that $|\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}}| \geq \varepsilon |\mathbf{E}\mathbf{w}^\varepsilon|$ and apply Theorem 1.45 to $\mathbf{w}^\varepsilon$ on $\check{\Omega}$. □

Let us consider the space of Kirchhoff-Love displacements on $\bar{\Omega}$ and $\check{\Omega}$

$$H_{KL}(\bar{\Omega}; \mathbb{R}^3) := \{\mathbf{v} \in H^1_\flat(\bar{\Omega}; \mathbb{R}^3) : (Ev)_{i2} = 0,\ i = 1,2,3,\},$$

$$H_{KL}(\check{\Omega}; \mathbb{R}^3) := \{\mathbf{v} \in H^1_\flat(\check{\Omega}; \mathbb{R}^3) : (Ev)_{i1} = 0,\ i = 1,2,3\},$$

and set

$$H_{KL} := H_{KL}(\bar{\Omega}; \mathbb{R}^3) \times H_{KL}(\check{\Omega}; \mathbb{R}^3).$$

In the remaining part of this section we assume $(\bar{\mathbf{u}}^\varepsilon, \check{\mathbf{u}}^\varepsilon)$ to be a sequence of functions in $H^1_\flat(\bar{\Omega}; \mathbb{R}^3) \times H^1_\flat(\check{\Omega}; \mathbb{R}^3)$ such that

$$\|\bar{\mathbf{E}}^\varepsilon \bar{\mathbf{u}}^\varepsilon\|_{L^2(\bar{\Omega}; \mathbb{R}^{3\times 3})} + \|\check{\mathbf{E}}^\varepsilon \check{\mathbf{u}}^\varepsilon\|_{L^2(\check{\Omega}; \mathbb{R}^{3\times 3})} \leq C\varepsilon, \tag{5.11}$$

for some constant C and for every $\varepsilon \in (0,1]$.

Theorem 5.4. *Let*

$$\mathcal{A}_\varepsilon := \left\{(\bar{\mathbf{u}}, \check{\mathbf{u}}) \in H^1_\flat(\bar{\Omega}; \mathbb{R}^3) \times H^1_\flat(\check{\Omega}; \mathbb{R}^3) : \text{ condition (5.8) is satisfied}\right\},$$

and $(\bar{\mathbf{u}}^\varepsilon, \check{\mathbf{u}}^\varepsilon)$ a sequence in $\mathcal{A}_\varepsilon$ which satisfies (5.11). Then, for any sequence of positive numbers ε_n converging to 0, there exist a subsequence (not relabelled) ε_n and functions $(\bar{\mathbf{v}}, \check{\mathbf{v}}) \in H_{KL}$ such that, as $n \to +\infty$,

$$(\frac{\bar{u}_1^{\varepsilon_n}}{\varepsilon_n}, \bar{u}_2^{\varepsilon_n}, \frac{\bar{u}_3^{\varepsilon_n}}{\varepsilon_n}) \rightharpoonup (\bar{v}_1, \bar{v}_2, \bar{v}_3) \text{ in } H^1(\bar{\Omega}; \mathbb{R}^3), \tag{5.12}$$

$$(\check{u}_1^{\varepsilon_n}, \frac{\check{u}_2^{\varepsilon_n}}{\varepsilon_n}, \frac{\check{u}_3^{\varepsilon_n}}{\varepsilon_n}) \rightharpoonup (\check{v}_1, \check{v}_2, \check{v}_3) \ \textit{in } H^1(\check{\Omega}; \mathbb{R}^3), \tag{5.13}$$

and

$$\bar{\mathbf{W}}^{\varepsilon_n}\bar{\mathbf{u}}^{\varepsilon_n} \rightharpoonup \bar{\mathbf{H}}\bar{\mathbf{v}} := \begin{pmatrix} 0 & -D_1\bar{v}_2 & 0 \\ D_1\bar{v}_2 & 0 & -D_2\bar{v}_3 \\ 0 & D_2\bar{v}_3 & 0 \end{pmatrix} \ \textit{in } L^2(\bar{\Omega}), \tag{5.14}$$

$$\check{\mathbf{W}}^{\varepsilon_n}\check{\mathbf{u}}^{\varepsilon_n} \rightharpoonup \check{\mathbf{H}}\check{\mathbf{v}} := \begin{pmatrix} 0 & D_2\check{v}_1 & -D_1\check{v}_3 \\ -D_2\check{v}_1 & 0 & 0 \\ D_1\check{v}_3 & 0 & 0 \end{pmatrix} \ \textit{in } L^2(\check{\Omega}). \tag{5.15}$$

Proof. We prove the theorem for $\check{\Omega}$, the case with $\bar{\Omega}$ is similar. It is convenient to set $\mathbf{w}^\varepsilon := (\check{u}_1^\varepsilon, \check{u}_2^\varepsilon/\varepsilon, \check{u}_3^\varepsilon/\varepsilon)$. Since $|\check{\mathbf{E}}^\varepsilon\check{\mathbf{u}}^\varepsilon| \geq \varepsilon|\mathbf{E}\mathbf{w}^\varepsilon|$, by (5.11), $\mathbf{E}\mathbf{w}^\varepsilon$ is uniformly bounded in $L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$ and by Korn's inequality $\mathbf{w}^\varepsilon$ is uniformly bounded in $H^1(\check{\Omega}; \mathbb{R}^3)$. It then exists a $\check{\mathbf{v}} \in H^1_\flat(\check{\Omega}; \mathbb{R}^3)$ and a subsequence ε_n such that $\mathbf{w}^{\varepsilon_n} \rightharpoonup \check{\mathbf{v}}$ in $H^1(\check{\Omega}; \mathbb{R}^3)$. Again, it is easy to check that $|(\check{E}^{\varepsilon_n}\check{u}^{\varepsilon_n})_{i1}| \geq |(Ew^{\varepsilon_n})_{i1}|$, thus, using (5.11) we deduce that $C\varepsilon_n \geq \|(Ew^{\varepsilon_n})_{i1}\|_{L^2(\check{\Omega})}$ and consequently, as $n \to \infty$, $(E\check{v})_{i1} = 0$ for $i = 1, 2, 3$. Hence $\check{\mathbf{v}} \in H_{KL}(\check{\Omega}; \mathbb{R}^3)$.
Using (5.11) and Theorem 5.3 we obtain that the sequence $\check{\mathbf{H}}^{\varepsilon_n}\check{\mathbf{u}}^{\varepsilon_n}$ is bounded in $L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$ so that, up to subsequences, it weakly converges in $L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$ to a matrix $\check{\mathbf{H}}\check{\mathbf{v}} \in L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$. Since, from inequality (5.11), $\check{\mathbf{E}}^{\varepsilon_n}\check{\mathbf{u}}^{\varepsilon_n} \to 0$ in $L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$, we have $\check{\mathbf{W}}^{\varepsilon_n}\check{\mathbf{u}}^{\varepsilon_n} \rightharpoonup \check{\mathbf{H}}\check{\mathbf{v}}$ in $L^2(\check{\Omega}; \mathbb{R}^{3\times 3})$. In particular, $\check{\mathbf{H}}\check{\mathbf{v}}$ is, almost everywhere, a skew-symmetric matrix. Since $(\check{H}^\varepsilon\check{u}^\varepsilon)_{12} = w_{1,2}^\varepsilon$, $(\check{H}^\varepsilon\check{u}^\varepsilon)_{31} = w_{3,1}^\varepsilon$ and $(\check{H}^\varepsilon\check{u}^\varepsilon)_{32} = \varepsilon w_{3,2}^\varepsilon$, we deduce that $(\check{H}\check{v})_{12} = \check{v}_{1,2}$, $(\check{H}\check{v})_{31} = \check{v}_{3,1}$ and $(\check{H}\check{v})_{32} = 0$. $\square$

By Theorem 5.4 the displacements $\bar{\mathbf{v}}$ and $\check{\mathbf{v}}$ are of Kirchhoff-Love type and therefore they can be written as (see, for instance, [39])

$$\begin{aligned} \bar{v}_1 &= \bar{\xi}_1(y_1, y_3) - y_2\bar{\xi}_{2,1}(y_1, y_3), \\ \bar{v}_2 &= \bar{\xi}_2(y_1, y_3), \\ \bar{v}_3 &= \bar{\xi}_3(y_1, y_3) - y_2\bar{\xi}_{2,3}(y_1, y_3), \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \check{v}_1 &= \check{\xi}_1(y_2, y_3) \\ \check{v}_2 &= \check{\xi}_2(y_2, y_3) - y_1\check{\xi}_{1,2}(y_2, y_3), \\ \check{v}_3 &= \check{\xi}_3(y_2, y_3) - y_1\check{\xi}_{1,3}(y_2, y_3), \end{aligned} \tag{5.17}$$

where $\bar{\xi}_1$, $\bar{\xi}_3 \in H^1_\flat((0,\bar{b}) \times (0,\ell))$, $\bar{\xi}_2 \in H^2_\flat((0,\bar{b}) \times (0,\ell))$, $\check{\xi}_2$, $\check{\xi}_3 \in H^1_\flat((0,\check{h}) \times (0,\ell))$ and $\check{\xi}_1 \in H^2_\flat((0,\check{h}) \times (0,\ell))$.

Lemma 5.5. *Under the same assumption and with the notation of Theorem 5.4 we have*

$$\begin{aligned}
&(\bar{E}\bar{v})_{11} = D_1\bar{v}_1, \qquad (\bar{E}\bar{v})_{33} = D_3\bar{v}_3,\\
&(\bar{E}\bar{v})_{13} = \frac{1}{2}(D_1\bar{v}_3 + D_3\bar{v}_1),\\
&(\check{E}\check{v})_{22} = D_2\check{v}_2, \qquad (\check{E}\check{v})_{33} = D_3\check{v}_3,\\
&(\check{E}\check{v})_{23} = \frac{1}{2}(D_2\check{v}_3 + D_3\check{v}_2),
\end{aligned} \tag{5.18}$$

where, up to subsequences, $(\bar{E}\bar{v})_{11}$, $(\bar{E}\bar{v})_{33}$ and $(\bar{E}\bar{v})_{13}$ are the limits of $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{11}/\varepsilon$, $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{33}/\varepsilon$ and $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{13}/\varepsilon$ in the weak convergence of $L^2(\bar{\Omega})$ and $(\check{E}\check{v})_{22}$, $(\check{E}\check{v})_{33}$ and $(\check{E}\check{v})_{23}$ are the limits of $(\check{E}^\varepsilon\check{u}^\varepsilon)_{22}/\varepsilon$, $(\check{E}^\varepsilon\check{u}^\varepsilon)_{33}/\varepsilon$ and $(\check{E}^\varepsilon\check{u}^\varepsilon)_{23}/\varepsilon$ in the weak convergence of $L^2(\check{\Omega})$.

Proof. To prove the lemma it suffices to notice that $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{11}/\varepsilon = D_1(\bar{u}_1^\varepsilon/\varepsilon)$, $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{33}/\varepsilon = D_3(\bar{u}_3^\varepsilon/\varepsilon)$, $(\bar{E}^\varepsilon\bar{u}^\varepsilon)_{13}/\varepsilon = 1/2(D_3(\bar{u}_1^\varepsilon/\varepsilon)+D_1(\bar{u}_3^\varepsilon/\varepsilon))$, $(\check{E}^\varepsilon\check{u}^\varepsilon)_{22}/\varepsilon = D_2(\check{u}_2^\varepsilon/\varepsilon)$, $(\check{E}^\varepsilon\check{u}^\varepsilon)_{33}/\varepsilon = D_3(\check{u}_3^\varepsilon/\varepsilon)$ and $(\check{E}^\varepsilon\check{u}^\varepsilon)_{23}/\varepsilon = 1/2(D_3(\check{u}_2^\varepsilon/\varepsilon)+D_2(\check{u}_3^\varepsilon/\varepsilon))$ and apply (5.12) and (5.13). □

5.5 Junction conditions

The present section is devoted to establish the relationship existing between the limit field $(\bar{\mathbf{v}}, \check{\mathbf{v}})$, introduced in Theorem 5.4. To find these relations, we now study the junction condition (5.8), that is

$$\bar{\mathbf{u}}^\varepsilon \circ \bar{p}_\varepsilon^{-1} = \check{\mathbf{u}}^\varepsilon \circ \check{p}_\varepsilon^{-1} \quad \text{in } \Gamma_\varepsilon.$$

The condition above can be equivalently written as

$$\bar{\mathbf{u}}^\varepsilon(\varepsilon z_1, z_2, z_3) = \check{\mathbf{u}}^\varepsilon(z_1, \varepsilon z_1, z_3), \tag{5.19}$$

where $\mathbf{z} = (z_1, z_2, z_3) \in \Gamma$ and $\Gamma := (-\check{s}/2, \check{s}/2) \times (0, \bar{s}/2) \times (0, \ell)$. We notice that in this way also the junction region, which originally depends on ε, has been trasformed into the fixed domain Γ.

Lemma 5.6. *Let $w \in H^1(\Gamma)$ and $w_\varepsilon \in H^1(\Gamma)$ be a sequence such that*

$$w_\varepsilon \rightharpoonup w \quad in \quad H^1(\Gamma).$$

Then the sequence of functions

$$(z_2, z_3) \mapsto \int_{-\check{s}/2}^{\check{s}/2} w_\varepsilon(\varepsilon z_1, z_2, z_3)\, dz_1$$

converges in $L^2((0,\bar{s}/2)\times(0,\ell))$ *to the trace on* $\{0\}\times(0,\bar{s}/2)\times(0,\ell)$ *of the function* w*, which we simply denote by* $w(0,z_2,z_3)$*.*

Proof. We have

$$\begin{aligned}
&\int_0^\ell \int_0^{\bar{s}/2} \Big|\fint_{-\check{s}/2}^{\check{s}/2} w_\varepsilon(\varepsilon z_1, z_2, z_3)\,dz_1 - w_\varepsilon(0,z_2,z_3)\Big|^2 dz_2\,dz_3 \\
&\quad = \int_0^\ell \int_0^{\bar{s}/2} \Big|\fint_{-\varepsilon\check{s}/2}^{\varepsilon\check{s}/2} (w_\varepsilon(z_1,z_2,z_3) - w_\varepsilon(0,z_2,z_3))\,dz_1\Big|^2 dz_2\,dz_3 \\
&\quad = \int_0^\ell \int_0^{\bar{s}/2} \Big|\fint_{-\varepsilon\check{s}/2}^{\varepsilon\check{s}/2} \int_0^{z_1} D_1 w_\varepsilon(t,z_2,z_3)\,dt\,dz_1\Big|^2 dz_2\,dz_3 \\
&\quad \le \varepsilon\check{s} \int_0^\ell \int_0^{\bar{s}/2} \fint_{-\varepsilon\check{s}/2}^{\varepsilon\check{s}/2} \int_{-\varepsilon\check{s}/2}^{\varepsilon\check{s}/2} |D_1 w_\varepsilon(t,z_2,z_3)|^2\,dt\,dz_1\,dz_2\,dz_3 \\
&\quad \le \varepsilon\check{s}\|D_1 w_\varepsilon\|^2_{L^2(\Gamma)} \le C\varepsilon,
\end{aligned}$$

and the claim follows by the continuity of the trace. □

Remark 5.7. With straightforward adaptions of Lemma 5.6, the sequence of functions

$$(z_1,z_3) \mapsto \fint_0^{\bar{s}/2} w_\varepsilon(z_1, \varepsilon z_2, z_3)\,dz_2$$

converges in $L^2((-\check{s}/2,\check{s}/2)\times(0,\ell))$ to $w(z_1,0,z_3)$ on $(-\check{s}/2,\check{s}/2)\times\{0\}\times(0,\ell)$.

Lemma 5.8. *Under the same assumption of Theorem 5.4, we have*

1. $\bar{\xi}_2(0,y_3) = 0$ *for almost every* $y_3 \in (0,\ell)$*;*
2. $\check{\xi}_1(0,y_3) = 0$ *for almost every* $y_3 \in (0,\ell)$*;*
3. $\check{\xi}_3(0,y_3) = \bar{\xi}_3(0,y_3)$ *for almost every* $y_3 \in (0,\ell)$*.*

Proof. As the pair $(\bar{\mathbf{u}}^{\varepsilon_n}, \check{\mathbf{u}}^{\varepsilon_n})$ satisfies (5.19), averaging the second component with rispect to z_1, we find

$$\fint_{-\check{s}/2}^{\check{s}/2} \bar{u}_2^{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\,dz_1 = \fint_{-\check{s}/2}^{\check{s}/2} \check{u}_2^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)\,dz_1.$$

Applying Lemma 5.6 to $\bar{u}_2^{\varepsilon_n}$ and using (5.12), we have that the left hand side of the above equation converges to $(z_2,z_3)\mapsto \bar{v}_2(0,z_2,z_3)$ in $L^2((0,\bar{s}/2)\times(0,\ell))$.

We prove that the right hand side converge to zero; indeed

$$\begin{aligned}
&\int_0^\ell \int_0^{\bar{s}/2} \Big|\fint_{-\check{s}/2}^{\check{s}/2} \check{u}_2^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)\, dz_1\Big|^2 dz_2\, dz_3 \\
&\qquad \le \int_0^\ell \fint_{-\check{s}/2}^{\check{s}/2} \int_0^{\bar{s}/2} \big|\check{u}_2^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)\big|^2 dz_2\, dz_1\, dz_3 \\
&\qquad = \int_0^\ell \fint_{-\check{s}/2}^{\check{s}/2} \frac{1}{\varepsilon_n} \int_0^{\varepsilon_n \bar{s}/2} \big|\check{u}_2^{\varepsilon_n}(z_1, z_2, z_3)\big|^2 dz_2\, dz_1\, dz_3 \\
&\qquad \le \frac{\varepsilon_n}{\check{s}} \int_{\check{\Omega}} \Big|\frac{\check{u}_2^{\varepsilon_n}}{\varepsilon_n}\Big|^2 dz,
\end{aligned}$$

and the last term of the previous inequality tends to zero due to (5.13). Taking into account (5.16), we obtain $\bar{v}_2(0, z_2, z_3) = \bar{\xi}_2(0, z_3) = 0$ for almost every $z_3 \in (0, \ell)$.

Averaging the first component of (5.19) with respect to z_2, we find

$$\fint_0^{\bar{s}/2} \bar{u}_1^{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\, dz_2 = \fint_0^{\bar{s}/2} \check{u}_1^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)\, dz_2.$$

Applying Remark 5.7 to $\check{u}_1^{\varepsilon_n}$ and using (5.13), we have that the right hand side of the above equation converges to $(z_1, z_3) \mapsto \check{v}_1(z_1, 0, z_3)$ in $L^2((-\check{s}/2, \check{s}/2) \times (0, \ell))$. We prove that the left hand side converge to zero; indeed

$$\begin{aligned}
&\int_0^\ell \int_{-\check{s}/2}^{\check{s}/2} \Big|\fint_0^{\bar{s}/2} \bar{u}_1^{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\, dz_2\Big|^2 dz_1\, dz_3 \\
&\qquad \le \int_0^\ell \fint_0^{\bar{s}/2} \int_{-\check{s}/2}^{\check{s}/2} \big|\bar{u}_1^{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\big|^2 dz_1\, dz_2\, dz_3 \\
&\qquad = \int_0^\ell \fint_0^{\bar{s}/2} \frac{1}{\varepsilon_n} \int_{-\varepsilon_n \check{s}/2}^{\varepsilon_n \check{s}/2} \big|\bar{u}_1^{\varepsilon_n}(z_1, z_2, z_3)\big|^2 dz_1\, dz_2\, dz_3 \\
&\qquad \le \frac{2\varepsilon_n}{\bar{s}} \int_{\bar{\Omega}} \Big|\frac{\bar{u}_1^{\varepsilon_n}}{\varepsilon_n}\Big|^2 dz,
\end{aligned}$$

and the last term of the previous inequality tends to zero due to (5.12). Taking into account (5.17), this proves statement *2.* of lemma.

In order to prove *3.*, we consider the following scaled average of the third component of (5.19)

$$\begin{aligned}
&\fint_0^{\bar{s}/2} \fint_{-\check{s}/2}^{\check{s}/2} \frac{\bar{u}_3^{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)}{\varepsilon_n}\, dz_1\, dz_2 = \\
&\qquad\qquad = \fint_{-\check{s}/2}^{\check{s}/2} \fint_0^{\bar{s}/2} \frac{\check{u}_3^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)}{\varepsilon_n}\, dz_2\, dz_1.
\end{aligned}$$

Applying Lemma 5.6 to $\bar{u}_3^{\varepsilon_n}$ and Remark 5.7 to $\check{u}_3^{\varepsilon_n}$, we deduce

$$\fint_0^{\bar{s}/2} \bar{v}_3(0, z_2, z_3)\, dz_2 = \fint_{-\check{s}/2}^{\check{s}/2} \check{v}_3(z_1, 0, z_3)\, dz_1.$$

Recalling (5.16) and (5.17), we have

$$\fint_0^{\bar{s}/2} \bar{v}_3(0, z_2, z_3)\, dz_2 = \fint_0^{\bar{s}/2} \left(\bar{\xi}_3(0, z_3) - z_2 \bar{\xi}_{2,3}(0, z_3)\right)\, dz_2$$

and

$$\fint_{-\check{s}/2}^{\check{s}/2} \check{v}_3(z_1, 0, z_3)\, dz_1 = \fint_{-\check{s}/2}^{\check{s}/2} \left(\check{\xi}_3(0, z_3) - z_1 \check{\xi}_{1,3}(0, z_3)\right)\, dz_1.$$

Taking into account the statements *1.* and *2.* of this lemma, we find $\bar{\xi}_3(0, z_3) = \check{\xi}_3(0, z_3)$ for almost every $z_3 \in (0, \ell)$. □

Following the lines traced by Le Dret in Lemma 4.8 of [34], we find a further junction condition.

Lemma 5.9. *Under the same assumption of Theorem 5.4, we have*

$$\bar{\xi}_{2,1}(0, z_3) = -\check{\xi}_{1,2}(0, z_3)$$

for almost every $z_3 \in (0, \ell)$*.*

To prove the lemma above, we recall the following result (for the proof see [34] and [35]).

Proposition 5.10. *Let* H *be a Hilbert space and* $(0, T) \subset \mathbb{R}$*. Then, for every* $\mathbf{u} \in H^1((0, T); H)$*, we have*

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_H \leq C \|\mathbf{u}\|_{H^1((0,T);H)} |t - s|^{1/2}, \tag{5.20}$$

where $t, s \in (0, T)$ *and* C *is a constant independent on* $\mathbf{u}$*.*

Proof. (of Lemma 5.9) As the pair $(\bar{\mathbf{u}}^{\varepsilon_n}, \check{\mathbf{u}}^{\varepsilon_n})$ satisfies (5.19), by differentiating equation (5.19) with respect to z_2, we obtain

$$\begin{aligned}
\frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3) &= \check{u}_{1,2}^{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3) \\
&= 2(\check{E}^{\varepsilon_n} \check{u}^{\varepsilon_n})_{12}(z_1, \varepsilon_n z_2, z_3) - \frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3),
\end{aligned}$$

for every $(z_1, z_2, z_3) \in \Gamma$. By multiplying the equation above with three test-functions $\varphi_1 \in C_c^\infty(-\check{s}/2, \check{s}/2)$, $\varphi_2 \in C_c^\infty(0, \bar{s}/2)$ and $\varphi_3 \in C_c^\infty(0, \ell)$ and then integrating, we find

$$\begin{aligned}\int_\Gamma \frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3) & \prod_{i=1}^3 \varphi_i(z_i)\, dz = \\ & = -\int_\Gamma \frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3) \prod_{i=1}^3 \varphi_i(z_i)\, dz + \\ & \quad + \int_\Gamma 2(\check{E}^{\varepsilon_n} \check{u}^{\varepsilon_n})_{12}(z_1, \varepsilon_n z_2, z_3) \prod_{i=1}^3 \varphi_i(z_i)\, dz.\end{aligned} \tag{5.21}$$

We note that

$$\frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n} \in H^1((-\check{s}/2, \check{s}/2); H^{-1}((0, \bar{s}/2) \times (0, \ell))),$$

indeed

$$\begin{cases} \dfrac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n} \in L^2((-\check{s}/2, \check{s}/2); L^2((0, \bar{s}/2) \times (0, \ell))), \\ D_1(\dfrac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}) = D_2(\dfrac{\bar{u}_{1,1}^{\varepsilon_n}}{\varepsilon_n}) \in L^2((-\check{s}/2, \check{s}/2); H^{-1}((0, \bar{s}/2) \times (0, \ell))). \end{cases}$$

Similarly

$$\frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n} \in H^1((0; \bar{s}/2); H^{-1}((-\check{s}/2, \check{s}/2) \times (0, \ell))).$$

Moreover, we have

$$\begin{aligned} \|\frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}\|_{H^1((-\check{s}/2,\check{s}/2);H^{-1}((0,\bar{s}/2)\times(0,\ell)))} &\le C\|\frac{\bar{u}_1^{\varepsilon_n}}{\varepsilon_n}\|_{H^1(\bar{\Omega})}, \\ \|\frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n}\|_{H^1((0,\bar{s}/2);H^{-1}((-\check{s}/2,\check{s}/2)\times(0,\ell)))} &\le C\|\frac{\check{u}_2^{\varepsilon_n}}{\varepsilon_n}\|_{H^1(\check{\Omega})}. \end{aligned} \tag{5.22}$$

From equations (5.12), (5.13) and (5.20), we obtain

$$\|\frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3) - \frac{\bar{u}_{1,2}^{\varepsilon_n}}{\varepsilon_n}(0, z_2, z_3)\|_{H^{-1}((0,\bar{s}/2)\times(0,\ell))} \le C\varepsilon_n^{1/2},$$

for every $z_1 \in (-\check{s}/2, \check{s}/2)$, and

$$\|\frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3) - \frac{\check{u}_{2,1}^{\varepsilon_n}}{\varepsilon_n}(z_1, 0, z_3)\|_{H^{-1}((-\check{s}/2,\check{s}/2)\times(0,\ell))} \le C\varepsilon_n^{1/2},$$

for every $z_2 \in (0, \bar{s}/2)$. Since $\varphi_2(z_2)\varphi_3(z_3) \in H^1_0((0,\bar{s}/2)\times(0,\ell))$, the left hand side of equation (5.21) can be written as

$$\begin{aligned}\int_\Gamma \frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz = \\ = \int_{-\check{s}/2}^{\check{s}/2}\varphi_1(z_1)\Big(\int_0^{\bar{s}/2}\int_0^\ell \frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\,\varphi_2(z_2)\,\varphi_3(z_3)\,dz_3dz_2\Big)dz_1 \\ = \int_{-\check{s}/2}^{\check{s}/2}\varphi_1(z_1)\Big(\int_0^{\bar{s}/2}\int_0^\ell \frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}(0, z_2, z_3)\,\varphi_2(z_2)\,\varphi_3(z_3)\,dz_3dz_2\Big)dz_1 + o(\varepsilon_n^{1/2}),\end{aligned}$$

where $o(\varepsilon_n^{1/2})$ is a sequence of functions such that $\lim_{n\to+\infty}\frac{o(\varepsilon_n^{1/2})}{\varepsilon_n^{1/2}} = 0$. Moreover, by equations (5.12) and (5.22), we have that $\frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}$ converges weakly in $H^1((-\check{s}/2,\check{s}/2);H^{-1}((0,\bar{s}/2)\times(0,\ell)))$. In particular

$$\frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}(0,z_2,z_3) \rightharpoonup \bar{v}_{1,2}(0,z_2,z_3) = -\bar{\xi}_{2,1}(0,z_3) \quad \text{in } H^{-1}((0,\bar{s}/2)\times(0,\ell)).$$

Then we find

$$\int_\Gamma \frac{\bar{u}^{\varepsilon_n}_{1,2}}{\varepsilon_n}(\varepsilon_n z_1, z_2, z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz \to -\int_\Gamma \bar{\xi}_{2,1}(0,z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz.$$

Similarly

$$\int_\Gamma \frac{\check{u}^{\varepsilon_n}_{2,1}}{\varepsilon_n}(z_1, \varepsilon_n z_2, z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz \to -\int_\Gamma \check{\xi}_{1,2}(0,z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz.$$

Hence, to prove the lemma, it sufficies to note that

$$\int_\Gamma (\check{E}^{\varepsilon_n}\check{u}^{\varepsilon_n})_{12}(z_1,\varepsilon_n z_2, z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz \to 0,$$

and this comes by noticing that, from Cauchy-Schwarz inequality and inequality (5.11), we have

$$|\int_\Gamma (\check{E}^{\varepsilon_n}\check{u}^{\varepsilon_n})_{12}(z_1,\varepsilon_n z_2, z_3)\prod_{i=1}^3 \varphi_i(z_i)\,dz| \le C\varepsilon_n^{1/2}\|\frac{\check{E}^{\varepsilon_n}\check{u}^{\varepsilon_n}_{12}}{\varepsilon_n}\|_{L^2(\check{\Omega})} \le k\varepsilon_n^{1/2}.$$

□

The study of the kinematics of the model in Section 5.4 and 5.5 is a first step to solve the Γ-convergence problem for the junction of two plates with residual stress. This will be the aim of a future work.

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