ADJOINT FORMS AND ALGEBRAIC FAMILIES

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ABSTRACT

In this thesis we study in details the theory of *adjoint forms* which was introduced by Collino and Pirola in [18] in the case of smooth curves and then generalized in higher dimension by Pirola and Zucconi in [55]. Useful generalizations are given, for example for Gorenstein curves, smooth projective hypersurfaces and fibrations over a smooth curve. The main applications of this theory concern infinitesimal Torelli problems and criteria which ensure that a family $\mathcal{X} \to B$ of algebraic varieties of general type and with Albanese morphism of degree 1 has birational fibers.

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INTRODUCTION

Given an $n \times n$ matrix $T \in \mathrm{Mat}(n, \mathbb{K})$, there always exists its *adjoint matrix*, $T^\vee$, such that by row-column product we obtain $T \cdot T^\vee = \det(T) \cdot I_n$, where $I_n \in \mathrm{Mat}(n, \mathbb{K})$ is the identity matrix. The theory of adjoint forms considers the above construction in the context of locally free sheaves. This theory was first introduced by Collino and Pirola in [18] in the case of smooth curves and later it was extended to any smooth algebraic variety by Pirola and Zucconi [55]. Since then these ideas have been fruitfully applied in [5], [17], [25], [26], [54], and [56].

The main idea is the following: let $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ be an extension class associated to the following exact sequence of locally free sheaves over an m-dimensional smooth variety X:

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0. \tag{1}$$

Assume that $\mathcal{F}$ is of rank n and that the kernel of the connecting homomorphism

$$\delta_\xi \colon H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X)$$

has dimension $\geqslant n + 1$. Take an $n + 1$-dimensional subspace $W \subset \mathrm{Ker}\, \delta_\xi$ and a basis $\mathcal{B} = \{\eta_1, \dots, \eta_{n+1}\}$ of W. By choosing a lifting $s_i \in H^0(X, \mathcal{E})$ of η_i, where $i = 1, \dots, n + 1$, we have a top form $\Omega \in H^0(X, \det \mathcal{E})$ from the element $s_1 \wedge \ldots \wedge s_{n+1} \in \bigwedge^{n+1} H^0(X, \mathcal{E})$. Since $\det \mathcal{E} = \det \mathcal{F}$ we actually obtain from Ω a top form ω of $\mathcal{F}$, which depends on the chosen liftings and on $\mathcal{B}$. We call such an $\omega \in H^0(X, \det \mathcal{F})$ an *adjoint form* of $\xi, W, \mathcal{B}$. Indeed consider the $n + 1$ top forms $\omega_1, \dots, \omega_{n+1} \in H^0(X, \det \mathcal{F})$ where ω_i is obtained by the element $\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1} \in \bigwedge^n H^0(X, \mathcal{F})$. It is easily seen that the subscheme of X where ω vanishes is locally given by the vanishing of the determinant of a suitable $(n + 1) \times (n + 1)$ matrix T and the local expressions of ω_i, $i = 1, \dots, n + 1$, give some entries of the *adjoint matrix* $T^\vee$. This explains the name of the theory. To the $n + 1$-dimensional subspace $W \subset \mathrm{Ker}\, \delta_\xi$ taken above, we associate D_W and Z_W which are respectively the fixed part and the base locus of the sublinear system given by $\lambda^n W \subset H^0(X, \det \mathcal{F})$, where $\lambda^n \colon \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F})$ is the natural homomorphism obtained by the wedge product. The theory of adjoint forms allows to relate the deformation ξ with the adjoint form ω via the following theorem

Theorem (Adjoint Theorem). *Let X be an m-dimensional compact complex smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf and let $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ be the class of the sequence (1). Let ω be an adjoint form associated to a subspace $W \subset H^0(X, \det \mathcal{F})$. If $\omega \in \lambda^n W$ then $\xi \in \mathrm{Ker}\,(\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{F}(-D_W), \mathcal{O}_X))$. Viceversa. Assume $\xi \in \mathrm{Ker}\,(\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{F}(-D_W), \mathcal{O}_X))$. If $h^0(X, \mathcal{O}_X(D_W)) = 1$, then $\omega \in \lambda^n W$.*

If $D_W = 0$, this theorem characterizes the vanishing of ξ.

The link between adjoint forms and extension classes gives us a useful tool to study Torelli type problems, in particular infinitesimal Torelli problems, which we briefly recall. Let $\pi\colon \mathcal{X} \to B$ be a family of algebraic varieties. Fix a point $0 \in B$. We can interpret $\pi\colon \mathcal{X} \to B$ as a deformation of complex structure on $X := \pi^{-1}(0)$. Indeed by Ehresmann's theorem, after possibly shrinking B, we have an isomorphism between $H^k(X_b, \mathbb{C})$ and $V := H^k(X, \mathbb{C})$, for every $b \in B$. Furthermore, $\dim F^p H^k(X_b, \mathbb{C}) = \dim F^p H^k(X, \mathbb{C}) =: b^{p,k}$, where $F^p H^k(X_b, \mathbb{C}) = \bigoplus_{r \geqslant p} H^{r,k-r}(X_b)$ gives the Hodge filtration on $H^k(X_b, \mathbb{C})$. The *period map*

$$\mathcal{P}^{p,k}\colon B \to \mathbb{G} = \mathrm{Grass}(b^{p,k}, V)$$

(cf. [38], [66]) is the map which to $b \in B$ associates the subspace $F^p H^k(X_b, \mathbb{C})$ of V. In [32], [33], P. Griffiths proved that $\mathcal{P}^{p,k}$ is holomorphic and that the image of the differential

$$d\mathcal{P}^{p,k} : T_{B,0} \to T_{\mathbb{G}, F^p H^k(X,\mathbb{C})}$$

is actually contained in

$$\mathrm{Hom}(F^p H^k(X, \mathbb{C}), F^{p-1} H^k(X, \mathbb{C}) / F^p H^k(X, \mathbb{C})).$$

Setting $q = k - p$ and using the canonical isomorphism

$$F^p H^k(X, \mathbb{C}) / F^{p+1} H^k(X, \mathbb{C}) \simeq H^q(X, \Omega^p_X)$$

he showed that $d\mathcal{P}^{p,k}$ is the composition of the Kodaira-Spencer map $T_{B,0} \to H^1(X, T_X)$ with the map given by the cup product:

$$d\pi^q_p\colon H^1(X, T_X) \to \mathrm{Hom}(H^q(X, \Omega^p_X), H^{q+1}(X, \Omega^{p-1}_X))$$

where T_X is the tangent sheaf of X.

Putting together the period maps $\mathcal{P}^{p,k}$ for $p = 1, \dots, k$, we obtain a map, which is also called period map

$$\mathcal{P}^k\colon B \to \prod_{p=1}^{k} \mathrm{Grass}(b^{p,k}, H^k(X, \mathbb{C}))$$
$$b \mapsto (\mathcal{P}^{1,k}(b), \dots, \mathcal{P}^{k,k}(b))$$

(as we will see in details, the appropriate codomain of this map is strictly contained in this product of Grassmannians). Its differential is again a composition

$$d\mathcal{P}^k\colon T_{B,0} \to H^1(X, T_X) \to \bigoplus_p \mathrm{Hom}(H^{k-p}(X, \Omega^p_X), H^{k-p+1}(X, \Omega^{p-1}_X)).$$

The k-*infinitesimal Torelli problem* asks if the differential of the period map $\mathcal{P}^k$ for the local Kuranishi family of X (see Definition 1.4.4, cf.[43], [49]) is injective. Hence take $\mathcal{F} = \Omega^1_X$, the cotangent sheaf on X, and consider the extension given by the Kuranishi family of X along a certain tangent vector to its base B

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0$$

together with its wedge products

$$0 \to \Omega^{i-1}_X \to \Omega^i_{\mathcal{X}|X} \to \Omega^i_X \to 0.$$

Since the Kodaira-Spencer map is bijective for the Kuranishi family, the infinitesimal Torelli problem can be translated in the following way: if all the coboundary maps

$$H^{k-p}(X, \Omega^p_X) \to H^{k-p+1}(X, \Omega^{p-1}_X)$$

coming from the wedge sequences are zero, can we deduce that the class ξ is zero? As we have seen, the adjoint forms give information on the vanishing of ξ, hence this theory can be naturally used for the study of such a problem.

This thesis is structured as follows. In the first chapter we recall some basic facts from Hodge theory and we give an overview on the Torelli problems.

The second chapter is focused on the adjoint theory for curves since this case is peculiar under many aspects and deserves a distinguished treatment. The case of smooth curves is discussed following the paper of Collino and Pirola [18]. The theory is extended also to irreducible Gorenstein curves (see [60]) and this is the first example where a generalization of adjoint forms is considered on a singular variety.

The main result is the following:

Theorem. *Let C be an irreducible Gorenstein curve and let $\mathcal{L}$ be a locally free sheaf of rank one on C. Consider the extension*

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

given by an element $\xi \in \mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^\vee)$. Call F the fixed part of the linear system associated to $\mathcal{L}$ and assume that F does not *contain singularities; call* M *its mobile part. Assume that the map* ϕ_M *given by* M *is of degree one and that* $l := \dim|M| \geqslant 3$*. If the cohomology map*

$$H^0(C, \mathcal{E}) \to H^0(C, \mathcal{L})$$

is surjective, then $\xi \in \operatorname{Ker}(\operatorname{Ext}^1(\mathcal{L}, \mathcal{O}_C) \to \operatorname{Ext}^1(\mathcal{L}(-F), \mathcal{O}_C))$.

See Theorem 2.5.4. The theory of adjoint forms for Gorenstein curves allows to prove an infinitesimal Torelli type theorem.

Theorem. *Let* C *be an irreducible Gorenstein curve of genus* 2 *or non-hyperelliptic of genus* $\geqslant 3$*,* $\xi \in H^1(C, \omega_C^\vee)$ *such that* $\delta_\xi = 0$*. Then* $\xi = 0$*.*

See Theorem 2.5.7.

The third chapter goes deeper along the direction indicated in [55]. The theory of adjoint forms is studied in details for a smooth variety X of arbitrary dimension and it is connected to the presence of particular quadrics, called *adjoint quadrics*, vanishing on the canonical image of X.

Definition. *An* adjoint quadric *for* ω *is a quadric in* $\mathbb{P}(H^0(X, \det \mathcal{F})^\vee)$ *of the form*

$$\omega^2 - \sum L_i \cdot \omega_i,$$

where ω *is an adjoint form of* $W \subset H^0(X, \mathcal{F})$*, the* ω_i*'s are as above and* $L_i \in H^0(X, \det \mathcal{F})$*,* $i = 1, \dots, n+1$*.*

We point out that adjoint quadrics have rank less then or equal to $2n+3$.

Denote by $|\det \mathcal{F}|$ the linear system $\mathbb{P}(H^0(X, \det \mathcal{F}))$ and by $D_{\det \mathcal{F}}$ its fixed part. Set $|\det \mathcal{F}| = D_{\det \mathcal{F}} + |M_{\det \mathcal{F}}|$. We prove a relation between liftability of adjoint forms and quadrics vanishing on the image of the map $\phi_{|M_{\det \mathcal{F}}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \det \mathcal{F})^\vee)$ associated to $|\det \mathcal{F}|$.

Theorem. *Let* X *be an* m*-dimensional smooth variety. Let* $\mathcal{F}$ *be a locally free sheaf of rank* n *such that* $h^0(X, \mathcal{F}) \geqslant n+1$*, let* $\xi \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ *and let* Y *be the schematic image of*

$$\phi_{|M_{\det \mathcal{F}}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \det \mathcal{F})^\vee).$$

If ξ *is such that* $\xi \cup \omega = 0$*, where* ω *is an adjoint form associated to an* $n+1$*-dimensional subspace* $W \subset \operatorname{Ker} \delta_\xi \subset H^0(X, \mathcal{F})$*, then* $\xi \in \operatorname{Ker}(\operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \operatorname{Ext}^1(\mathcal{F}(-D_W), \mathcal{O}_X))$*, provided that there are no* ω*-adjoint quadrics vanishing on* Y*.*

See Theorem 3.5.2 and Corollary 3.5.3. Note that this theorem applies directly if $Y := \phi_{|M_{\det\mathcal{F}}|}(X)$ is a hypersurface of $\mathbb{P}(H^0(X,\det\mathcal{F})^\vee)$ of degree > 2; see Corollary 3.5.6 for a more general claim.

In this chapter we also prove a criterion for a family $\pi\colon \mathcal{X} \to B$ of algebraic varieties of general type and with Albanese morphism of degree 1 to have birational fibers.

We say that $\pi\colon \mathcal{X}\to B$ satisfies *extremal liftability conditions over* B if the following maps are surjective

(i) $H^0(\mathcal{X},\Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b,\Omega^1_{X_b})$;

(ii) $H^0(\mathcal{X},\Omega^n_{\mathcal{X}}) \twoheadrightarrow H^0(X_b,\Omega^n_{X_b})$.

The families $\pi\colon \mathcal{X} \to B$ which satisfy extremal liftability conditions and such that $H^0(X_b,\Omega^1_{X_b}) \neq 0$, that is with irregular fibers, have the advantage that all the fibers have the same Albanese variety. Now by a result called *the Volumetric theorem*, see: [55, Theorem 1.5.3], cf. Theorem 3.6.10, we obtain:

Theorem. *Let $\pi\colon \mathcal{X} \to B$ be a family of n-dimensional irregular varieties which satisfies extremal liftability conditions. Assume that for every fiber X the irregularity is $\geqslant n+1$, the Albanese map is of degree 1 and that there are no adjoint quadrics containing the canonical image of X. Then the fibers of $\pi\colon \mathcal{X} \to B$ are birational. In particular the claim holds if there are no quadrics of rank less then or equal to $2n+3$ passing through the canonical image of X.*

See Theorem 3.6.11.

In Chapter 3 the theory is developed to deal with sequences like (1) and this can be quite restrictive sometimes. For this reason in Chapter 4 the results of Chapter 3 are extended to sequences of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{L}$ is a line bundle possibly different from the structure sheaf.

In this case $\det\mathcal{E} = \mathcal{L}\otimes\det\mathcal{F}$ and liftings $s_1,\dots,s_{n+1} \in H^0(X,\mathcal{E})$ of $\eta_1,\dots,\eta_{n+1} \in H^0(X,\mathcal{F})$ determine a section $\Omega \in H^0(X,\det\mathcal{E})$ corresponding to $s_1\wedge s_2\wedge\dots\wedge s_{n+1}$. This section is called a *generalized adjoint form*. We define the sections ω_i, $i=1,\dots,n+1$, as before and we characterize the case where Ω belongs to the image of

$$H^0(X,\mathcal{L})\otimes\lambda^n W \to H^0(X,\det\mathcal{E})$$

by the natural tensor product map. The game is more complicated than before because the linear system $|\lambda^n W|$ is inside $\mathbb{P}H^0(X, \det \mathcal{F})$ and we have to relate the fixed divisor D_W of $|\lambda^n W|$ and the base locus Z_W of the moving part M_W to forms which are not anymore inside $H^0(X, \det \mathcal{F})$. Nevertheless the result is analogous to the Adjoint theorem.

Theorem. *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\mathcal{L}$ an invertible sheaf. Consider an extension $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ corresponding to $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})$. Let $W = \langle \eta_1, \dots, \eta_{n+1} \rangle$ be an $n+1$-dimensional sublinear system of $\mathrm{Ker}\,(\delta_\xi) \subset H^0(X, \mathcal{F})$. Let $\Omega \in H^0(X, \det \mathcal{E})$ be a generalized adjoint form associated to W as above. It holds that if $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$ then $\xi \in \mathrm{Ker}\,(\mathrm{Ext}^1(\mathcal{F}, \mathcal{L}) \to \mathrm{Ext}^1(\mathcal{F}(-D_W), \mathcal{L}))$. Viceversa assume that ξ is in this kernel. If $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$, then $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$.*

See Theorem 4.1.12 and Theorem 4.1.13.

Using this result we can study extension classes of sheaves via adjoint forms. Indeed even if $\mathcal{F}$ has no global sections we can always take the tensor product with a sufficiently ample linear system $\mathcal{M}$ such that $\mathcal{F} \otimes \mathcal{M}$ has enough global sections in order to apply the theory of adjoint forms.

Applications are given in the following chapter, Chapter 5, in the case of smooth projective hypersurfaces and of smooth sufficiently ample divisors of a projective variety. See [61]. By applying the above idea to the case where $X \subset \mathbb{P}^n$, $n > 2$, is a hypersurface of degree $d > 3$ and $\mathcal{F} := \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(2)$ we have a reformulation of the infinitesimal Torelli theorem for X in the setting of generalized adjoint theory. Recall that given a degree d form $F \in \mathbb{C}[x_0, \dots, x_n]$ the Jacobian ideal of F is the ideal $\mathcal{J}$ generated by the partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 0, \dots, n$ and by [34][Theorem 9.8], any infinitesimal deformation $\xi \in H^1(X, T_X)$, where $X = (F = 0)$, is given by a class $[R]$ in the quotient $\mathbb{C}[x_0, \dots, x_n]/\mathcal{J}$ where R is a homogeneous form of degree d.

Theorem. *For a smooth hypersurface X of degree d in $\mathbb{P}^n$ with $n \geqslant 3$ and $d > 3$ the following are equivalent:*

i) the differential of the period map is zero on the infinitesimal deformation

$$[R] \in (\mathbb{C}[x_0, \dots, x_n]/\mathcal{J})_d \cong H^1(X, T_X)$$

ii) R is an element of the Jacobian ideal $\mathcal{J}$

iii) $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{O}_X(2)) \otimes \lambda^n W \to H^0(X, \mathcal{O}_X(n+d-1)))$ *for the generic generalized adjoint* Ω

iv) The generic generalized adjoint Ω *lies in* $\mathcal{J}$.

See Theorem 5.1.8. An analogous statement holds for smooth sufficiently ample divisors of a projective variety, see Theorem 5.2.9.

In the last chapter, following González-Alonso [26] we extend the theory of generalized adjoint forms seen in Chapter 4 to the case of a fibration over a smooth curve of genus g. The idea is to construct "global" objects over the base of this fibration in order to control what happens on the general fiber.

1 PRELIMINARIES

In this chapter we recall some basic facts from the theory of Hodge structures and we give an overview on the so called Torelli problems. For Hodge theory, the main references for details and missing proofs are Voisin's books [66],[67] and Peters and Steenbrink [52]. For period maps and Torelli problems we refer also to Carlson, Müller-Stach and Peters [10].

1.1 HODGE STUCTURES AND POLARIZATIONS

We start with the definition of Hodge structure:

Definition 1.1.1. *A* pure Hodge structure of weight k *is given by a free abelian group* $V_{\mathbb{Z}}$ *of finite type and a decomposition of its complexification*

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q} \tag{2}$$

where $V^{p,q}$ *are complex vector subspaces of* $V_{\mathbb{C}}$ *satisfying* $\overline{V^{p,q}} = V^{q,p}$.

The numbers $h^{p,q} := \dim V^{p,q}$ *are the* Hodge numbers *of the Hodge structure.*

A Hodge structure defines the associated *Hodge filtration* $F^{\cdot}V_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by

$$F^pV_{\mathbb{C}} := \bigoplus_{r \geqslant p} V^{r,k-r}. \tag{3}$$

The Hodge filtration is a decreasing filtration and satisfies the following property:

$$V_{\mathbb{C}} = F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}}. \tag{4}$$

Conversely, a decreasing filtration

$$V_{\mathbb{C}} \supset \cdots \supset F^pV_{\mathbb{C}} \supset F^{p+1}V_{\mathbb{C}} \cdots \tag{5}$$

which satisfies (4) determines a weight k Hodge decomposition on $V_{\mathbb{C}}$ by

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}. \tag{6}$$

Consider two Hodge structures $V_{\mathbb{Z}}$ and $W_{\mathbb{Z}}$ of weight n and $m = n + 2r$ respectively.

Definition 1.1.2. *A* morphism of Hodge structures *is a morphism of groups* $\phi\colon V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ *whose complexification* $\phi_{\mathbb{C}}\colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ *maps* $V^{p,q}$ *to* $W^{p+r,q+r}$*. Equivalently we ask that* $\phi_{\mathbb{C}}(F^pV_{\mathbb{C}}) \subset F^{p+r}W_{\mathbb{C}}$*. We say that* ϕ *is a morphism of type* (r,r)*.*

It is not difficult to see that ϕ induces Hodge structures on $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and $\operatorname{Coker}\phi$ compatible with those on V and W.

Definition 1.1.3. *An* integral polarized Hodge structure of weight k *is given by a Hodge structure of weight* k *together with a* $\mathbb{Z}$*-valued bilinear form*

$$Q\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z} \tag{7}$$

which is symmetric if k *is even, alternating otherwise, and its complexification* $Q_{\mathbb{C}}$ *satisfies*

1. $Q_{\mathbb{C}}(u,v) = 0$ *for* $u \in V^{p,q}, v \in V^{r,s}$ *and* $(p,q) \neq (s,r)$
2. $i^{p-q}Q_{\mathbb{C}}(u,\overline{u}) > 0$ *for* $u \in V^{p,q}, u \neq 0$.

The multiplication by i^{p-q} on $V^{p,q}$ defines an operator on $V_{\mathbb{C}}$ called the Weil operator and usually denoted by C. Conditions 1 and 2 can be equivalently reformulated in terms of the Hodge filtration and the Weil operator:

1. the orthogonal complement of $F^pV_{\mathbb{C}}$ is $F^{k-p+1}V_{\mathbb{C}}$
2. the hermitian form on $V_{\mathbb{C}}$ defined by $Q(Cu,\overline{v})$ is positive definite.

The example of Hodge structure we are interested in is given by the cohomology of an n-dimensional compact Kähler manifold. For the basic notions we refer again to [66]. Recall the well known Hodge decomposition

Theorem 1.1.4 (Hodge decomposition)**.** *Let* X *be an* n*-dimensional compact Kähler manifold. Let* $H^{p,q}(X)$ *be the space of cohomology classes representable by a closed form of type* (p,q)*. Then there is a direct sum decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X). \tag{8}$$

Moreover $H^{p,q}(X) = \overline{H^{q,p}(X)}$*.*

Hence if we take $V_{\mathbb{Z}} = H^k(X,\mathbb{Z})$ (modulo torsion) we have that $V_{\mathbb{C}} = H^k(X,\mathbb{C})$ and the Hodge decomposition defines a Hodge structure of weight k.

We recall now the definition of primitive cohomology.

Definition 1.1.5. *The* primitive cohomology *is defined as*

$$H^k_{\text{prim}}(X,\mathbb{C}) = \operatorname{Ker}\left(L^{n-k+1}\colon H^k(X,\mathbb{C}) \to H^{2n-k+2}(X,\mathbb{C})\right), \quad k \leqslant n \tag{9}$$

where

$$L\colon H^k(X,\mathbb{C}) \to H^{k+2}(X,\mathbb{C})$$

is the Lefschetz operator given by multiplication with the Kähler form ω*.*

There is an induced Hodge decomposition on the space of primitive cohomology:

$$H^k_{\text{prim}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\text{prim}}(X). \tag{10}$$

Here $H^{p,q}_{\text{prim}}(X) = \operatorname{Ker}\left(L^{n-p-q+1}\colon H^{p,q}(X) \to H^{n-q+1,n-p+1}(X)\right)$ is the space of primitive (p,q)-forms.

Moreover we have a natural bilinear form given by integration. Call $\epsilon(k) := (-1)^{\frac{1}{2}k(k+1)}$ and

$$Q(\alpha,\beta) = \epsilon(k)\int_X \alpha \wedge \beta \wedge \omega^{n-k} \quad [\alpha],[\beta] \in H^k(X,\mathbb{C}). \tag{11}$$

This form is called the Hodge-Riemann form and is symmetric if k is even, alternating otherwise. It can be proved that

$$Q(H^{p,q}(X), H^{r,s}(X)) = 0 \quad \text{if } (r,s) \neq (q,p) \tag{12}$$

(hence this holds also for primitive (p,q)-forms) and

$$Q(Cu,\overline{u}) > 0 \quad \text{for } u \in H^{p,q}_{\text{prim}}(X), u \neq 0. \tag{13}$$

When the class of the Kähler form $[\omega]$ is integral, i.e. $[\omega] \in H^2(X,\mathbb{Z})$, Q as defined in (11) takes integral values on the integral classes, hence there is an integral polarized Hodge structure on the primitive cohomology of X. This is the case when X is a projective algebraic variety. The converse is also true, that is if $[\omega]$ is integral, then by Kodaira embedding theorem, X is projective.

A polarized Hodge structure can be defined also on the non-primitive cohomology $H^k(X,\mathbb{C})$ using the Lefschetz decomposition. For details see [52, Example 2.10] and [69, Corollary page 77].

1.2 VARIATIONS OF HODGE STRUCTURES

Let X be a complex manifold.

Definition 1.2.1. *A* local system *on* X *is a sheaf of abelian groups locally isomorphic to a constant sheaf of stalk* G, *where* G *is a fixed abelian group.*

Given an open cover U_i of X where the local system trivializes, the transition morphisms are given by $M_{i,j} \in \mathrm{Aut}(G)$. If H is a local system of $\mathbb{C}$-vector spaces, the tensor product $\mathcal{H} := H \otimes_{\mathbb{C}} \mathcal{O}_X$ is a sheaf of locally free $\mathcal{O}_X$-modules. The associated holomorphic vector bundle is equipped with a natural flat connection. Recall that

Definition 1.2.2. *A* connection *on* $\mathcal{H}$ *is a* $\mathbb{C}$*-linear map* $\nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X$ *satisfying the Leibniz rule*

$$\nabla(f \cdot \sigma) = f\nabla(\sigma) + df \otimes \sigma \tag{14}$$

for f *a local section of* $\mathcal{O}_X$ *and* σ *a local section of* $\mathcal{H}$.

The connection ∇ gives a map

$$\nabla \colon \mathcal{H} \otimes \Omega^1_X \to \mathcal{H} \otimes \Omega^2_X \tag{15}$$

defined by

$$\nabla(\sigma \otimes \alpha) = \nabla(\sigma) \wedge \alpha + \sigma \otimes d\alpha. \tag{16}$$

Definition 1.2.3. *The* curvature *of the connection is defined by*

$$\Theta = \nabla \circ \nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega^2_X. \tag{17}$$

If $\Theta = 0$ *we say that the connection is* flat.

For $\sigma \in \mathcal{H}$, $\sigma = \sum \alpha_i \sigma_i$ in a local trivialization of H, we can define a connection in the following way:

$$\nabla(\sigma) = \sum \sigma_i \otimes d\alpha_i \in \mathcal{H} \otimes \Omega^1_X; \tag{18}$$

it is easy to see that this definition does not depend on the choice of the trivialization.

Definition 1.2.4. *This connection is the* Gauss-Manin *connection associated to the local system* H.

The Gauss-Manin connection associated to a local system is flat. Conversely, given a holomorphic vector bundle $\mathcal{H}$ equipped with a flat connection, the kernel $\mathrm{Ker}\,\nabla \subset \mathcal{H}$ is a local system on X. We have the following result:

Theorem 1.2.5. *The construction sketched above gives a bijective correspondence between isomorphism classes of holomorphic vector bundles equipped with a flat connection and isomorphism classes of local systems of vector spaces.*

Definition 1.2.6. *A* variation of Hodge structure of weight k *on a complex manifold* X *is given by the following data:*

1. *a local system* $H_{\mathbb{Z}}$ *of finitely generated abelian groups on* X
2. *a finite decreasing filtration* $\mathcal{F}^{\cdot}$ *of the holomorphic vector bundle* $\mathcal{H} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X$ *by holomorphic subbundles (called the* Hodge filtration*).*

These data should satisfy:

1. *for each* $x \in X$ *the filtration induced by* $\mathcal{F}^{\cdot}\mathcal{H}$ *on* $H_{\mathbb{Z},x} \otimes \mathbb{C}$ *defines a Hodge structure of weight* k*.*
2. *the Gauss-Manin connection* $\nabla\colon \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X$ *satisfies the* Griffiths' transversality condition

$$\nabla(\mathcal{F}^p\mathcal{H}) \subset \mathcal{F}^{p-1}\mathcal{H} \otimes \Omega^1_X. \tag{19}$$

In other words a variation of Hodge structure is a family of Hodge structures parametrized by a complex manifold.

A morphism of variations of Hodge structures is defined in the obvious way.

Denote by $\underline{\mathbb{Z}}$ the constant sheaf on X of stalk $\mathbb{Z}$.

Definition 1.2.7. *A* polarized variation of Hodge structure of weight k *is given by a morphism of local systems on* X

$$Q\colon H \otimes H \to \underline{\mathbb{Z}} \tag{20}$$

which induces on each fiber a polarized Hodge structure of weight k*.*

In this thesis we are interested in the geometric case which arises considering a family of complex manifolds.

Definition 1.2.8. *A* family of complex manifolds *is a proper submersive holomorphic map* $\phi\colon \mathcal{X} \to B$ *between complex manifolds.* B *is called* base of the family.

We denote by X_b the fiber $\phi^{-1}(b)$ over a point $b \in B$. If B is connected we fix a reference point $0 \in B$ and we see X_b as a deformation of the fiber $X := X_0$. In this case we say that $\mathcal{X}$ is a *family of deformations* of X. Now consider the algebra of dual numbers $\mathbb{C}[\epsilon] := \mathbb{C}[t]/(t^2)$.

Definition 1.2.9. *We say that* $X_\epsilon \to \mathrm{Spec}(\mathbb{C}[\epsilon])$ *is a* first order *(or* infinitesimal*)* deformation *of* X *if we have the following commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_\epsilon \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[\epsilon]). \end{array} \tag{21}$$

First order deformations arise naturally in the following way: let $\phi\colon \mathcal{X} \to B$ be a family of complex manifolds and

$$\mathrm{Spec}(\mathbb{C}[\epsilon]) \to B$$

a tangent vector in the origin $0 \in B$. The base change

$$\begin{array}{ccc} X_\epsilon & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \phi \\ \mathrm{Spec}(\mathbb{C}[\epsilon]) & \longrightarrow & B \end{array} \tag{22}$$

defines a first order deformation $X_\epsilon \to \mathrm{Spec}(\mathbb{C}[\epsilon])$ of X. It is not true, however, that all the first order deformations of X come from a first order neighborhood of a family $\mathcal{X} \to B$. Such deformations are said to be obstructed.

We have the following important result:

Proposition 1.2.10. *The first order deformations of* X *are parametrized by* $H^1(X, T_X)$.

The differential $\phi_*\colon T_{\mathcal{X}} \to \phi^*(T_B)$ of a family gives an exact sequence of vector bundles over X

$$0 \to T_X \to T_{\mathcal{X}|X} \to \phi^*(T_B)_{|X} \to 0. \tag{23}$$

Since $\phi^*(T_B)_{|X} = T_{B,0} \otimes \mathcal{O}_X$, we have that this sequence gives in cohomology a map

$$\rho\colon H^0(X, T_{B,0} \otimes \mathcal{O}_X) = T_{B,0} \to H^1(X, T_X). \tag{24}$$

Definition 1.2.11. *The map* $\rho\colon T_{B,0} \to H^1(X, T_X)$ *is called the* Kodaira-Spencer map *at* 0 *of the family* $\phi\colon \mathcal{X} \to B$.

The Kodaira-Spencer map associates to a tangent vector $\frac{\partial}{\partial \epsilon} \in T_{B,0}$ the corresponding first order deformation $X_\epsilon \to \mathrm{Spec}(\mathbb{C}[\epsilon])$.

The following theorem is well known and it will be essential from now on.

Theorem 1.2.12 (Ehresmann). *Let $\phi\colon \mathcal{X} \to B$ a proper submersion between differentiable manifolds, where B is a contractible manifold equipped with a base point $0 \in B$. There exists a diffeomorphism*

$$T\colon \mathcal{X} \to X_0 \times B \tag{25}$$

over B.

In the case of a family of complex manifolds this means that for every point $b \in B$ there exists an open neighborhood $U \subset B$ of b such that $\phi^{-1}(U)$ is diffeomorphic to $U \times X_b$. Now consider a ring of coefficients A (usually A will be $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) and the shaves $R^k\phi_*A$, where $R^k\phi_*$ is the k-th derived functor of ϕ_*. $R^k\phi_*A$ is the sheaf on B associated to the presheaf

$$V \to H^k(\phi^{-1}(V), A).$$

Since we can choose U to be contractible we have

$$H^k(\phi^{-1}(V), A) = H^k(V \times X_b, A) = H^k(X_b, A)$$

for a fundamental system of neighborhoods of b, hence $R^k\phi_*A$ is a local system on B locally isomorphic to the constant sheaf of stalk $H^k(X_b, A)$.

Now take $A = \mathbb{C}$ and consider a family of Kähler manifolds. Let $\mathcal{H}^k = R^k\phi_*\mathbb{C} \otimes \mathcal{O}_B$ be the holomorphic vector bundle associated to the local system $R^k\phi_*\mathbb{C}$. This vector bundle has fiber $\mathcal{H}^k_b = H^k(X_b, \mathbb{C})$ and the Hodge filtration $F^pH^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C})$ on each fiber defines holomorphic subbundles $\mathcal{F}^p\mathcal{H}^k \subset \mathcal{H}^k$.

Definition 1.2.13. *The holomorphic bundles $\mathcal{F}^p\mathcal{H}^k$ are called* Hodge bundles.

Consider the Gauss-Manin connection ∇.

Theorem 1.2.14. *The Hodge bundles satisfy the Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p\mathcal{H}^k) \subset \mathcal{F}^{p-1}\mathcal{H}^k \otimes \Omega^1_B. \tag{26}$$

It immediately follows that these data define a variation of Hodge structure on the local system $R^k\phi_*\mathbb{Z}$.

The quotients

$$\mathcal{H}^{p,q} := \mathcal{F}^p\mathcal{H}^k/\mathcal{F}^{p+1}\mathcal{H}^k \tag{27}$$

have fibers

$$\begin{aligned}\mathcal{H}_b^{p,q} &= (\mathcal{F}^p\mathcal{H}^k)_b/(\mathcal{F}^{p+1}\mathcal{H}^k)_b = F^pH^k(X_b,\mathbb{C})/F^{p+1}H^k(X_b,\mathbb{C}) \\ &= H^{p,q}(X), \quad p+q=k.\end{aligned} \tag{28}$$

Thanks to Theorem 1.2.14, the Gauss-Manin connection induces maps

$$\overline{\nabla}^{p,q}\colon \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_B. \tag{29}$$

As ∇ satisfies the Leibniz rule (14), we have that for $\sigma \in \mathcal{F}^p\mathcal{H}^k$ and $f \in \mathcal{O}_B$

$$\nabla(f\cdot\sigma) = f\nabla(\sigma) \mod \mathcal{F}^p\mathcal{H}^k \otimes \Omega^1_B,$$

hence

$$\overline{\nabla}^{p,q}(f\cdot\sigma) = f\overline{\nabla}^{p,q}(\sigma)$$

for $\sigma \in \mathcal{H}^{p,q}$. In particular we have that $\overline{\nabla}^{p,q}$ is a morphism of $\mathcal{O}_B$-modules which gives on the fibers

$$\overline{\nabla}_b^{p,q}\colon H^{p,q}(X) \to H^{p-1,q+1}(X) \otimes \Omega^1_{B,b}.$$

If we think of $\overline{\nabla}_b^{p,q}$ as a map

$$\overline{\nabla}_b^{p,q}\colon H^{p,q}(X) \otimes T_{B,b} \to H^{p-1,q+1}(X),$$

we have an explicit description given by contraction:

$$\overline{\nabla}_b^{p,q}(\sigma\otimes v) = \sigma\cdot\rho(v) \tag{30}$$

where ρ is the Kodaira-Spencer map introduced in 1.2.11.

Definition 1.2.15. *The maps* $\overline{\nabla}_b^{p,q}$ *are called* infinitesimal variations of Hodge structure.

1.3 THE PERIOD MAP AND THE PERIOD DOMAIN

Let X be a Kähler manifold and $\phi\colon \mathcal{X} \to B$ a family of deformations of X. Up to restricting B we may assume that all the fibers X_b are Kähler manifolds by [66, Theorem 9.23] and that the Hodge numbers are constant by [66, Section 9.3.2]. We can also assume that B is contractible, hence by Ehresmann theorem 1.2.12, we have

$$H^k(X,\mathbb{C}) \cong H^k(X_b,\mathbb{C})$$

for every $b \in B$.

Definition 1.3.1. *The map*

$$\begin{aligned} \mathcal{P}^{p,k} \colon B &\to \mathrm{Grass}(b^{p,k}, H^k(X, \mathbb{C})) \\ b &\mapsto F^p H^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C}) \cong H^k(X, \mathbb{C}), \end{aligned} \tag{31}$$

where $b^{p,k} = \dim F^p H^k(X_b, \mathbb{C})$*, is called* period map.

Theorem 1.3.2 (Griffiths)**.** *The period map* $\mathcal{P}^{p,k}$ *is holomorphic for all* $p, k, p \leqslant k$.

Consider now the differential of the period map at a point $b \in B$. The tangent space to the Grassmannian $\mathrm{Grass}(r, V)$ at a point W is $\mathrm{Hom}(W, V/W)$, hence

$$d\mathcal{P}^{p,k} \colon T_{B,b} \to \mathrm{Hom}(F^p H^k(X_b, \mathbb{C}), H^k(X, \mathbb{C})/F^p H^k(X_b, \mathbb{C})). \tag{32}$$

Actually it holds

Proposition 1.3.3. *The image of* $d\mathcal{P}^{p,k}$ *is contained in*

$$\mathrm{Hom}\left(\frac{F^p H^k(X_b, \mathbb{C})}{F^{p+1} H^k(X_b, \mathbb{C})}, \frac{F^{p-1} H^k(X_b, \mathbb{C})}{F^p H^k(X_b, \mathbb{C})}\right).$$

By the identification

$$\frac{F^i H^k(X_b, \mathbb{C})}{F^{i+1} H^k(X_b, \mathbb{C})} \cong H^{i,k-i}(X)$$

we have that

$$d\mathcal{P}^{p,k} \colon T_{B,b} \to \mathrm{Hom}(H^{p,k-p}(X_b), H^{p-1,k-p+1}(X_b)). \tag{33}$$

The differential of the period map is explicitly described by contraction:

$$d\mathcal{P}^{p,k}(\nu)(\omega) = \omega \cdot \rho(\nu) \tag{34}$$

where $\nu \in T_{B,b}$, $\omega \in H^{p,k-p}(X_b)$ and ρ is the Kodaira-Spencer map. Hence the infinitesimal variation of Hodge structure can be regarded as the differential of the period map (cf. (30)).

Alternatively, using the Dolbeault isomorphism

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X), \tag{35}$$

see [66, Corollary 4.38], the differential of the period map is

$$d\mathcal{P}^{p,k} \colon T_{B,b} \to \mathrm{Hom}(H^{k-p}(X_b, \Omega^p_X), H^{k-p+1}(X_b, \Omega^{p-1}_X)). \tag{36}$$

Definition 1.3.4. *The* flag space $F_{b^\cdot} H^k(X, \mathbb{C})$ *parametrizes the decreasing filtrations* $F^\cdot$ *on* $H^k(X, \mathbb{C})$ *such that* $\dim F^p H^k(X, \mathbb{C}) = b^{p,k}$.

Of course if X is Kähler, the Hodge filtration

$$0 = F^{k+1}H^k(X,\mathbb{C}) \subset \cdots \subset F^pH^k(X,\mathbb{C}) \subset \\ \subset F^{p-1}H^k(X,\mathbb{C}) \subset \cdots \subset F^0H^k(X,\mathbb{C}) = H^k(X,\mathbb{C}) \tag{37}$$

is an element of the flag space.

The flag space can be realized as a subspace of the product of Grassmannian

$$\prod_{p=1}^{k} \mathrm{Grass}(b^{p,k}, H^k(X,\mathbb{C}))$$

consisting of the k-tuples $(W^1,\dots,W^k)$ such that $W^i \subset W^{i-1}$. In this way it can be seen that the flag space is a complex manifold.

Putting together the period maps $\mathcal{P}^{p,k}$ for $p = 1,\dots,k$, we obtain a map, which is also called period map,

$$\begin{aligned} \mathcal{P}^k \colon B &\to F_{b^\cdot}H^k(X,\mathbb{C}) \\ b &\mapsto (\mathcal{P}^{1,k}(b),\dots,\mathcal{P}^{k,k}(b)). \end{aligned} \tag{38}$$

By Theorem 1.3.2 we immediately have that $\mathcal{P}^k$ is holomorphic. By (34), its differential is

$$d\mathcal{P}^k \colon T_{B,b} \to \bigoplus_p \mathrm{Hom}(H^{p,k-p}(X_b), H^{p-1,k-p+1}(X_b)). \tag{39}$$

Since we have assumed at the beginning of this section that X_b is Kähler for every $b \in B$, it follows that the filtration $\mathcal{P}^k(b)$ satisfies also

$$F^pH^k(X_b,\mathbb{C}) \oplus \overline{F^{k-p+1}H^k(X_b,\mathbb{C})} = H^k(X_b,\mathbb{C}). \tag{40}$$

This condition defines an open set $\mathcal{D} \subset F_{b^\cdot}H^k(X,\mathbb{C})$ and the image of $\mathcal{P}^k$ is contained in $\mathcal{D}$.

Definition 1.3.5. *$\mathcal{D}$ is called* period domain.

A similar construction can be done in the case of primitive cohomology. Consider a family $\phi\colon \mathcal{X} \to B$ and assume that there exists a form $\omega \in H^2(\mathcal{X},\mathbb{Z})$ which restricted to X_b is a Kähler class for every $b \in B$. Under these hypotheses, the primitive cohomology $H^k_{\mathrm{prim}}(X_b,\mathbb{C})$ admits a polarized Hodge structure, as we have seen in the previous sections. Furthermore the intersection form Q is compatible with the identification $H^k(X_b,\mathbb{C}) \cong H^k(X,\mathbb{C})$.

The Hodge filtration on the primitive cohomology $H^k_{\mathrm{prim}}(X_b,\mathbb{C})$ satisfies

1. $F^pH^k_{prim}(X_b,\mathbb{C}) \oplus \overline{F^{k-p+1}H^k_{prim}(X_b,\mathbb{C})} = H^k_{prim}(X_b,\mathbb{C})$ as in the non-polarized case
2. the orthogonal complement of $F^pH^k_{prim}(X_b,\mathbb{C})$ with respect to Q is $F^{k-p+1}H^k_{prim}(X_b,\mathbb{C})$
3. $Q(Cu,\overline{u}) > 0 \quad$ for $u \in H^{p,q}_{prim}(X_b), u \neq 0$.

See Definition 1.1.3.

Definition 1.3.6. *The filtrations of* $H^k_{prim}(X_b,\mathbb{C})$ *which satisfy 1,2 and 3 define a set* $\mathcal{D}_{pol} \subset F_{b^\cdot}H^k_{prim}(X,\mathbb{C})$ *called* polarized period domain.

We remark that condition 2 is described by holomorphic equations on $F_{b^\cdot}H^k_{prim}(X,\mathbb{C})$; conditions 1 and 3 are open conditions on the set of filtrations described by condition 2.

Definition 1.3.7. *The map*

$$\mathcal{P}^k_{pol} \colon B \to \mathcal{D}_{pol} \tag{41}$$

which to $b \in B$ *associates the Hodge filtration on the primitive cohomology* $H^k_{prim}(X_b,\mathbb{C}) \cong H^k_{prim}(X,\mathbb{C})$ *is called* polarized period map.

As in the non-polarized case, the polarized period map is holomorphic. Its differential is given by contraction on the primitive cohomology:

$$d\mathcal{P}^k_{pol} \colon T_{B,b} \to \bigoplus_p \mathrm{Hom}(H^{p,k-p}_{prim}(X_b), H^{p-1,k-p+1}_{prim}(X_b)). \tag{42}$$

The period maps we have introduced so far are often called local, because of the local nature of the base B. In this thesis we are mostly interested in local period maps, but for the sake of completeness we recall the notion of global period map. The difficulty of constructing global period maps lies in the monodromy of B. In fact different homotopy classes of paths in B may induce distinct isomorphisms of cohomology groups $H^k(X,\mathbb{Z}) \cong H^k(X_b,\mathbb{Z})$. More precisely to the local system $R^k\phi_*\mathbb{Z}$ is associated a representation

$$\pi^1(B,0) \to \mathrm{Aut}(H^k(X,\mathbb{Z})). \tag{43}$$

For details see [66, Corollary 3.10].

Definition 1.3.8. *This representation is called* monodromy representation *and its image is usually denoted by* $\Gamma \subset \mathrm{Aut}(H^k(X, \mathbb{Z}))$.

The period map $\mathcal{P}^k \colon B \to \mathcal{D}$ as defined above associates to $b \in B$ the Hodge filtration on $H^k(X_b, \mathbb{C})$ seen as a filtration on $H^k(X, \mathbb{C})$ via the isomorphism $H^k(X, \mathbb{C}) \cong H^k(X_b, \mathbb{C})$. Since this isomorphism is no longer unique, we have that such a map is multivalued. The multivaluedness is controlled exactly by the monodromy representation, hence

$$\mathcal{P}^k \colon B \to \mathcal{D}/\Gamma \tag{44}$$

is univalued.

Definition 1.3.9. *The map* $\mathcal{P}^k \colon B \to \mathcal{D}/\Gamma$ *is called* global period map.

In the polarized case, the monodromy representation takes values in the orthogonal group of the quadratic form Q:

$$\pi^1(B, 0) \to \mathrm{Aut}(H^k(X, \mathbb{Q}), Q). \tag{45}$$

Also in this case a global period map as (44) can be constructed. Such maps are *locally liftable* and holomorphic, that is for every point $b \in B$ there is a open neighborhood U and a holomorphic lifting

$$\widetilde{\mathcal{P}^k} \colon U \to \mathcal{D} \tag{46}$$

of $\mathcal{P}^k$. Furthermore any such lifting satisfies Griffiths' transversality conditions. Conversely, any map $B \to \mathcal{D}/\Gamma$ which is holomorphic, locally liftable and whose local liftings satisfy Griffiths' transversality conditions defines a variation of Hodge structure with monodromy contained in Γ. See [10, Lemma-Definition 4.5.3].

1.4 TORELLI PROBLEMS

Before introducing the so called Torelli problems we need to recall a few more facts about families of deformations.

Consider a family $\phi \colon \mathcal{X} \to B$ with central fiber $X_0 = X$ and a holomorphic map $B' \to B$ preserving base points. Then the fiber product $\mathcal{X} \times_B B'$ yields a new deformation of X.

Definition 1.4.1. *The family* $\mathcal{X} \times_B B' \to B'$ *is called* pullback family *or* family induced by ϕ.

Definition 1.4.2. *A family* $\phi\colon \mathcal{X} \to B$ *is called* complete *if any other deformation of* X_0 *is induced by* ϕ*. If the inducing map is unique we say that the deformation is* universal*. If its differential at the base point is unique it is called* versal*.*

Versal families are unique up to isomorphism.

We need the following theorem

Theorem 1.4.3 (Kuranishi)**.** *For any compact complex manifold* X *there exists a versal deformation with a bijective Kodaira-Spencer map.*

Definition 1.4.4. *This family is called* Kuranishi family*.*

Under some hypotheses, for example $H^2(X, T_X) = 0$, we can assume that the base of the Kuranishi family is smooth.

The classical Torelli theorem for curves states that a smooth curve of genus $g > 1$ is determined up to isomorphism by its Jacobian (as a polarized abelian variety) or, equivalently, by the polarized Hodge structure on the first cohomology.

For higher-dimensional varieties we can formulate, in analogy with the case of curves, the *global Torelli problem*. Denote by $\mathcal{M}(X)$ the set of equivalence classes of complex structures on a differentiable manifold X. We have a global period map $\mathcal{M}(X) \to \mathcal{D}/\Gamma$ where $\mathcal{D}$ denotes as usual the classifying space of polarized Hodge structures and Γ is the subgroup of the orthogonal group $\mathrm{Aut}(H^k(X,\mathbb{Q}), Q)$ given by the elements that fix $H^k(X,\mathbb{Z})$. The global Torelli problem asks if this map is injective, that is if we can reconstruct the complex structure on X starting from its Hodge structure. If $\mathcal{M}(X)$ can be endowed with the structure of a complex analytic space in a functorial way, we have that the period map is holomorphic.

The *local Torelli problem* asks if $\mathcal{M}(X) \to \mathcal{D}/\Gamma$ is a local embedding and the *generic Torelli problem* asks whether the period map is generically injective. See [11] and [20].

The k*-infinitesimal Torelli problem* asks if the differential of the local period map $\mathcal{P}^k\colon B \to \mathcal{D}$ for the Kuranishi family $\phi\colon \mathcal{X} \to B$ is injective at the point $0 \in B$. In this case we say that the k-infinitesimal Torelli theorem holds for X_0.

Assume that the base B of the Kuranishi family is smooth, we have seen in (34) that the differential of the period map is the composition of the Kodaira-Spencer map and the map given by cup product:

$$d\mathcal{P}^k\colon T_{B,0} \to H^1(X, T_X) \to \bigoplus_p \mathrm{Hom}(H^{k-p}(X, \Omega_X^p), H^{k-p+1}(X, \Omega_X^{p-1})). \tag{47}$$

Since by definition of Kuranishi family the Kodaira-Spencer map is bijective, the infinitesimal Torelli problem is reduced to the study of the map given by the cup product

$$H^1(X, T_X) \to \bigoplus_p \mathrm{Hom}(H^{k-p}(X, \Omega_X^p), H^{k-p+1}(X, \Omega_X^{p-1})). \tag{48}$$

We now describe another useful way to interpret this problem. If X is a smooth variety, the cohomology space $H^1(X, T_X)$ is isomorphic to $\mathrm{Ext}^1(\Omega_X^1, \mathcal{O}_X)$ which is the group that classifies up to isomorphism exact sequences of the form

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \Omega_X^1 \to 0, \tag{49}$$

where $\mathcal{E}$ is a locally free sheaf of rank $\dim X + 1$. Such a sequence is also called extension. The idea to prove this isomorphism is the following: take sequence (49) and dualize it to obtain

$$0 \to T_X \to \mathcal{E}^\vee \to \mathcal{O}_X \to 0. \tag{50}$$

From the associated long exact sequence in cohomology, we obtain a map

$$\delta\colon H^0(X, \mathcal{O}_X) \to H^1(X, T_X).$$

Call ξ the image of $1 \in H^0(X, \mathcal{O}_X)$ via δ, that is $\xi := \delta(1)$. The class $\xi \in H^1(X, T_X)$ is exactly the element of $H^1(X, T_X)$ corresponding to the exact sequence (49). We have that $\xi = 0$ if and only if the extension (49) splits, that is $\mathcal{E} \cong \mathcal{O}_X \oplus \Omega_X^1$. The space $H^1(X, T_X)$ also parametrizes the first order deformations of X (see Proposition 1.2.10). In fact sequence (50) is isomorphic to the tangent sequence

$$0 \to T_X \to T_{X_\epsilon|X} \to \mathcal{O}_X \to 0 \tag{51}$$

where $X_\epsilon \to \mathrm{Spec}(\mathbb{C}[\epsilon])$ is the first order deformation associated to ξ. Consequently sequence (49) is

$$0 \to \mathcal{O}_X \to \Omega^1_{X_\epsilon|X} \to \Omega_X^1 \to 0. \tag{52}$$

Furthermore the p-th wedge product of (49) is the exact sequence

$$0 \to \Omega_X^{p-1} \to \bigwedge^p \mathcal{E} \to \Omega_X^p \to 0. \tag{53}$$

This sequence is associated to an element of $\mathrm{Ext}^1(\Omega_X^p, \Omega_X^{p-1})$. It can be proved that this group is again isomorphic to $\mathrm{Ext}^1(\Omega_X^1, \mathcal{O}_X)$, that is we have

$$\mathrm{Ext}^1(\Omega_X^p, \Omega_X^{p-1}) \cong \mathrm{Ext}^1(\Omega_X^1, \mathcal{O}_X) \cong H^1(X, T_X). \tag{54}$$

The class associated to (53) via these isomorphisms is again ξ.

The coboundary maps arising in cohomology by these exact sequences

$$H^q(X, \Omega_X^p) \to H^{q+1}(X, \Omega_X^{p-1})$$

are all given by contraction by ξ.

Hence the infinitesimal Torelli problem for X, that is the injectivity of (48), is reduced to the following question: if all the coboundary maps

$$H^{k-p}(X, \Omega_X^p) \to H^{k-p+1}(X, \Omega_X^{p-1}) \tag{55}$$

are zero, can we deduce that the class ξ is zero? In this thesis we will mostly face the problem in this form.

1.5 INFINITESIMAL TORELLI FOR SMOOTH PROJECTIVE HYPERSURFACES

Take a smooth hypersurface $V \subset \mathbb{P}^n$ defined by a homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ of degree d. Denote by S the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ and by $\mathcal{J}$ the *Jacobian ideal* of F, that is the ideal of S generated by the partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 0, \dots, n$. Both are graded by the degree: $S = \bigoplus S^k$ and $\mathcal{J} = \bigoplus \mathcal{J}^k$, where $S^k = \mathbb{C}[x_0, \dots, x_n]_k$ and $\mathcal{J}^k = \mathcal{J} \cap S^k$. The *Jacobian ring* is by definition the quotient $R = S/\mathcal{J}$. It is also graded in the obvious way.

Consider $U_{n,d}$ the open subset of $\mathbb{P}(S^d)$ parametrizing smooth hypersurfaces of degree d. Our hypersurface V is a point of $U_{n,d}$, with tangent space

$$T_{U_{n,d},V} = S^d/\mathbb{C} \cdot F. \tag{56}$$

Explicitly, every element of the tangent space is the equivalence class of a curve $F + tG$ and two such curves $F + tG$ and $F + tG'$ are equivalent if and only if $G - G'$ is a multiple of F. It is not difficult to see that the tangent vectors to the $GL(n+1)$ orbit passing through V are exactly given by the Jacobian ideal

$$\mathcal{J}^d/\mathbb{C} \cdot F \subset S^d/\mathbb{C} \cdot F.$$

Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_V(d) \to 0. \tag{57}$$

Since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ we have an isomorphism

$$H^0(V, \mathcal{O}_V(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))/\mathbb{C} \cdot F = T_{U_{n,d},V}.$$

The incidence variety $I := \{(U, x) \in U_{n,d} \times \mathbb{P}^n \mid x \in U\}$ with the natural projection onto $U_{n,d}$ gives a family $I \to U_{n,d}$. The Kodaira-Spencer map of this family is given by the coboundary of the normal exact sequence

$$0 \to T_V \to T_{\mathbb{P}^n|V} \to \mathcal{O}_V(d) \to 0, \tag{58}$$

that is

$$H^0(V, \mathcal{O}_V(d)) \to H^1(V, T_V). \tag{59}$$

It associates to $G \in H^0(V, \mathcal{O}_V(d))$ the first order deformation of equation $F + tG = 0$, $t^2 = 0$. By Bott's Vanishing Theorem, $H^1(V, T_{\mathbb{P}^n|V}) = 0$ for $n \geqslant 4$ or $n = 3$ and $d \neq 4$, hence the Kodaira-Spencer map is surjective in these cases. Its kernel is given by the degree d part of the Jacobian ideal $\mathcal{J}^d$. In particular we have the following

Lemma 1.5.1. *If $n \geqslant 4$ or $n = 3$ and $d \neq 4$ there is an isomorphism*

$$H^1(V, T_V) \cong R^d. \tag{60}$$

It turns out that the Kodaira-Spencer map restricted to the slice transversal to the $GL(n+1)$ orbit is an isomorphism and we obtain a universal family for our smooth hypersurface.

Furthermore we have the following result (see [67, Section 6])

Theorem 1.5.2. *There is an isomorphism for every p*

$$H^{n-p,p-1}_{\text{prim}}(V) \cong R^{dp-n-1}. \tag{61}$$

Under the previous identifications, the infinitesimal variation of Hodge structure

$$H^1(V, T_V) \otimes H^{n-p,p-1}_{\text{prim}}(V) \to H^{n-p-1,p}_{\text{prim}}(V) \tag{62}$$

is given by the polynomial multiplication

$$R^d \otimes R^{dp-n-1} \to R^{d(p+1)-n-1}. \tag{63}$$

The study of the infinitesimal Torelli problem in the case of smooth hypersurfaces of degree d is then reduced to the study of the injectivity of the map

$$R^d \to \bigoplus_p \text{Hom}(R^{(n-p)d-n-1}, R^{(n-p+1)d-n-1}) \tag{64}$$

given by the standard polynomial multiplication. Macaulay's theorem gives an answer to this problem.

Theorem 1.5.3 (Macaulay). *Set* $N = (n+1)(d-2)$. *Then* $R^N \cong \mathbb{C}$ *and for every integer* k, *the pairing*

$$R^k \times R^{N-k} \to R^N \tag{65}$$

is perfect.

As a corollary we obtain

Corollary 1.5.4. *For integers* a, b *such that* $b \geqslant 0$ *and* $a + b \leqslant N$, *the map given by the product*

$$R^a \to \mathrm{Hom}(R^b, R^{a+b}) \tag{66}$$

is injective.

Remark 1.5.5. *Macaulay's theorem actually holds in a more general setting. Take* $S = \mathbb{C}[x_0, \ldots, x_n]$ *as before and* $\mathcal{J}$ *the ideal of* S *generated by a sequence of* $n+1$ *homogeneous polynomials* G_i *of degree* $d_i > 0$ *without common zeroes. We say that such a sequence is a* regular sequence. *Taking* $R = S/\mathcal{J}$ *and* $N = \sum_{i=0}^{n} d_i - n - 1$, *Theorem 1.5.3 and its Corollary still hold.*

Using Corollary 1.5.4, the Infinitesimal Torelli theorem holds for smooth hypersurfaces if there exist $p \geqslant 0$ such that $(n-p)d - n - 1 \geqslant 0$ and $(n-p+1)d - n - 1 \leqslant N = (n+1)(d-2)$. This is always the case except for $d = 2$ (quadratic hypersurfaces of any dimension) and $d = 3, n = 3$ (cubic surfaces). This theorem is due to Griffiths, see [34].

2 THE ADJOINT THEORY FOR CURVES

In this chapter we discuss the theory of adjoint forms in the case of curves. First we deal with smooth curves following the lines of the original paper by Collino and Pirola [18], then we give a generalization of this theory for singular curves; in particular we are concerned with Gorenstein curves (see [60]). Some applications for Torelli-type problems are given.

2.1 INFINITESIMAL TORELLI PROBLEM FOR CURVES

Let $\phi\colon \mathcal{C} \to B$ be a smooth family of smooth curves of genus g over a complex polydisk B and let 0 be a reference point in B. As in the previous chapter, we will denote by C the fiber $C_0 = \phi^{-1}(0)$.

In the case of a curve we have that the Hodge decomposition is given by $H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$, hence the (local) period map $\mathcal{P}^1$ is simply given by

$$\begin{aligned} \mathcal{P}^1\colon B &\to \mathrm{Grass}(g, H^1(C,\mathbb{C})) \\ b &\to H^{1,0}(C_b) \subset H^1(C_b,\mathbb{C}) \cong H^1(C,\mathbb{C}). \end{aligned} \tag{67}$$

As we have seen in the previous chapter, the differential is given by the composition

$$T_{B,0} \to H^1(C, T_C) \to \mathrm{Hom}(H^0(C,\omega_C), H^1(C,\mathcal{O}_C)),$$

where ω_C is the canonical sheaf of the curve C. If ϕ is the Kuranishi family of C, then the Kodaira-Spencer map is an isomorphism and the infinitesimal Torelli problem asks if

$$H^1(C, T_C) \to \mathrm{Hom}(H^0(C,\omega_C), H^1(C,\mathcal{O}_C)) \tag{68}$$

is injective. This can be dually stated as follows. Write (68) as

$$H^1(C, T_C) \to H^0(C,\omega_C)^\vee \otimes H^1(C,\mathcal{O}_C). \tag{69}$$

The injectivity of this map is equivalent to the surjectivity of the dual map

$$H^0(C,\omega_C) \otimes H^1(C,\mathcal{O}_C)^\vee \to H^1(C,T_C)^\vee. \tag{70}$$

By Serre duality we have that

$$H^1(C, \mathcal{O}_C)^\vee \cong H^0(C, \omega_C)$$

and

$$H^1(C, T_C)^\vee \cong H^0(C, \omega_C^{\otimes 2}),$$

hence the infinitesimal Torelli problem is equivalent to the surjectivity of

$$H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2}). \tag{71}$$

This map is the natural map given by the product of sections (see [66, Lemma 10.22]).

The answer to the infinitesimal Torelli problem for curves is well known (cf. [1], [51], [65], [68]).

Theorem 2.1.1 (Infinitesimal Torelli for curves)**.** *Let C be a smooth curve of genus $g = 1, 2$ or $g \geqslant 3$ and C not hyperelliptic. Then the local period map*

$$\mathcal{P}^1 \colon B \to \mathrm{Grass}(g, H^1(C, \mathbb{C})) \tag{72}$$

for the Kuranishi family $\phi \colon \mathcal{C} \to B$ is an embedding at the point 0 corresponding to C.

Reformulating this result in terms of (71) we have the Noether Theorem

Theorem 2.1.2 (Noether)**.** *Let C be a smooth curve of genus $g = 1, 2$ or $g \geqslant 3$ and C not hyperelliptic. Then the product map*

$$H^0(C, \omega_C)^{\otimes 2} \to H^0(C, \omega_C^{\otimes 2}) \tag{73}$$

is surjective.

Actually in the case of smooth curves the answer to the global Torelli theorem is known, as we have recalled in the previous chapter. Call $\mathcal{M}_g$ the coarse moduli space of complete nonsingular curves of genus g and $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. Then the Torelli map $\tau_g \colon \mathcal{M}_g \to \mathcal{A}_g$ which sends the isomorphism class of a curve to the isomorphism class of its Jacobian is injective.

Concerning singular curves, there is a large literature about the problem of extending the Torelli map to a morphism

$$\overline{\tau_g} \colon \overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}$$

where $\overline{\mathcal{M}_g}$ is the Deligne-Mumford compactification of $\mathcal{M}_g$ (see [50] and [9]) and $\overline{\mathcal{A}_g}$ is a suitable compactification of $\mathcal{A}_g$. Clearly the problem varies according to the chosen compactification of $\mathcal{A}_g$: see [9].

In the case of irreducible stable curves Namikawa proved that the canonical map from the open set of irreducible stable curves to the normalization of the Satake compactification of $\mathcal{A}_g$ is injective; see: [50, Theorem 7 page 245]. He proved that if C is an irreducible stable curve of genus g whose normalization is a non-hyperelliptic curve of genus > 2 then C is uniquely determined by its generalized Jacobian [50, Proposition 9 page 245]. He also showed that the above map can't be injective over the divisor $\mathcal{N} = \cup_{i=1}^{[\frac{g}{2}]} \mathcal{N}_i$ where $\mathcal{N}_i$ is the divisor whose general points correspond to stable curves with two non-singular irreducible components C_1, C_2 with genus i and $g-i$ meeting at one point.

Note that in the singular case the dualizing sheaf ω_C and the sheaf of one forms Ω^1_C are different as we will see in more details. Furthermore we do not have the isomorphism

$$\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \cong H^1(C, T_C)$$

and we have to work directly with the group $\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$. The map

$$\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Hom}(H^0(C, \Omega^1_C), H^1(C, \mathcal{O}_C)), \tag{74}$$

corresponding to (68) in the singular case, is constructed considering the short exact sequence associated to an element of $\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ and taking the coboundary map in cohomology. Hence, (74) fails to be injective if there exists $\xi \in \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$, $\xi \neq 0$ such that for one of the associated extensions

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \Omega^1_C \to 0 \tag{75}$$

the coboundary homomorphism $\delta_\xi \colon H^0(C, \Omega^1_C) \to H^1(C, \mathcal{O}_C)$ is trivial.

In this chapter we study the injectivity of (74). In particular we focus on the general case of Gorenstein curves.

2.2 ADJOINT THEORY FOR SMOOTH CURVES

In this section we introduce the theory of adjoint forms in the case of smooth curves. We recall the basic definitions and constructions as they were introduced by Collino and Pirola in

[18]. This theory can be generalized in higher dimension and is suitable for the study of Torelli type problems and variations of Hodge structure, as we will see in the following chapters. We will also present a generalization of the adjoint theory for Gorenstein curves with the aim of tackling the problem given by the injectivity of (74).

Let C be a smooth curve of genus g and consider a first order deformation of C. As we have seen in Proposition 1.2.10, it corresponds to an element $\xi \in H^1(C, T_C)$ and to an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \omega_C \to 0 \tag{76}$$

via the isomorphism $H^1(C, T_C) \cong \mathrm{Ext}^1(\omega_C, \mathcal{O}_C)$.

From (76) we have the long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(\mathcal{O}_C) = \mathbb{C} \to H^0(\mathcal{E}) \to H^0(\omega_C) \xrightarrow{\delta_\xi} H^1(\mathcal{O}_C) \to \\ \to H^1(\mathcal{E}) \to H^1(\omega_C) \to 0. \end{aligned} \tag{77}$$

We assume that the kernel of δ_ξ is of dimension at least 2. Since the idea is to use this theory for Torelli type problems, this assumption is not restrictive, because usually δ_ξ is the zero map.

Fix now a subspace $W \subset \mathrm{Ker}\,\delta_\xi$ of dimension 2 and a basis $\mathcal{B} = \{\eta_1, \eta_2\}$ of W. Since η_1 and η_2 are in the kernel of δ_ξ, by the exactness of (77), we can pick liftings $s_1, s_2 \in H^0(C, \mathcal{E})$ of η_1, η_2.

Following [18], we denote by α the composition

$$\alpha\colon \bigwedge^2 H^0(C, \mathcal{E}) \to H^0(C, \bigwedge^2 \mathcal{E}) \to H^0(C, \omega_C).$$

The first arrow $\bigwedge^2 H^0(C, \mathcal{E}) \to H^0(C, \bigwedge^2 \mathcal{E})$ is simply given by the wedge product on sections of $\mathcal{E}$. The second one follows from the isomorphism

$$\bigwedge^2 \mathcal{E} \cong \omega_C \tag{78}$$

arising from sequence (76). Since $\mathcal{E}$ is a locally free sheaf of rank 2 on C, $\bigwedge^2 \mathcal{E}$ is its determinant sheaf, hence we will denote it also by $\det \mathcal{E}$.

Definition 2.2.1. *The form*

$$\omega_{\xi, W, \mathcal{B}} := \alpha(s_1 \wedge s_2) \in H^0(C, \omega_C) \tag{79}$$

is called adjoint form *of* ξ, W *and* $\mathcal{B}$.

Problem 2.2.2. *We want to decide whether this adjoint form $\omega_{\xi,W,\mathcal{B}}$ is in the space W. In other words, starting from η_1, η_2 we have constructed another global differential form $\omega_{\xi,W,\mathcal{B}} \in H^0(C, \omega_C)$. Is this form really different from our initial data? Or is it just a linear combination of η_1 and η_2?*

The adjoint form depends on the extension class ξ, on the space W and on the chosen basis of W as the notation suggests. It also depends on the choice of the liftings s_1 and s_2, but as we will see in a moment, this choice is not relevant when looking at Problem 2.2.2. Since we want to understand if $\omega_{\xi,W,\mathcal{B}}$ is in W, we give the following natural definition:

Definition 2.2.3. *The class $[\omega_{\xi,W,\mathcal{B}}] \in H^0(C, \omega_C)/W$ is called* adjoint image.

Remark 2.2.4. *If we choose another lifting s_1' of η_1, the forms $\alpha(s_1 \wedge s_2)$ and $\alpha(s_1' \wedge s_2)$ differ by a multiple of η_2. In fact $s_1' \wedge s_2 - s_1 \wedge s_2 = (s_1' - s_1) \wedge s_2 = c \wedge s_2$, where c is a section of $H^0(C, \mathcal{E})$ coming from $H^0(C, \mathcal{O}_C)$ (see exact sequence (77)). $\alpha(c \wedge s_2)$ is of course a multiple of η_2 since s_2 is a lifting of η_2 and c leads to a constant. Viceversa if we choose another lifting s_2' of η_2 the adjoint forms differ by a multiple of η_1. This means that the answer to Problem 2.2.2 does not depend on the choice of the liftings s_1 and s_2. In other words while the adjoint form of Definition 2.2.1 depends on the liftings, the adjoint image of Definition 2.2.3 does not.*

Remark 2.2.5. *If we choose another basis $\mathcal{B}'$ of W, both the adjoint form and the adjoint image change. Let $\mathcal{B}' = \{\eta_1', \eta_2'\}$ with*

$$\eta_i' = a_{i1}\eta_1 + a_{i2}\eta_2 \quad \textit{for } i = 1, 2.$$

Then, if s_i is a lifting of η_i, we can choose $s_i' := a_{i1}s_1 + a_{i2}s_2$ as lifting of η_i', for $i = 1, 2$. We have then

$$s_1' \wedge s_2' = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} s_1 \wedge s_2.$$

Therefore

$$[\omega_{\xi,W,\mathcal{B}'}] = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} [\omega_{\xi,W,\mathcal{B}'}]$$

and the adjoint images related to different basis differ by a nonzero multiplicative constant. Hence the adjoint image relative to $\mathcal{B}$ is zero if and only if the adjoint image relative to $\mathcal{B}'$ is zero, which means that also the choice of a basis of W is not relevant for Problem 2.2.2.

Definition 2.2.6. *Denote by* $|W| \subset \mathbb{P}H^0(C, \omega_C)$ *the linear system associated to* W. *We call* D_W *its fixed divisor.*

We want to give a local description of the adjoint form. Restrict sequence (76) to an open subset U of C where $\mathcal{E}$ and ω_C become trivial. Call dz the local generator of ω_C, and dz, dt the local generators of $\mathcal{E}$. Locally we have

$$\eta_{i|U} = a_i dz \quad \text{for } i = 1, 2 \tag{80}$$

and hence

$$s_{i|U} = a_i dz + b_i dt \quad \text{for } i = 1, 2.$$

The wedge product $s_1 \wedge s_2$ is then

$$s_1 \wedge s_{2|U} = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} dz \wedge dt$$

and the adjoint form $\omega_{\xi,W,\mathcal{B}}$

$$\omega_{\xi,W,\mathcal{B}|U} = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} dz.$$

It follows immediately that $\omega_{\xi,W,\mathcal{B}}$ vanishes on D_W.

Lemma 2.2.7. *We have an exact sequence*

$$0 \to T_C(D_W) \to \mathcal{O}_C \oplus \mathcal{O}_C \to \omega_C(-D_W) \to 0. \tag{81}$$

Proof. The sections $\eta_1, \eta_2 \in H^0(C, \omega_C)$ vanish on D_W, hence can be lifted to $H^0(C, \omega_C(-D_W))$. We call $\overline{\eta_1}$ and $\overline{\eta_2}$ the corresponding liftings. The map $T_C(D_W) \to \mathcal{O}_C \oplus \mathcal{O}_C$ is given exactly by $-\overline{\eta_2}, \overline{\eta_1}$, while the map $\mathcal{O}_C \oplus \mathcal{O}_C \to \omega_C(-D_W)$ is defined by

$$(a, b) \mapsto a\overline{\eta_1} + b\overline{\eta_2}. \tag{82}$$

The exactness of sequence (81) is straightforward. □

Lemma 2.2.8. *Sequence (81) fits into a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & T_C(D_W) & \to & \mathcal{O}_C \oplus \mathcal{O}_C & \to & \omega_C(-D_W) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_C & \to & \mathcal{E} & \to & \omega_C & \to & 0. \end{array} \tag{83}$$

Proof. Call $\overline{\omega_{\xi,W,\mathcal{B}}}$ the lifting of $\omega_{\xi,W,\mathcal{B}}$ in $H^0(C, \omega_C(-D_W))$. The first vertical map $T_C(D_W) \to \mathcal{O}_C$ is given by contraction with $\overline{\omega_{\xi,W,\mathcal{B}}}$. The second vertical map is given by s_1 and s_2.

The commutativity of the second square is immediate.

We prove locally the commutativity of the first square. For η_i and s_i we use the same notation of (80) and following. Take local expression $\overline{a_i}$ of $\overline{\eta_i}$, that is $a_i = \overline{a_i}d$, where d is a local equation of D_W. We call $f \cdot \sigma$ a local section of $T_C(D_W)$. The composition

$$T_C(D_W) \to \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{E}$$

is given by

$$f \cdot \sigma \mapsto (-\overline{a_2}f, \overline{a_1}f) \mapsto -\overline{a_2}fs_1 + \overline{a_1}fs_2 = (-\overline{a_2}b_1f + \overline{a_1}b_2f)dt.$$

On the other hand

$$T_C(D_W) \to \mathcal{O}_C \to \mathcal{E}$$

is given by

$$f \cdot \sigma \mapsto f(-\overline{a_2}b_1 + \overline{a_1}b_2) \mapsto (-\overline{a_2}b_1f + \overline{a_1}b_2f)dt.$$

□

Definition 2.2.9. *We say that the* $\xi \in H^1(C, T_C)$ *is* supported on a divisor D *if its image via* $H^1(C, T_C) \to H^1(C, T_C(D))$ *is zero.*

This can be interpreted as follows. Call ρ_D the map

$$\rho_D \colon H^1(C, T_C) \cong \mathrm{Ext}^1(\omega_C, \mathcal{O}_C) \to H^1(C, T_C(D)) \cong \mathrm{Ext}^1(\omega_C(-D), \mathcal{O}_C).$$

$\rho_D(\xi)$ corresponds to an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \omega_C(-D) \to 0 \tag{84}$$

which fits in the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_C & \to & \mathcal{E}' & \to & \omega_C(-D_W) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_C & \to & \mathcal{E} & \to & \omega_C & \to & 0. \end{array} \tag{85}$$

If $\rho_D(\xi) = 0$, it means that the top row of this diagram splits. Of course this does not imply that also the bottom row splits.

We can prove the main theorem of this section. It is called *Adjoint Theorem* (cf. [18, Theorem 1.1.8]) and relates the condition of ξ being supported on D_W and Problem 2.2.2.

Theorem 2.2.10 (Adjoint Theorem for smooth curves)**.** *Let* C *be a smooth curve. Consider* $\xi \in H^1(C, T_C)$ *associated to the extension (76). Define* $W = \langle \eta_1, \eta_2 \rangle \subset \mathrm{Ker}\,(\delta_\xi) \subset H^0(C, \omega_C)$ *and* $\omega_{\xi,W,\mathcal{B}}$ *as above. Call* ξ_{D_W} *the image of* ξ *via the morphism* ρ_{D_W}*. We have that* $[\omega_{\xi,W,\mathcal{B}}] = 0$ *if and only if* $\xi_{D_W} = 0$*.*

Proof. Take the dual of diagram (83)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_C & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_C(D_W) & \longrightarrow & \mathcal{O}_C \oplus \mathcal{O}_C & \longrightarrow & \omega_C(-D_W) & \longrightarrow & 0. \end{array}$$

In cohomology we have

$$\begin{array}{ccccc} H^0(\mathcal{E}^\vee) & \longrightarrow & H^0(\mathcal{O}_C) = \mathbb{C} & \xrightarrow{\alpha} & H^1(T_C) \\ \downarrow & & \downarrow \beta & & \downarrow \rho \\ H^0(\mathcal{O}_C)^{\oplus 2} = \mathbb{C}^2 & \xrightarrow{\nu} & H^0(\omega_C(-D_W)) & \xrightarrow{\gamma} & H^1(T_C(D_W)). \end{array}$$

Since $\alpha(1) = \xi$, as we have seen in the previous chapter, we have that $\rho(\alpha(1)) = \xi_{D_W}$. On the other side $\beta(1) = \overline{\omega_{\xi,W,\mathcal{B}}}$ (see Lemma 2.2.8). By commutativity it follows that $\gamma(\overline{\omega_{\xi,W,\mathcal{B}}}) = \xi_{D_W}$, hence $\xi_{D_W} = 0$ if and only if $\overline{\omega_{\xi,W,\mathcal{B}}} \in \operatorname{Im} \nu$. Recall that ν is given by $\overline{\eta_1}$ and $\overline{\eta_2}$. Hence $\xi_{D_W} = 0$ if and only if $\overline{\omega_{\xi,W,\mathcal{B}}} \in \langle \overline{\eta_1}, \overline{\eta_2} \rangle$ and this is obviously equivalent to $\omega_{\xi,W,\mathcal{B}} \in \langle \eta_1, \eta_2 \rangle = W$. □

We have an immediate corollary

Corollary 2.2.11. *If* $D_W = 0$, $[\omega_{\xi,W,\mathcal{B}}] = 0$ *if and only if* $\xi = 0$.

The Adjoint Theorem and its corollary are suitable for the study of Torelli problems because they may be used to prove that (68) is injective. Actually one can reprove, using this techniques, the infinitesimal Torelli theorem 2.1.1. This is done in [15]. In this thesis we skip this proof in the case of smooth curves as we will give a generalization in the case of Gorenstein curves.

2.3 GORENSTEIN CURVES

In this section we recall some useful preliminaries on Gorenstein curves.

Definition 2.3.1. *A* Gorenstein *curve is a reduced connected projective scheme* C *of pure dimension* 1 *such that the dualizing sheaf* ω_C *is invertible.*

Given a Gorenstein curve C, we can define the sheaf of Kähler differentials Ω^1_C; see for instance [40, Chapter II Section 8]. This sheaf is not locally free as in the smooth case and in general it has a torsion part. On the other hand the dualizing sheaf ω_C is, by definition of Gorenstein curve, a locally free invertible sheaf on the curve C.

The sheaves Ω^1_C and ω_C are isomorphic outside the singular locus of C and this isomorphism can be completed to a morphism $\rho\colon \Omega^1_C \to \omega_C$, see [31, Page 244]. The morphism ρ fits into the following exact sequence:

$$0 \to K \to \Omega^1_C \xrightarrow{\rho} \omega_C \to N \to 0. \tag{86}$$

The kernel and the cokernel of ρ, denoted by K and N respectively, are torsion sheaves supported on the singularities.

Notation 2.3.2. *As customary, we denote by* $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ *the group of morphisms between* $\mathcal{F}$ *and* $\mathcal{G}$*, and by* $\mathcal{H}om(\mathcal{F},\mathcal{G})$ *the sheaf given by*

$$U \mapsto \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U}).$$

In the same fashion, $\mathrm{Ext}^1(\mathcal{F},\mathcal{G})$ *is the right derived functor of* $\mathrm{Hom}(\mathcal{F},\mathcal{G})$*, while* $\mathcal{E}xt^1(\mathcal{F},\mathcal{G})$ *is the right derived functor of* $\mathcal{H}om(\mathcal{F},\mathcal{G})$*.*

As in the smooth case, the group $\mathrm{Ext}^1(\Omega^1_C, O_C)$ parametrizes the first order deformations of C (cf. [63, Corollary 1.1.11]). The following result gives some important information on this group:

Proposition 2.3.3. *We have an exact sequence*

$$0 \to H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \xrightarrow{\mu} R \to 0 \tag{87}$$

where

$$R \cong H^0(C, \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)) \cong \mathrm{Ext}^1(K, \mathcal{O}_C). \tag{88}$$

Furthermore $\mathrm{Ext}^1(\omega_C, \mathcal{O}_C)$ *surjects onto the kernel of* μ*.*

Proof. This sequence can be obtained by the five term sequence associated to the local to global spectral sequence of Ext's. This spectral sequence relates the sheaf $\mathcal{E}xt$ with the group Ext:

$$E_2^{p,q} = H^p(\mathcal{E}xt^q(\Omega^1_C, \mathcal{O}_C)) \Rightarrow H^{p+q} = \mathrm{Ext}^{p+q}(\Omega^1_C, \mathcal{O}_C). \tag{89}$$

The five term exact sequence associated to this spectral sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2 \tag{90}$$

gives in our case

$$0 \to H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \\ \to H^0(C, \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)) \to 0.$$

Here we also give an alternative proof because we will use some of its steps throughout the next sections.

Sequence (86) splits into the following short exact sequences:

$$0 \to K \to \Omega^1_C \to \hat{\omega} \to 0 \tag{91}$$

and

$$0 \to \hat{\omega} \to \omega_C \to N \to 0. \tag{92}$$

Dualizing (91) we have

$$0 \to \mathcal{H}om(\hat{\omega}, \mathcal{O}_C) \to \mathcal{H}om(\Omega^1_C, \mathcal{O}_C) \to 0 \to \mathcal{E}xt^1(\hat{\omega}, \mathcal{O}_C) \to \\ \to \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C) \to \mathcal{E}xt^1(K, \mathcal{O}_C) \to 0 \tag{93}$$

and in particular we deduce that the dual of the sheaf Ω^1_C is isomorphic to the dual of the sheaf $\hat{\omega}$. Taking the dual of (92) we obtain the long exact sequence

$$0 \to \mathcal{H}om(\omega_C, \mathcal{O}_C) \to \mathcal{H}om(\hat{\omega}, \mathcal{O}_C) \to \mathcal{E}xt^1(N, \mathcal{O}_C) \to \\ \to \mathcal{E}xt^1(\omega_C, \mathcal{O}_C) \to \mathcal{E}xt^1(\hat{\omega}, \mathcal{O}_C) \to 0.$$

Since the dualizing sheaf is locally free we have that

$$\mathcal{E}xt^1(\omega_C, \mathcal{O}_C) \cong \mathcal{E}xt^1(\mathcal{O}_C, \omega_C^\vee) = 0,$$

see [40, Chapter III, Propositions 6.3 and 6.7]. Hence we deduce that

$$\mathcal{E}xt^1(\hat{\omega}, \mathcal{O}_C) = 0;$$

see also [39, Lemma 1.1].

Now we apply the functor $\mathrm{Hom}(-, \mathcal{O}_C)$ to (91) and we have the exact sequence

$$0 \to \mathrm{Hom}(\hat{\omega}, \mathcal{O}_C) \to \mathrm{Hom}(\Omega^1_C, \mathcal{O}_C) \to 0 \to \mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C) \to \\ \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Ext}^1(K, \mathcal{O}_C) \to 0.$$

In particular

$$0 \to \mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathrm{Ext}^1(K, \mathcal{O}_C) \to 0 \tag{94}$$

is exact. Note that the sheaf K is supported on the singularities. On the other hand, the kernel $\mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C)$ parametrizes the extensions of the sheaf $\hat{\omega}$ by the structure sheaf $\mathcal{O}_C$, that is isomorphism classes of exact sequences of the form

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \hat{\omega} \to 0. \tag{95}$$

The sheaves of this sequence are torsion free and hence reflexive (see [39, Lemma 1.1]). This implies that taking the dual sequence gives a bijective correspondence between extensions as (95) and their dual

$$0 \to \hat{\omega}^{\vee} \to \mathcal{E}^{\vee} \to \mathcal{O}_C \to 0.$$

This means that the group $\mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C)$ is isomorphic to

$$\mathrm{Ext}^1(\mathcal{O}_C, \hat{\omega}^{\vee}) \cong H^1(C, \hat{\omega}^{\vee}).$$

Since

$$\hat{\omega}^{\vee} \cong \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)$$

as seen in (93), we conclude that

$$H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) \cong \mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C). \tag{96}$$

Sequence (94) is then our desired sequence

$$0 \to H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to R \to 0.$$

For the last statement, apply the functor $\mathrm{Hom}(-, \mathcal{O}_C)$ to (92) to obtain

$$\begin{aligned} 0 \to \mathrm{Hom}(\omega_C, \mathcal{O}_C) \to \mathrm{Hom}(\hat{\omega}, \mathcal{O}_C) \to \mathrm{Ext}^1(N, \mathcal{O}_C) \to \\ \to \mathrm{Ext}^1(\omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C) \to 0. \end{aligned}$$

□

2.3.1 *Nodal curves*

Nodal curves are an interesting example of Gorenstein curves.

Definition 2.3.4. *A point in a projective curve is a* node *if it has a neighborhood in the analytic topology which is isomorphic to a neighborhood of the origin in the space* $(xy = 0) \subset \mathbb{C}^2$. *A* nodal curve *is a curve with only nodes as singularities.*

Given a nodal curve C, we denote by $\nu \colon \widetilde{C} \to C$ its normalization. In this case the situation described above is more explicit.

Around a node P given locally by $xy = 0$, the sheaf of Kähler differentials Ω^1_C is generated by dx and dy with the relation $y dx + x dy = 0$; see [2, Chapter X].

On the other hand the dualizing sheaf is defined as follows. Consider $P_1, \ldots, P_n$ the nodes of the curve and $Q_1, Q_1', \ldots, Q_n, Q_n'$

their preimages in the normalization $\widetilde{C}$ of C. Then ω_C is the subsheaf of

$$\nu_*(\omega_{\widetilde{C}}(\sum Q_i + Q_i'))$$

given by the sections σ with opposite residues in Q_i and Q_i', that is

$$\mathrm{Res}_{Q_i}(\sigma) + \mathrm{Res}_{Q_i'}(\sigma) = 0.$$

A local generator for ω_C in a neighborhood of a node is by adjunction $\frac{dx \wedge dy}{F}$, where F is a local equation for the curve.

Locally near a node P, the map $\rho \colon \Omega^1_C \to \omega_C$ is given by

$$\rho(dx) = x\frac{dx \wedge dy}{F}$$

and

$$\rho(dy) = -y\frac{dx \wedge dy}{F}.$$

The stalk K_P of the kernel of ρ is the $\mathbb{C}$-vector space generated by $x dy = -y dx$. To understand the cokernel N_P, we note that the image of ρ, denoted by $\hat{\omega}$ as in the general case, is generated by the ideal (x, y) in ω_C, that is $\hat{\omega} = (x, y) \cdot \omega_C$. N_P is then $\omega_C / (x, y) \cdot \omega_C$.

By the explicit description of K in the nodal case, (87) becomes

$$0 \to H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \bigoplus_i \mathbb{C}_{P_i} \to 0; \tag{97}$$

see for instance [2, Chapter XI].

The meaning of this sequence is that there are two kinds of first order deformations of C. The deformations coming from the direct sum $\bigoplus_i \mathbb{C}_{P_i}$ are those that give the smoothing of the nodes. More precisely the generator of $\mathbb{C}_{P_i}$ corresponds to the infinitesimal deformation given by $xy = \epsilon$ around the node P_i glued together with the trivial deformation outside a neighborhood of P_i.

On the other hand the deformations coming from

$$H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C))$$

are locally trivial around the nodes. They can also be seen as deformations of the pointed curve $(\widetilde{C}, Q_1, Q_1', \ldots, Q_n, Q_n')$. This comes from the fact that

$$\mathcal{H}om(\Omega^1_C, \mathcal{O}_C) \cong \nu_*(T_{\widetilde{C}}(-\sum(Q_i + Q_i'))). \tag{98}$$

We want now to give a Torelli-type theorem for the deformations coming from the group $H^1(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C))$. For this purpose we introduce the adjoint theory in the case of a Gorenstein curve.

2.4 ADJOINT THEORY FOR IRREDUCIBLE GORENSTEIN CURVES

As a first generalization of the adjoint theory seen in Section 2.2, we work in the case of irreducible Gorenstein curves and moreover we consider extensions of arbitrary locally free sheaves of rank one.

Let $\mathcal{F}$ and $\mathcal{L}$ be two locally free sheaves of rank one on C. Consider the exact sequence of locally free sheaves

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{99}$$

associated to an element $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{L})$. Since $\mathcal{L}$ is locally free of rank one, if we tensor this sequence by $\mathcal{L}^\vee$ and take the dual, we obtain again an exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{F}^\vee \to \mathcal{L} \otimes \mathcal{E}^\vee \to \mathcal{O}_C \to 0;$$

see [40, Chapter III, Section 6]. As in the smooth case the image of $1 \in H^0(C, \mathcal{O}_C)$ via the morphism

$$H^0(C, \mathcal{O}_C) \to H^1(C, \mathcal{L} \otimes \mathcal{F}^\vee)$$

characterizes the extension and gives an isomorphism

$$\text{Ext}^1(\mathcal{F}, \mathcal{L}) \cong H^1(C, \mathcal{L} \otimes \mathcal{F}^\vee).$$

Let $\delta_\xi \colon H^0(C, \mathcal{F}) \to H^1(C, \mathcal{L})$ be the connecting homomorphism of (99), and let $W \subset \text{Ker}(\delta_\xi)$ be a vector subspace of dimension 2. Choose a basis $\mathcal{B} := \{\eta_1, \eta_2\}$ of W. By definition we can take liftings $s_1, s_2 \in H^0(C, \mathcal{E})$ of the sections η_1, η_2.

Consider the base locus of W with its natural scheme structure. It consists of a finite number of smooth points and singularities. Denote by D_W the Cartier divisor associated to the smooth points in the base locus, and by $\mathcal{I}_W$ the ideal of the singular points contained in the base locus. The ideal $\mathcal{I}_W$ is an *effective generalized divisor* on the curve C.

The theory of generalized divisor on Gorenstein curves can be found in [39].

Definition 2.4.1. *A* generalized divisor *on* C *is a nonzero subsheaf of the constant sheaf of the function field* $\mathcal{K}$ *which is also a coherent*

$\mathcal{O}_C$-module. A generalized divisor is effective *if it is a nonzero ideal of $\mathcal{O}_C$, that is if it corresponds to a 0-dimensional closed subscheme of* C.

The inverse of the generalized divisor $\mathcal{I}$ is locally given by $\mathcal{I}^{-1} := \{f \in \mathcal{K} \mid f \cdot \mathcal{I} \subset \mathcal{O}_C\}$.

Lemma 2.4.2. *There is a short exact sequence*

$$0 \to \mathcal{F}^{\vee}(D_W) \otimes \mathcal{I}_W^{-1} \to W \otimes \mathcal{O}_C \to \mathcal{F}(-D_W) \otimes \mathcal{I}_W \to 0. \quad (100)$$

Proof. This is basically Lemma 2.2.7, but here the proof is more delicate and we spell out some details. Call $\overline{\eta_1}$ and $\overline{\eta_2}$ the liftings of η_1 and η_2 in $H^0(\mathcal{F}(-D_W) \otimes \mathcal{I}_W)$. The map

$$f\colon \mathcal{F}^{\vee}(D_W) \otimes \mathcal{I}_W^{-1} \to W \otimes \mathcal{O}_C$$

is given by $-\overline{\eta_2}, \overline{\eta_1}$, while

$$g\colon W \otimes \mathcal{O}_C \to \mathcal{F}(-D_W) \otimes \mathcal{I}_W$$

is given by

$$(a, b) \mapsto a\overline{\eta_1} + b\overline{\eta_2}.$$

The surjectivity of g and the injectivity of f come from the fact that $\overline{\eta_1}, \overline{\eta_2}$ generate the sheaf $\mathcal{F}(-D_W) \otimes \mathcal{I}_W$. Obviously it holds that $\operatorname{Im} f \subset \operatorname{Ker} g$. To see the opposite inclusion take $(a, b) \in W \otimes \mathcal{O}_C$ such that $a\overline{\eta_1} + b\overline{\eta_2} = 0$, and denote by l_1, l_2 local equations of $\overline{\eta_1}, \overline{\eta_2}$. This means that $al_1 = -bl_2$. Given a point $P \in C$, possibly singular, at least one between l_1 and l_2 does not vanish in P. Assume for example that l_1 is invertible. Then

$$a = -\frac{bl_2}{l_1} \quad (101)$$

and since $\left(-\frac{b}{l_1}\right) \cdot \mathcal{F}(-D_W) \otimes \mathcal{I}_W \subset \mathcal{O}_C$, $-\frac{b}{l_1}$ is a local equation for an element of $\mathcal{F}^{\vee}(D_W) \otimes \mathcal{I}_W^{-1}$. We conclude that $(a, b) = f(\frac{-b}{l_1})$. □

Sequence (100) of Lemma 2.4.2 fits into the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}^{\vee}(D_W) \otimes \mathcal{I}_W^{-1} & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & \mathcal{F}(-D_W) \otimes \mathcal{I}_W & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle (s_1, s_2)} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0.
\end{array} \quad (102)$$

We can complete the diagram with a morphism $\omega\colon \mathcal{F}^\vee(D_W) \otimes \mathcal{I}_W^{-1} \to \mathcal{L}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^\vee(D_W)\otimes\mathcal{I}_W^{-1} & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & \mathcal{F}(-D_W)\otimes\mathcal{I}_W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle (s_1,s_2)} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0, \end{array} \tag{103}$$

that is

$$\omega \in \mathrm{Hom}(\mathcal{F}^\vee(D_W) \otimes \mathcal{I}_W^{-1}, \mathcal{L}).$$

Definition 2.4.3. *The morphism* $\omega \in \mathrm{Hom}(\mathcal{F}^\vee(D_W) \otimes \mathcal{I}_W^{-1}, \mathcal{L})$ *is called an* adjoint *of* W *and* ξ*.*

We want to study the problem:

Problem 2.4.4. *Define*

$$\Phi_{\mathcal{B}}\colon \mathrm{Hom}(W \otimes \mathcal{O}_C, \mathcal{L}) \to \mathrm{Hom}(\mathcal{F}^\vee(D_W) \otimes \mathcal{I}_W^{-1}, \mathcal{L}) \tag{104}$$

the map obtained applying the functor $\mathrm{Hom}(-,\mathcal{L})$ *to the first row of diagram (103). Is* $\omega \in \mathrm{Im}\,\Phi_{\mathcal{B}}$*?*

The morphism ω is the generalization of the adjoint form of Definition 2.2.1 in the case of a Gorenstein curve. In fact in the smooth case the adjoint form $\omega_{\xi,W,\mathcal{B}}$ gives by contraction the first vertical arrow of diagram (83) while now ω is by definition the first vertical arrow of diagram (103). Furthermore if the adjoint image $[\omega_{\xi,W,\mathcal{B}}]$ is zero, that is $\omega_{\xi,W,\mathcal{B}} = a\eta_1 + b\eta_2$ for $a, b \in \mathbb{C}$, then there is a diagonal map in diagram (83)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_C(D_W) & \longrightarrow & \mathcal{O}_C\oplus\mathcal{O}_C & \longrightarrow & \omega_C(-D_W) & \longrightarrow & 0 \\ & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & \omega_C & \longrightarrow & 0. \end{array} \tag{105}$$

given by a and b which makes the upper triangle commutative. Viceversa if such a triangle is commutative and the diagonal map is given by a, b, then $\omega_{\xi,W,\mathcal{B}} = a\eta_1 + b\eta_2$. This means that Problem 2.2.2 can be reduced to Problem 2.4.4.

Remark 2.4.5. *As in the smooth case, the morphism* ω *depends on the choice of the liftings* s_1, s_2*, whereas the condition* $\omega \in \mathrm{Im}\,\Phi_{\mathcal{B}}$ *does not. To be more precise if we change liftings* s_1', s_2' *and construct the corresponding* ω'*, we have that* $\omega \neq \omega'$ *in general, but* $\omega - \omega' \in \mathrm{Im}\,\Phi_{\mathcal{B}}$.

Remark 2.4.6. *Consider another basis* $\mathcal{B}' := \{\eta_1', \eta_2'\}$ *of* W *and let* A *be the matrix of the basis change. The sections* s_1', s_2' *obtained from* s_1, s_2 *through the matrix* A *are liftings of* η_1', η_2'*. It is easy to see composing with* A *that* $\omega \in \operatorname{Im}\Phi_{\mathcal{B}}$ *if and only if* $\omega' \in \operatorname{Im}\Phi_{\mathcal{B}'}$ *where* ω' *is the adjoint constructed from* η_1', η_2'*.*

Theorem 2.4.7 (Adjoint Theorem for Gorenstein curves)**.** *Let* C *be an irreducible Gorenstein curve. Let* $\mathcal{F}$*,* $\mathcal{L}$ *be invertible sheaves on* C*. Consider* $\xi \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{L})$ *associated to the extension (99). Assume that there exists* $W = \langle \eta_1, \eta_2 \rangle \subset \operatorname{Ker}(\delta_\xi) \subset H^0(C, \mathcal{F})$ *and define* ω *as above. Call* ξ_{D_W} *the image of* ξ *via the morphism*

$$\operatorname{Ext}^1(\mathcal{F}, \mathcal{L}) \xrightarrow{\rho} \operatorname{Ext}^1(\mathcal{F}(-D_W) \otimes \mathcal{I}_W, \mathcal{L}).$$

We have that $\omega \in \operatorname{Im}\Phi_{\mathcal{B}}$ *if and only if* $\xi_{D_W} = 0$*.*

Proof. Take diagram (103) and apply the functor $\operatorname{Hom}(-, \mathcal{L})$. We obtain

$$\begin{array}{ccccc} \operatorname{Hom}(\mathcal{E}, \mathcal{L}) & \longrightarrow & \operatorname{Hom}(\mathcal{L}, \mathcal{L}) & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}, \mathcal{L}) \\ \downarrow & & \downarrow \beta & & \downarrow \rho \\ \operatorname{Hom}(W \otimes \mathcal{O}_C, \mathcal{L}) & \xrightarrow{\Phi_{\mathcal{B}}} & \operatorname{Hom}(\mathcal{F}^\vee(D_W) \otimes \mathcal{I}_W^{-1}, \mathcal{L}) & \xrightarrow{\delta} & \operatorname{Ext}^1(\mathcal{F}(-D_W) \otimes \mathcal{I}_W, \mathcal{L}). \end{array} \tag{106}$$

Obviously $\beta(\mathrm{id}) = \omega$ and, by commutativity, $\delta(\beta(\mathrm{id})) = \xi_{D_W}$. We have then $\xi_{D_W} = 0$ if and only if $\omega \in \operatorname{Im}\Phi_{\mathcal{B}}$. □

2.4.1 *Non-singular base locus*

If the base locus of η_1 and η_2 does not contain singular points, then the situation is easier and it can be more explicitly described following [18] and Section 2.2.

Consider again the liftings $s_1, s_2 \in H^0(C, \mathcal{E})$. The natural map

$$\Lambda^2 \colon \bigwedge^2 H^0(C, \mathcal{E}) \to H^0(C, \bigwedge^2 \mathcal{E})$$

defines the section

$$\omega := \Lambda^2(s_1 \wedge s_2). \tag{107}$$

It is easy to see that ω is in the image of the natural injection $\det \mathcal{E}(-D_W) \to \det \mathcal{E}$.

Since our sheaves are locally free and the base locus does not contain the singular points, sequence (100) is

$$0 \to \mathcal{F}^\vee(D_W) \xrightarrow{i} \mathcal{O}_C \oplus \mathcal{O}_C \xrightarrow{\nu} \mathcal{F}(-D_W) \to 0. \tag{108}$$

Diagram (103) is now

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^\vee(D_W) & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & \mathcal{F}(-D_W) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle (s_1,s_2)} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0, \end{array} \qquad (109)$$

where the first vertical arrow is exactly given by contraction with ω. In fact we have that $\det \mathcal{E} \cong \mathcal{F} \otimes \mathcal{L}$, and

$$\omega \in H^0(C, \det \mathcal{E}(-D_W)) \cong H^0(C, \mathcal{F} \otimes \mathcal{L}(-D_W))$$

gives a map $\mathcal{F}^\vee(D_W) \to \mathcal{L}$.

Therefore condition $\omega \in \operatorname{Im} \Phi_{\mathcal{B}}$ can be written as

$$\omega \in \operatorname{Im}(H^0(C, \mathcal{L}) \otimes \langle s_1, s_2 \rangle \to H^0(C, \det \mathcal{E})) \qquad (110)$$

or, equivalently,

$$\omega \in \operatorname{Im}(H^0(C, \mathcal{L}) \otimes W \to H^0(C, \det \mathcal{E})). \qquad (111)$$

The first map is given by the wedge product, the second one by the fact that $\det \mathcal{E} \cong \mathcal{L} \otimes \mathcal{F}$. Note that if $H^0(C, \mathcal{L}) = 0$ this condition is equivalent to $\omega = 0$.

Remark 2.4.8. *If $\mathcal{L} = \mathcal{O}_C$, then $\det \mathcal{E} \cong \mathcal{F}$ and we can work as in Section 2.2 with the only difference that we have replaced ω_C by an arbitrary locally free sheaf $\mathcal{F}$. We will see ω as an element of $H^0(C, \mathcal{F})$ and write condition (111) as $\omega \in W$.*

From the natural map

$$\mathcal{F}^\vee \otimes \mathcal{L} \to \mathcal{F}^\vee \otimes \mathcal{L}(D_W)$$

we have a homomorphism

$$H^1(C, \mathcal{F}^\vee \otimes \mathcal{L}) \xrightarrow{\rho} H^1(C, \mathcal{F}^\vee \otimes \mathcal{L}(D_W));$$

in analogy with Theorem 2.2.10 and Theorem 2.4.7 we call $\xi_{D_W} := \rho(\xi)$.

In the case where there are no singularities in the base locus, Theorem 2.4.7 is formulated in the following easier way:

Corollary 2.4.9. *Let C be an irreducible Gorenstein curve. Let $\mathcal{F}$, $\mathcal{L}$ be invertible sheaves on C. Consider $\xi \in H^1(C, \mathcal{F}^\vee \otimes \mathcal{L})$ associated to the extension (99). Assume that there exists $W = \langle \eta_1, \eta_2 \rangle \subset \operatorname{Ker}(\delta_\xi) \subset H^0(C, \mathcal{F})$ and define ω as above. Assume also that the base locus of W does not contain singularities. We have that $\omega \in \operatorname{Im}(H^0(C, \mathcal{L}) \otimes W \to H^0(C, \det \mathcal{E}))$ if and only if $\xi_{D_W} = 0$.*

Remark 2.4.10. *If $D_W = 0$, then under the same hypothesis we have $\omega \in \operatorname{Im}(H^0(C, \mathcal{L}) \otimes W \to H^0(C, \det \mathcal{E}))$ if and only if $\xi = 0$.*

2.5 TORELLI-TYPE THEOREM

We want to study the extensions of the dualizing sheaf ω_C of a Gorenstein curve C. We need a version of the Castelnuovo theorem in the case of Gorenstein curves.

Theorem 2.5.1. *Let* C *be an irreducible Gorenstein curve,* $|D|$ *a base point free linear system of dimension* $r \geqslant 3$ *and assume that the map*

$$\phi_D \colon C \to \mathbb{P}^r$$

is birational onto the image. Then the natural map

$$\mathrm{Sym}^l H^0(C, \mathcal{O}(D)) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C(lD))$$

is surjective for $l \geqslant 0$*.*

Since $|D|$ is base point free then its general member does not contain singular points and D is a Cartier divisor. Following [3] page 151, we proceed by steps.

Lemma 2.5.2 (Castelnuovo)**.** *Let* $\mathcal{L}$ *be an invertible sheaf over* C *and* $\mathcal{D} \subset |D|$ *a base point free pencil given by* $V \subset H^0(C, \mathcal{O}(D))$*. If*

$$H^1(C, \mathcal{O}(-D) \otimes \mathcal{L}) = 0,$$

then the map

$$V \otimes H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}(D))$$

is surjective.

Proof. By the same arguments that give (108), we have the short exact sequence

$$0 \to \mathcal{O}(-D) \to V \otimes \mathcal{O} \to \mathcal{O}(D) \to 0. \tag{112}$$

Tensoring by the locally free sheaf $\mathcal{L}$ and taking the corresponding sequence in cohomology, we are done. □

Lemma 2.5.3. *Let* $\mathcal{D} \subset |D|$ *and* V *be as before. The image of the natural map*

$$\mathrm{Sym}^l V \otimes H^0(C, \omega_C) \to H^0(C, \omega_C(lD)) \tag{113}$$

has codimension $lr - 1$*.*

Proof. We proceed by induction on l.

In the case $l = 1$ the result can be obtained by a simple application of the base point free pencil trick and the Riemann-Roch theorem for Gorenstein curves.

For the case $l = 2$ consider the sequence

$$0 \to \omega_C \to V \otimes \omega_C(D) \to \omega_C(2D) \to 0 \tag{114}$$

(see the proof of the previous Lemma). Taking cohomology, it fits into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\omega_C) & \longrightarrow & V \otimes H^0(\omega_C(D)) & \longrightarrow & H^0(\omega_C(2D)) & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^0(\omega_C) & \longrightarrow & V \otimes V \otimes H^0(\omega_C) & \longrightarrow & \mathrm{Sym}^2 V \otimes H^0(\omega_C) & \longrightarrow & 0. & & \end{array}$$

We are interested in the codimension of the image of the third vertical map. It can be easily computed since we know the codimension of the image of the second vertical map by the case $l = 1$. The codimension is then $2(r-1)+1 = 2r-1$.

For the general case consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\omega_C((l-1)D)) & \longrightarrow & V \otimes H^0(\omega_C(lD)) & \longrightarrow & H^0(\omega_C((l+1)D)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Sym}^{l-1} V \otimes H^0(\omega_C) & \longrightarrow & V \otimes \mathrm{Sym}^l V \otimes H^0(\omega_C) & \longrightarrow & \mathrm{Sym}^{l+1} V \otimes H^0(\omega_C) & \longrightarrow & 0 \end{array}$$

obtained again by the same methods. The surjectivity of the first row comes from the previous lemma with $\mathcal{L} = \omega_C(lD)$. Hence we have that the codimension of the image of the last vertical map is given by the difference between the other two codimensions. By induction this means that the wanted codimension is $2(rl-1) - (r(l-1)-1) = rl + l - 1 = r(l+1) - 1$, as desired. ☐

We can now prove Theorem 2.5.1.

Proof. Take a general element $P_1 + \ldots + P_d$ of $|D|$ and define $E = P_1 + \ldots + P_{r-1}$. $|D - E|$ is a base point free pencil which gives the exact sequence

$$0 \to \mathcal{O}_C(D-E) \to \mathcal{O}_C(D) \to \mathcal{O}_C(D) \otimes \mathcal{O}_E \to 0. \tag{115}$$

By Riemann-Roch and the General position theorem ([3, Page 109]), it is easy to see that $h^1(C, \mathcal{O}_C(D-E)) = h^1(C, \mathcal{O}_C(D))$, hence have that the cohomology sequence

$$0 \to H^0(C, \mathcal{O}_C(D-E)) \to H^0(C, \mathcal{O}_C(D)) \to H^0(C, \mathcal{O}_C(D) \otimes \mathcal{O}_E) \to 0 \tag{116}$$

is exact. This sequence, tensored by with $V := H^0(C, \omega_C(lD))$, fits into the following diagram

$$\begin{array}{ccccccccc} 0 & \to & V \otimes H^0(\mathcal{O}(D-E)) & \to & V \otimes H^0(\mathcal{O}(D)) & \to & V \otimes H^0(\mathcal{O}_E(D)) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(\omega_C((l+1)D-E)) & \to & H^0(\omega_C((l+1)D)) & \to & H^0(\mathcal{O}_E(D) \otimes \omega_C(lD)) & \to & 0. \end{array}$$

We proceed by induction on l. If $l = 0$ the previous lemma tells us that the dimension of the image of $H^0(\omega_C) \otimes H^0(\mathcal{O}(D-E))$ in $H^0(\omega_C(D))$ is $g + d - r$ which is exactly the dimension of $H^0(\omega_C(D-E))$. Hence the first vertical map is surjective. Since also the third vertical map is surjective we have proved the claim for $l = 0$.

Now for $l > 0$ we proceed in the same way to prove that the middle vertical map is surjective. To prove that the first one is surjective it is enough to note that the cokernel is $H^1(\omega_C((l-1)D + E))$ which is zero. Hence we have that

$$V \otimes H^0(\mathcal{O}(D)) \to H^0(\omega_C((l+1)D)) \tag{117}$$

is onto for $l \geqslant 0$. We have the following diagram

$$\begin{array}{ccc} H^0(\mathcal{O}(D)) \otimes \mathrm{Sym}^{l-1} H^0(\mathcal{O}(D)) \otimes H^0(\omega_C) & \twoheadrightarrow & \mathrm{Sym}^l H^0(\mathcal{O}(D)) \otimes H^0(\omega_C) \\ \downarrow \alpha & & \downarrow \gamma \\ H^0(\mathcal{O}(D)) \otimes H^0(\omega_C((l-1)D)) & \overset{\beta}{\twoheadrightarrow} & H^0(\omega_C(lD)) \end{array}$$

α is surjective by induction, while β is surjective by (117). The surjectivity of γ follows and the theorem is proved. □

We can prove the following theorem. The version for smooth curves can be found in [15].

Theorem 2.5.4. *Let* C *be an irreducible Gorenstein curve as above, and let* $\mathcal{L}$ *be a locally free sheaf of rank one on* C*. Consider the extension*

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0 \tag{118}$$

given by an element $\xi \in \mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_C) \cong H^1(C, \mathcal{L}^\vee)$*. Call* F *the fixed part of the linear system associated to* $\mathcal{L}$ *and assume that* F *does not contain singularities; call* M *its mobile part. Assume that the map* ϕ_M *given by* M *is of degree one and that* $l := \dim |M| \geqslant 3$*. If the cohomology map*

$$H^0(C, \mathcal{E}) \to H^0(C, \mathcal{L}) \tag{119}$$

is surjective, then $\xi_F = 0$*.*

Proof. Take $P_1, \ldots, P_{l-1}$ points in general position in $\phi_M(C)$ and call $D := \sum P_i$. The hyperplanes passing through the points P_i form a pencil and we take generators η_1 and η_2 of $H^0(C, \mathcal{L}(-D-F))$.

We have the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \hat{\mathcal{E}} & \longrightarrow & \mathcal{L}(-D-F) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{L} \otimes \mathcal{O}_{D+F} & = & \mathcal{L} \otimes \mathcal{O}_{D+F} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{120}$$

The top row is an extension associated to an element

$$\hat{\xi} \in H^1(C, \mathcal{L}^\vee(D+F)).$$

The surjectivity of $H^0(C, \mathcal{E}) \to H^0(C, \mathcal{L})$ implies the surjectivity $H^0(C, \hat{\mathcal{E}}) \to H^0(C, \mathcal{L}(-D-F))$ as can be seen by the above diagram, thus the sections η_1 and η_2 can be lifted to $H^0(C, \hat{\mathcal{E}})$. Furthermore the space $W := \langle \eta_1, \eta_2 \rangle$ coincides with the whole space $H^0(C, \mathcal{L}(-D-F))$, hence an adjoint ω constructed from η_1 and η_2 is forced to be in the space W. Hence Remark 2.4.8 and Corollary 2.4.9 can be used to deduce that $\hat{\xi} = 0$, since W is base point free by the General Position Theorem; see [3, Page 109].

Thus ξ is in the kernel of the map

$$H^1(C, \mathcal{L}^\vee) \to H^1(C, \mathcal{L}^\vee(D+F)) \tag{121}$$

for every D as above. We want to show that this implies that ξ is in the kernel of

$$H^1(C, \mathcal{L}^\vee) \to H^1(C, \mathcal{L}^\vee(F)), \tag{122}$$

that is $\xi_F = 0$.

We have the injective map

$$H^0(C, \omega_C \otimes \mathcal{L}(-D-F)) \to H^0(C, \omega_C \otimes \mathcal{L}), \tag{123}$$

which factors as

$$H^0(C, \omega_C \otimes \mathcal{L}(-D-F)) \stackrel{\Phi_D}{\to} H^0(C, \omega_C \otimes \mathcal{L}(-F)) \to H^0(C, \omega_C \otimes \mathcal{L}). \tag{124}$$

We consider now ξ_F as an element of the dual of $H^0(C, \omega_C \otimes \mathcal{L}(-F))$; the fact that ξ is in the kernel of (121) means that $\operatorname{Im} \Phi_D \subset \ker \xi_F$.

Define

$$K := \left\langle \bigcup_{\substack{D \in \mathrm{Div}^+(C) \\ \deg D = l-1 \\ D \text{ general}}} \operatorname{Im} \Phi_D \right\rangle \subset H^0(C, \omega_C \otimes \mathcal{L}(-F)). \tag{125}$$

We will show that $K = H^0(C, \omega_C \otimes \mathcal{L}(-F))$ and this will prove our thesis. To do this define

$$H^0(C, \mathcal{L}(-D-F)) \stackrel{\hat{\Phi}_D}{\to} H^0(C, \mathcal{L}(-F)) \tag{126}$$

and

$$K' := H^0(C, \omega_C) \otimes \left\langle \bigcup_{\substack{D \in \mathrm{Div}^+(C) \\ \deg D = l-1 \\ D \text{ general}}} \operatorname{Im} \hat{\Phi}_D \right\rangle \subset H^0(C, \omega_C) \otimes H^0(C, \mathcal{L}(-F)). \tag{127}$$

Since $\left\langle \bigcup_{\substack{D \in \mathrm{Div}^+(C) \\ \deg D = l-1 \\ D \text{ general}}} \operatorname{Im} \hat{\Phi}_D \right\rangle$ is equal to $H^0(C, \mathcal{L}(-F))$ we have

$$K' = H^0(C, \omega_C) \otimes H^0(C, \mathcal{L}(-F)). \tag{128}$$

Now it is easy to see that the natural morphism $K' \to H^0(C, \omega_C \otimes \mathcal{L}(-F))$ factors through K as in the following diagram

$$\begin{array}{ccc} K & \hookrightarrow & H^0(C, \omega_C \otimes \mathcal{L}(-F)) \\ \uparrow & \nearrow & \\ K' & & \end{array} \tag{129}$$

The diagonal arrow is surjective by Theorem 2.5.1 and we deduce that $K = H^0(C, \omega_C \otimes \mathcal{L}(-F))$, as wanted. □

2.5.1 *Extension classes of ω_C: the Gorenstein case*

Recall that an irreducible Gorenstein curve C of arithmetic genus $p_a \geqslant 2$ is *hyperelliptic* if there exist a finite morphism $C \to \mathbb{P}^1$ of degree 2. It is well known that if $p_a \geqslant 1$, then ω_C is base point free and if $p_a \geqslant 3$, then C is not-hyperelliptic if and only if ω_C is very ample; see for example [39, Theorem 1.6].

We will need the following

Theorem 2.5.5. *If C is a non-hyperelliptic Gorenstein curve, then the homomorphism*

$$\mathrm{Sym}^l H^0(C, \omega_C) \to H^0(C, \omega_C^l) \tag{130}$$

is surjective for $l \geqslant 1$.

Proof. This is the Max Noether theorem for Gorenstein curves. It follows from Theorem 2.5.1. See also [46]. □

Theorem 2.5.6. *Let C be a non-hyperelliptic Gorenstein curve and $\xi \in H^1(C, \omega_C^\vee)$. Consider the morphism*

$$\Psi_P \colon H^1(C, \omega_C^\vee) \to H^1(C, \omega_C^\vee(P)) \tag{131}$$

associated to a smooth point $P \in C$. If $\xi_P := \Psi_P(\xi) = 0$ for every P, then $\xi = 0$.

Proof. The proof follows closely the proof of Theorem 2.5.4.

Via Serre duality applied to Ψ_P we obtain

$$\Psi_P^\vee := \Phi_P \colon H^0(C, \omega_C^{\otimes 2}(-P)) \to H^0(C, \omega_C^{\otimes 2}).$$

Following the proof of Theorem 2.5.4 we set $K := \langle \bigcup_{P \in C} \operatorname{Im} \Phi_P \rangle \subset H^0(C, \omega_C^{\otimes 2})$ and diagram (129) becomes

$$\begin{array}{ccc} K & \hookrightarrow & H^0(C, \omega_C^{\otimes 2}) \\ \uparrow & \nearrow & \\ H^0(C, \omega_C) \otimes H^0(C, \omega_C) & & \end{array} \tag{132}$$

Here the diagonal arrow is surjective by Max Noether theorem. Hence, as in the proof of Theorem 2.5.4, $K = H^0(C, \omega_C^{\otimes 2})$ and the claim follows. □

The next theorem gives the injectivity of the restriction of (74) to $H^1(C, \omega_C^\vee)$:

Theorem 2.5.7. *Let C be an irreducible Gorenstein curve of genus 2 or non-hyperelliptic of genus $\geqslant 3$, $\xi \in H^1(C, \omega_C^\vee)$ such that $\delta_\xi = 0$. Then $\xi = 0$.*

Proof. Case $p_a = 2$. Take $\xi \in H^1(C, \omega_C^\vee)$. Since $h^0(C, \omega_C) = 2$ we have that $W = \langle \eta_1, \eta_2 \rangle = H^0(C, \omega_C)$. This means that an adjoint ω constructed from η_1 and η_2 is forced to be in the space W. By Remark 2.4.8 and Corollary 2.4.9 we have immediately that $\xi_{D_W} = 0$. Since ω_C is base point free, $D_W = 0$ and we are done.

Case $p_a = 3$. Note that we can't apply Theorem 2.5.4. Assume that $p_a = 3$ and take a point $P \in C$. By [39, Proposition 1.5] we have

$$h^0(C, \omega_C(-P)) = h^0(C, \omega_C) - 1 = 2.$$

Take $\xi \in H^1(C, \omega_C^\vee)$ and consider the extension associated to ξ_P, that is the first row of the diagram

$$\begin{CD}
@. @. 0 @. 0 @. \\
@. @. @VVV @VVV @. \\
0 @>>> \mathcal{O}_C @>>> \hat{\mathcal{E}} @>>> \omega_C(-P) @>>> 0\\
@. @| @VVV @VVV @.\\
0 @>>> \mathcal{O}_C @>>> \mathcal{E} @>>> \omega_C @>>> 0\\
@. @. @VVV @VVV @.\\
@. @. \omega_C \otimes \mathcal{O}_P @= \omega_C \otimes \mathcal{O}_P @.\\
@. @. @VVV @VVV @.\\
@. @. 0 @. 0. @.
\end{CD} \tag{133}$$

By the fact that $\omega_C(-P)$ has only two linearly independent global sections it follows that if we construct the adjoint ω starting from a base η_1, η_2 of $H^0(C, \omega_C(-P))$, then the adjoint is forced to be a linear combination of η_1, η_2. Hence Corollary 2.4.9 can be applied and $\xi_P = 0$ for every $P \in C$. Note that ω_C is very ample and in particular this means that, for any $P \in C$, $\omega_C(-P)$ has no base points. By Theorem 2.5.6, $\xi = 0$ and we are done.

Case $p_a > 3$. If $p_a > 3$ we can apply Theorem 2.5.4 with $\mathcal{L} = \omega_C$. In fact ω_C is very ample and its associated map is an embedding. Also in this case the base locus is zero, hence $\xi = 0$. Note that the same proof given for $p_a = 3$ does not work in this case since $h^0(C, \omega_C(-P)) > 2$ and an adjoint ω constructed from $\eta_1, \eta_2 \in H^0(C, \omega_C(-P))$ is not forced to be a linear combination of η_1, η_2. □

2.5.2 *Infinitesimal deformations of Gorenstein curves*

The above analysis of the extensions of the dualizing sheaf ω_C of a Gorenstein curve C gives information also on the infinitesimal deformations of C. Take $\xi \in \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ in the kernel of μ in (87). By Proposition 2.3.3, we can find a lifting $\tilde{\xi} \in \mathrm{Ext}^1(\omega_C, \mathcal{O}_C)$ of ξ. Of course if $\tilde{\xi}$ is zero, then also ξ is zero. We have proved the following

Theorem 2.5.8. *Let C be an irreducible Gorenstein curve and let $\xi \in \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ be an infinitesimal deformation of C. Assume that $\xi \in \ker(\mu \colon \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to R)$, see Proposition 2.3.3. Then ξ can be lifted to an element $\tilde{\xi}$ of $\mathrm{Ext}^1(\omega_C, \mathcal{O}_C)$. Moreover if C is of genus 2 or non-hyperelliptic of genus $\geqslant 3$ and $\delta_{\tilde{\xi}} = 0$ then $\xi = 0$.*

Remark 2.5.9. *In general it seems not a trivial problem to use the condition $\delta_{\tilde{\xi}} = 0$ to obtain $\delta_\xi = 0$.*

2.6 REDUCIBLE NODAL CURVES

Unlike the irreducible case, for reducible Gorenstein curves there is no hope to have, in general, an infinitesimal Torelli-type theorem. Consider C a nodal curve with two components meeting transversely in n nodes. Note that all the results of Section 2.3 still hold in this case.

Sequence (97) and isomorphism (98) give the short exact sequence

$$\begin{aligned} 0 \to H^1(C_1, T_{C_1}(-\textstyle\sum P_i)) \oplus H^1(C_2, T_{C_2}(-\textstyle\sum P_i)) \to \\ \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \bigoplus_{i=1}^{n} \mathbb{C}_{P_i} \to 0 \end{aligned} \tag{134}$$

where C_1 and C_2 are the disjoint components of the normalization of C. From now on call $D := \sum P_i$ the divisor on C_i, $i = 1, 2$, induced by the intersection points.

As in the irreducible case we study the first order deformations coming from the kernel of this sequence, that is the deformations $\xi \in \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ which can be written as $\xi = \xi_1 \oplus \xi_2$, with $\xi_i \in H^1(C_i, T_{C_i}(-D))$ associated to an exact sequence

$$0 \to \mathcal{O}_{C_i} \to \mathcal{E}_i \to \omega_{C_i}(D) \to 0. \tag{135}$$

Note that the curves C_i are smooth, hence $\Omega^1_{C_i}$ is the dualizing sheaf ω_{C_i} of C_i.

Definition 2.6.1. *We say that* ξ *satisfies the* split liftability condition *if the deformations* ξ_i, $i = 1, 2$, *lift all the corresponding global sections in (135), that is* $\delta_{\xi_1} = \delta_{\xi_2} = 0$.

In the final subsections of this chapter we present explicit cases where even if ξ satisfies the split liftability condition, $\xi \neq 0$.

To see which kind of deformations can be used to provide such examples note that the image of ξ_i in $H^1(C_i, T_{C_i})$ gives an extension of ω_{C_i} which fits into the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{C_i} & \to & \widehat{\mathcal{E}}_i & \to & \omega_{C_i} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{C_i} & \to & \mathcal{E}_i & \to & \omega_{C_i}(D) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \omega_{C_i} \otimes \mathcal{O}_D & = & \omega_{C_i} \otimes \mathcal{O}_D & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \qquad (136)$$

Now assume that the map $H^0(C_i, \mathcal{E}_i) \to H^0(C_i, \omega_{C_i}(D))$ is surjective, that is $\delta_{\xi_i} = 0$, then it is easy to see that $H^0(C_i, \widehat{\mathcal{E}}_i) \to H^0(C_i, \omega_{C_i})$ is also surjective. If, for example, C_i is of genus 2 or non-hyperelliptic of genus $\geqslant 3$, then we deduce by the infinitesimal Torelli theorem for smooth curves that the first row of (136) splits. Hence ξ_i is in the kernel of

$$H^1(C_i, T_{C_i}(-D)) \to H^1(C_i, T_{C_i}).$$

By the exact sequence

$$0 \to T_{C_i}(-D) \to T_{C_i} \to T_{C_i|D} \to 0 \qquad (137)$$

we know that this kernel is not zero if $g(C_i) > 1$.

Thus the natural place to look for non-trivial split-liftable deformations is the kernel of (137).

2.6.1 *Split liftability and nonzero deformations*

Consider $C = R \cup \hat{C}$ with R a smooth rational curve and $\hat{C}$ an arbitrary smooth curve meeting transversely in n points. We want to study how the situation varies with n.

If $n = 1$, sequence (134) is

$$0 \to H^1(\hat{C}, T_{\hat{C}}(-P)) \oplus H^1(R, T_R(-P)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathbb{C}_P \to 0.$$

We write $\xi = \xi_{\hat{C}} \oplus \xi_R$. In this case $\omega_{\hat{C}}(P)$ has the base point P hence even if the extension $\xi_{\hat{C}}$ lifts everything, it may be different from zero and supported on P; see Theorem 2.5.4.

If $n = 2$, the situation is more complicated. Sequence (134) is

$$0 \to H^1(\hat{C}, T_{\hat{C}}(-P-Q)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \mathbb{C}_P \oplus \mathbb{C}_Q \to 0$$

because $H^1(R, T_R(-P-Q)) = g(R) = 0$ since R is rational. This means that our deformation $\xi \in \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ comes from an element $\xi_{\hat{C}} \in H^1(\hat{C}, T_{\hat{C}}(-P-Q))$ (recall that we are studying the deformations in the kernel of (134)). Note that the sheaf $\omega_{\hat{C}}(P+Q)$ is base point free. Now if $\hat{C}$ is of genus 2 or non-hyperelliptic of genus $\geqslant 3$ the map induced by the linear system $H^0(\hat{C}, \omega_{\hat{C}}(P+Q))$ is of degree one since the canonical map of $\hat{C}$ is already of degree one. Hence in this case Theorem 2.5.4 applies and the lifting hypothesis $\delta_{\xi_{\hat{C}}} = 0$ implies $\xi_{\hat{C}} = 0$. If $\hat{C}$ is hyperelliptic, the same conclusions holds if $|P+Q|$ is not the g^1_2 linear system. In fact the map given by $H^0(C, \omega_{\hat{C}}(P+Q))$ has degree $\leqslant 2$ since the canonical map has degree 2 and we prove that it has degree 2 only if $|P+Q|$ is the g^1_2. In fact assume that the degree is 2, then the image of $\hat{C}$ in $\mathbb{P}(H^0(\omega_{\hat{C}}(P+Q))^\vee) = \mathbb{P}^g$ is a (nondegenerate) curve of degree g. Hence it is projectively isomorphic to the rational normal curve; see [37, Page 179]. The morphism factors as

$$\hat{C} \xrightarrow{2:1} \mathbb{P}^1 \to \mathbb{P}^g$$

and $|P+Q|$ is exactly the g^1_2.

If $n = 3$, sequence (134) is

$$0 \to H^1(\hat{C}, T_{\hat{C}}(-\sum_{i=1}^{3} P_i)) \to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \bigoplus_{i=1}^{3} \mathbb{C}_{P_i} \to 0$$

since $H^1(R, T_R(-\sum P_i)) = 0$ by duality. Hence as in the previous case we deal only with $\xi_{\hat{C}}$. Assume that $\delta_{\xi_{\hat{C}}} = 0$. Since $\omega_{\hat{C}}(\sum P_i)$ is very ample, by Theorem 2.5.4 we have that $\xi_{\hat{C}} = 0$ and we deduce that $\xi = 0$. This means that in this case the split liftability condition implies that $\xi = 0$.

On the other hand assume that $n > 3$. Sequence (134) is

$$0 \to H^1(\hat{C}, T_{\hat{C}}(-\sum P_i)) \oplus H^1(R, T_R(-\sum P_i)) \to$$
$$\to \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to \bigoplus_{i=1}^{n} \mathbb{C}_{P_i} \to 0.$$

Now $H^1(R, T_R(-\sum P_i)) \neq 0$, so take a nonzero element ξ_R. It corresponds to a sequence

$$0 \to \mathcal{O}_R \to \mathcal{E}_R \to \omega_R(\sum P_i) \to 0.$$

By the fact that $H^1(R, \mathcal{O}_R) = 0$, we immediately deduce that $\delta_{\xi_R} = 0$, whereas $\xi_R \neq 0$. Hence taking $\xi = 0 \oplus \xi_R$ we have that in this case the split liftability condition does not imply that $\xi = 0$.

2.6.2 *Liftability of global sections and nonzero deformations*

Take a plane curve $C = C_1 + C_2$ where C_1 is a smooth quadric and C_2 is a smooth cubic. As before we are interested in the deformations $\xi \in \mathrm{Ext}^1(\hat{\omega}, \mathcal{O}_C) = H^1(C, \hat{\omega}^\vee)$. To show that infinitesimal Torelli theorem does not hold in this case we have to show that the map

$$H^1(C, \hat{\omega}^\vee) \to H^0(C, \hat{\omega})^\vee \otimes H^1(C, \mathcal{O}_C) \tag{138}$$

is not injective. This map is exactly (74) restricted to $H^1(C, \hat{\omega}^\vee)$. Dualizing, we have to show that

$$H^0(C, \omega_C) \otimes H^0(C, \hat{\omega}) \to H^0(C, \hat{\omega} \otimes \omega_C) \tag{139}$$

is not surjective.

By [2, Sequence 2.13 page 91] we have that $\hat{\omega} \cong \nu_*(\omega_{\tilde{C}})$, so we can easily compute the dimensions of the vector spaces appearing in (139). We obtain that $h^0(C, \omega_C) = 6$ and $h^0(C, \hat{\omega}) = 1$. On the other hand $\nu^*(\omega_C) = \omega_{\tilde{C}}(D)$ where D is the divisor induced by the intersection points on the normalization $\tilde{C}$; see [2, Page 101]. Hence we have by the projection formula

$$h^0(\hat{\omega} \otimes \omega_C) = h^0(\nu_*(\omega_{\tilde{C}}) \otimes \omega_C) = h^0(\nu_*(\omega_{\tilde{C}} \otimes \omega_{\tilde{C}}(D))) = 9.$$

Hence (139) is not surjective.

Another example is given by $C = C_1 + C_2$ with C_1 a smooth quartic and C_2 a line. We consider again the map

$$H^0(C, \omega_C) \otimes H^0(C, \hat{\omega}) \to H^0(C, \hat{\omega} \otimes \omega_C).$$

This map fits into the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ H^0(C, \omega_C) \otimes H^0(C, \hat{\omega}) & \longrightarrow & H^0(C, \hat{\omega} \otimes \omega_C) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^2 H^0(C, \omega_C) & \longrightarrow & H^0(C, \omega_C^{\otimes 2}). \end{array} \tag{140}$$

The second row is surjective by [4, Theorem 1] and [23, Theorem 3.3] and it is easy to compute that its kernel, which we denote by K, has dimension 6. Call V the image of $H^0(C, \omega_C) \otimes H^0(C, \hat{\omega})$ in $\mathrm{Sym}^2 H^0(C, \omega_C)$. V has dimension 15 whereas $h^0(C, \hat{\omega} \otimes \omega_C) = 11$, hence it will be enough to prove that $K' := \mathrm{Ker}\,(V \to H^0(C, \hat{\omega} \otimes \omega_C))$ has dimension $\geqslant 5$.

We have that $h^0(C, \hat{\omega}) = 3$ and $h^0(C, \omega_C) = 6$, hence take a basis $\{l_1, l_2, l_3\}$ of $H^0(C, \hat{\omega})$ and complete it to a basis

$$\{l_1, l_2, l_3, b_1, b_2, b_3\}$$

of $H^0(C, \omega_C)$. The sheaf ω_C is by adjunction $\mathcal{O}_C(2)$ and its global sections can be seen as quadrics restricted to C. Recall by Section 2.3 that $\hat{\omega}$ is the subsheaf of ω_C consisting of the sections vanishing on the nodes of C, therefore l_1, l_2, l_3 are quadrics vanishing on the four nodes of C, and hence on the line C_2.

Now the elements of K are quadrics vanishing on the canonical image of $C \subset \mathbb{P}^5$ and an element of K is in K' if it is zero when restricted on the plane $(l_1 = l_2 = l_3 = 0) \subset \mathbb{P}^5$. The image of C_2 in $\mathbb{P}^5$ is a smooth rational curve contained in this plane and it is easy to see by the linear independence of $l_1, l_2, l_3, b_1, b_2, b_3$ that the dimension of K' is at most one less than the dimension of K, hence we are done.

Remark 2.6.2. *As we have seen at the beginning of this chapter, in the case of smooth curves the infinitesimal Torelli theorem is equivalent to the Noether theorem. This is no longer true for singular curves as these examples clearly show. Look at diagram (140), the second row is surjective (this is the Noether theorem) while the first row is not (hence the infinitesimal Torelli does not hold).*

The Noether theorem still holds in many (singular) cases. For irreducible curves see [46], for reducible curves see [4] and [23].

3 THE ADJOINT THEORY IN HIGHER DIMENSION

The theory of adjoint forms for higher dimensional varieties was introduced by Pirola and Zucconi in [55]. In this chapter we present a slight generalization (see [59]) and we point out an interesting correlation among adjoint forms, quadrics vanishing on the canonical image of a variety and Torelli problems.

3.1 THE ADJOINT FORM

Let X be a compact complex smooth variety of dimension m and let $\mathcal{F}$ be a locally free sheaf of rank n on X. Since this theory was created with the purpose of studying the sheaf of holomorphic 1-forms on X, that is $\mathcal{F} = \Omega^1_X$, in [55] there is the additional hypothesis that $m = n$. Here we want to point out that this assumption is not necessary, hence we will allow the case $m \neq n$. Fix an extension class $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ associated to the exact sequence:

$$0 \to \mathcal{O}_X \stackrel{d\epsilon}{\to} \mathcal{E} \to \mathcal{F} \to 0. \tag{141}$$

Giving a morphism $\mathcal{O}_X \to \mathcal{E}$ is equivalent to the choice of a global section $d\epsilon \in H^0(X, \mathcal{E})$, hence we denote the morphism and the associated section in the same way. The Koszul resolution associated to the section $d\epsilon \in H^0(X, \mathcal{E})$ is given by taking the wedge product with this section:

$$0 \to \mathcal{O}_{X_0} \stackrel{d\epsilon}{\to} \mathcal{E} \stackrel{\wedge d\epsilon}{\to} \bigwedge^2 \mathcal{E} \stackrel{\wedge d\epsilon}{\to} \dots \stackrel{\wedge d\epsilon}{\to} \det \mathcal{E} \stackrel{\wedge d\epsilon}{\to} 0.$$

This long exact sequence splits into $n+1$ short exact sequences,

$$0 \to \mathcal{O}_X \stackrel{d\epsilon}{\to} \mathcal{E} \stackrel{\rho_1}{\to} \mathcal{F} \to 0, \tag{142}$$

$$0 \to \mathcal{F} \stackrel{d\epsilon}{\to} \bigwedge^2 \mathcal{E} \stackrel{\rho_2}{\to} \bigwedge^2 \mathcal{F} \to 0, \tag{143}$$

$$\dots$$

$$0 \to \bigwedge^{n-1} \mathcal{F} \stackrel{d\epsilon}{\to} \bigwedge^{n} \mathcal{E} \stackrel{\rho_n}{\to} \det \mathcal{F} \to 0, \tag{144}$$

$$0 \to \det \mathcal{F} \stackrel{d_\xi}{\to} \det \mathcal{E} \to 0 \to 0, \tag{145}$$

each corresponding to ξ via the natural isomorphism

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong \mathrm{Ext}^1(\bigwedge^i \mathcal{F}, \bigwedge^{i-1} \mathcal{F})$$

where $i = 0, \dots, n$. In particular (145) gives an isomorphism $\det \mathcal{F} \cong \det \mathcal{E}$ corresponding to (78) of the previous chapter. As in the case of curves we have an isomorphism

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{F}^\vee) \cong H^1(X, \mathcal{F}^\vee)$$

and we still denote by ξ the element of $H^1(X, \mathcal{F}^\vee)$ which corresponds to our extension (141). The coboundary homomorphisms

$$\delta_\xi^{i,j} \colon H^i(X, \bigwedge^j \mathcal{F}) \to H^{i+1}(X, \bigwedge^{j-1} \mathcal{F})$$

are computed by the cup product with ξ. We denote $\delta_\xi^{0,1}$ simply by δ_ξ.

Call Λ^{n+1} the natural map given by wedge product

$$\Lambda^{n+1} \colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{E}). \tag{146}$$

Composing with the isomorphism (145) we have a map

$$\Lambda \colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{E}) \to H^0(X, \det \mathcal{F}). \tag{147}$$

Let $W \subset \mathrm{Ker}\,(\delta_\xi) \subset H^0(X, \mathcal{F})$ be a vector subspace of dimension $n+1$ and let $\mathcal{B} := \{\eta_1, \dots, \eta_{n+1}\}$ be a basis of W. By definition we can take liftings $s_1, \dots, s_{n+1} \in H^0(X, \mathcal{E})$ such that $\rho_1(s_i) = \eta_i$. If we consider the natural map

$$\lambda^n \colon \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F}),$$

we can define the subspace $\lambda^n W \subset H^0(X, \det \mathcal{F})$ generated by

$$\omega_i := \lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1})$$

for $i = 1, \dots, n+1$.

Definition 3.1.1. *The section*

$$\omega_{\xi,W,\mathcal{B}} := \Lambda(s_1 \wedge \dots \wedge s_{n+1}) \in H^0(X, \det \mathcal{F}).$$

is an adjoint form *of* $\xi, W, \mathcal{B}$.

Definition 3.1.2. *The class*

$$[\omega_{\xi,W,\mathcal{B}}] \in \frac{H^0(X, \det \mathcal{F})}{\lambda^n W}$$

is an adjoint image *of* W *by* ξ*.*

Notation 3.1.3. *Sometimes, when there is no ambiguity, we will denote* $\omega_{\xi,W,\mathcal{B}}$ *simply by* ω*.*

The problem we want to study is the analogue of Problem 2.2.2:

Problem 3.1.4. *Is the adjoint form* $\omega_{\xi,W,\mathcal{B}}$ *in the subspace* $\lambda^n W$*? Equivalently, is the adjoint image* $[\omega_{\xi,W,\mathcal{B}}]$ *equal to* 0*?*

Concerning the well-posedness of this problem we have the following remarks:

Remark 3.1.5. *The class* $[\omega_{\xi,W,\mathcal{B}}]$ *depends on* ξ*,* W *and* $\mathcal{B}$ *only. The form* $\omega_{\xi,W,\mathcal{B}}$ *depends also on the choice of the liftings* $s_1, \dots, s_{n+1}$*. Hence the answer to Problem 3.1.4 does not depend on the choice of the liftings. Cf. Remark 2.2.4.*

Remark 3.1.6. *If we consider another basis* $\mathcal{B}' = \{\eta'_1, \dots, \eta'_{n+1}\}$ *of* W*, then* $[\omega_{\xi,W,\mathcal{B}}] = k[\omega_{\xi,W,\mathcal{B}'}]$ *where* k *is the determinant of the matrix of the change of basis. In particular* $[\omega_{\xi,W,\mathcal{B}}] = 0$ *if and only if* $[\omega_{\xi,W,\mathcal{B}'}] = 0$*. Cf. Remark 2.2.5.*

Lemma 3.1.7. *If* $[\omega_{\xi,W,\mathcal{B}}] = 0$ *then we can find liftings* $t_i \in H^0(X, \mathcal{E})$ *of* η_i*,* $i = 1, \dots, n+1$*, such that* $\Lambda^{n+1}(t_1 \wedge \dots \wedge t_{n+1}) = 0$ *in* $H^0(X, \det \mathcal{E})$*. In particular with this choice we have* $\omega_{\xi,W,\mathcal{B}} = 0$ *in* $H^0(X, \det \mathcal{F})$*.*

Proof. Take $s_1, \dots, s_{n+1} \in H^0(X, \mathcal{E})$ liftings of $\eta_1, \dots, \eta_{n+1}$. By hypothesis there exist $a_i \in \mathbb{C}$ such that

$$\omega_{\xi,W,\mathcal{B}} = \sum_{i=1}^{n+1} a_i \omega_i = \sum_{i=1}^{n+1} a_i \cdot \lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1}). \tag{148}$$

This means that

$$\Lambda^{n+1}(s_1 \wedge \dots \wedge s_{n+1}) = \sum_{i=1}^{n+1} a_i \Lambda^{n+1}(s_1 \wedge \dots \wedge \widehat{s_i} \wedge \dots \wedge s_{n+1} \wedge d\epsilon)$$

since $\Lambda^{n+1}(s_1 \wedge \dots \wedge \widehat{s_i} \wedge \dots \wedge s_{n+1} \wedge d\epsilon)$ corresponds to $\lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1})$ via the isomorphism $\det \mathcal{F} \cong \det \mathcal{E}$. Define a new lifting for the element η_i:

$$t_i := s_i + (-1)^{n-i} a_i \cdot d\epsilon.$$

We have

$$\Lambda^{n+1}(t_1 \wedge \cdots \wedge t_{n+1}) = \Lambda^{n+1}(s_1 \wedge \cdots \wedge s_{n+1}) - \\ - \sum_{i=1}^{n+1} a_i \Lambda^{n+1}(s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_{n+1} \wedge d\epsilon) = 0.$$

□

Definition 3.1.8. *If* $\lambda^n W$ *is nontrivial we denote by*

$$|\lambda^n W| \subset \mathbb{P}(H^0(X, \det \mathcal{F}))$$

the induced sublinear system. We call D_W *the fixed divisor of this linear system and* Z_W *the base locus of its moving part*

$$|M_W| \subset \mathbb{P}(H^0(X, \det \mathcal{F}(-D_W))).$$

Definition 3.1.9. *We say that an extension* $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ *is* sup-ported on a divisor D *if*

$$\xi \in \mathrm{Ker}\, \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{F}(-D), \mathcal{O}_X).$$

See Definition 2.2.9.

3.2 LOCAL EXPRESSION OF THE ADJOINT FORM

Take an open neighborhood $U \subset X$ where the sheaves $\mathcal{E}$ and $\mathcal{F}$ of sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

trivialize. Take local generators $(\sigma_1, \ldots, \sigma_n)$ of $\mathcal{F}$. Hence $\mathcal{E}$ is generated on U by $(\sigma_1, \ldots, \sigma_n, d\epsilon)$. We can write the sections η_i

$$\eta_{i|U} = \sum_{j=1}^{n} b^i_j \cdot \sigma_j$$

and

$$s_{i|U} = \sum_{j=1}^{n} b^i_j \cdot \sigma_j + c^i \cdot d\epsilon.$$

The wedge product $\Lambda^{n+1}(s_1 \wedge \cdots \wedge s_{n+1})$ is then

$$\Lambda^{n+1}(s_1 \wedge \cdots \wedge s_{n+1})_{|U} = \begin{vmatrix} b^1_1 & \ldots & b^1_n & c^1 \\ \vdots & \ddots & \vdots & \vdots \\ b^{n+1}_1 & \ldots & b^{n+1}_n & c^{n+1} \end{vmatrix} \cdot \sigma_1 \wedge \cdots \wedge \sigma_n \wedge d\epsilon.$$

The isomorphism $\det \mathcal{E} \cong \det \mathcal{F}$ is locally given by dropping $d\epsilon$, hence

$$\omega_{\xi,W,\mathcal{B}|U} = \begin{vmatrix} b_1^1 & \dots & b_n^1 & c^1 \\ \vdots & \ddots & \vdots & \vdots \\ b_1^{n+1} & \dots & b_n^{n+1} & c^{n+1} \end{vmatrix} \cdot \sigma_1 \wedge \dots \wedge \sigma_n. \tag{149}$$

The forms ω_i can be locally computed similarly and we have that

$$\omega_{i|U} = \lambda^{n+1}(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1}) = \\ = \begin{vmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^{i-1} & \dots & b_n^{i-1} \\ b_1^{i+1} & \dots & b_n^{i+1} \\ \vdots & \ddots & \vdots \\ b_1^{n+1} & \dots & b_n^{n+1} \end{vmatrix} \cdot \sigma_1 \wedge \dots \wedge \sigma_n \tag{150}$$

hence we always have a relation

$$\omega_{\xi,W,\mathcal{B}|U} = c^1 \cdot \omega_{1|U} + \dots + (-1)^n c^{n+1} \cdot \omega_{n+1|U}$$

but this is only local on U and the c^i's are not necessarily complex numbers.

As we have pointed out in the Introduction, the local expressions of the sections ω_i give some entries of the adjoint matrix of (149).

3.3 THE ADJOINT THEOREM

There is a natural map of Ext groups

$$\mathrm{Ext}^1(\det \mathcal{F}, \bigwedge^{n-1} \mathcal{F}) \to \mathrm{Ext}^1(\det \mathcal{F}(-D_W), \bigwedge^{n-1} \mathcal{F}) \tag{151}$$

coming from the morphism $\mathcal{F}(-D_W) \to \mathcal{F}$. Under the isomorphisms

$$H^1(X, \mathcal{F}^\vee) \cong \mathrm{Ext}^1(\det \mathcal{F}, \bigwedge^{n-1} \mathcal{F})$$

and

$$H^1(X, \mathcal{F}^\vee(D_W)) \cong \mathrm{Ext}^1(\det \mathcal{F}(-D_W), \bigwedge^{n-1} \mathcal{F}),$$

(151) is

$$H^1(X, \mathcal{F}^\vee) \stackrel{\rho}{\to} H^1(X, \mathcal{F}^\vee(D_W));$$

following the notation of the previous chapter we call $\xi_{D_W} = \rho(\xi)$. Hence we have an extension $\mathcal{E}^{(n)}$ corresponding to ξ_{D_W} and fitting the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{n-1}\mathcal{F} & \to & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det\mathcal{F}(-D_W) & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \cdot D_W} & & \\
0 & \to & \bigwedge^{n-1}\mathcal{F} & \to & \bigwedge^{n}\mathcal{E} & \xrightarrow{\rho_n} & \det\mathcal{F} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & = & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{152}$$

We prove a slightly more general version of [55, Theorem 1.5.1]. The proof is also slightly different because it does not use the Grothendieck duality. This theorem is the analogue of Theorem 2.2.10 for varieties of arbitrary dimension but without the assumption $\mathcal{F} = \Omega^1_X$.

Theorem 3.3.1 (Adjoint Theorem). *Let X be an m-dimensional compact complex smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\xi \in H^1(X, \mathcal{F}^\vee)$ the extension class of the exact sequence (141). Let $W \subset \operatorname{Ker}(\delta_\xi) \subset H^0(X, \mathcal{F})$ and $[\omega]$ one of its adjoint images. If $[\omega] = 0$ then $\xi \in \operatorname{Ker}(H^1(X, \mathcal{F}^\vee) \to H^1(X, \mathcal{F}^\vee(D_W)))$.*

Proof. By Remark 3.1.6, the vanishing of $[\omega]$ does not depend on the choice of a particular basis of W, hence choose an arbitrary basis $\mathcal{B} = \{\eta_1, \dots, \eta_{n+1}\}$ of W.

By hypothesis, $\omega \in \lambda^n W$, hence by Lemma 3.1.7 we can choose liftings $s_i \in H^0(X, \mathcal{E})$ of η_i with $\Lambda^{n+1}(s_1 \wedge \dots \wedge s_{n+1}) = 0$. Consider

$$s_1 \wedge \dots \wedge \widehat{s_i} \wedge \dots \wedge s_{n+1} \in \bigwedge^n H^0(X, \mathcal{E})$$

and define Ω_i its image in $H^0(X, \bigwedge^n \mathcal{E})$ through the natural map. By construction $\rho_n(\Omega_i) = \omega_i$, where we remind the reader that $\omega_i := \lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1})$. Since D_W is the fixed divisor of the linear system $|\lambda^n W|$ and the sections ω_i generate this linear system, then the ω_i's are in the image of

$$\det \mathcal{F}(-D_W) \xrightarrow{\cdot D_W} \det \mathcal{F},$$

so we can find sections $\tilde{\omega}_i \in H^0(X, \det \mathcal{F}(-D_W))$ such that

$$d \cdot \tilde{\omega}_i = \omega_i, \tag{153}$$

where d is a global section of $\mathcal{O}_X(D_W)$ with $(d) = D_W$. Hence, using the commutativity of (152), we can find liftings $\tilde{\Omega}_i \in H^0(X, \mathcal{E}^{(n)})$ of the sections Ω_i, $i = 1, \dots, n+1$.

The evaluation map

$$\bigoplus_{i=1}^{n+1} \mathcal{O}_X \xrightarrow{\tilde{\mu}} \mathcal{E}^{(n)}$$

given by the global sections $\tilde{\Omega}_i$, composed with the map α of (152), induces a map μ which fits into the following diagram

$$\begin{array}{ccccccccc} & & & & \bigoplus_{i=1}^{n+1} \mathcal{O}_X & = & \bigoplus_{i=1}^{n+1} \mathcal{O}_X & & \\ & & & & \downarrow \tilde{\mu} & & \downarrow \mu & & \\ 0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F} & \longrightarrow & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det \mathcal{F}(-D_W) & \longrightarrow & 0. \end{array} \tag{154}$$

The morphism μ is given by the multiplication by $\tilde{\omega}_i$ on the i-th component. Consider the sheaf $\operatorname{Im} \tilde{\mu}$. Locally, on an open subset U, we write

$$s_{i|U} = \sum_{j=1}^{n} b^i_j \cdot \sigma_j + c^i \cdot d\epsilon, \tag{155}$$

where $(\sigma_1, \dots, \sigma_n, d\epsilon)$ is a family of local generators of $\mathcal{E}$. By the fact that $\Lambda^{n+1}(s_1 \wedge \dots \wedge s_{n+1}) = 0$ we have

$$\begin{vmatrix} b^2_1 & \dots & b^2_n \\ \vdots & \ddots & \vdots \\ b^{n+1}_1 & \dots & b^{n+1}_n \end{vmatrix} c^1 + \dots + (-1)^n \begin{vmatrix} b^1_1 & \dots & b^1_n \\ \vdots & \ddots & \vdots \\ b^n_1 & \dots & b^n_n \end{vmatrix} c^{n+1} = 0.$$

Obviously also

$$\begin{vmatrix} b^2_1 & \dots & b^2_n \\ \vdots & \ddots & \vdots \\ b^{n+1}_1 & \dots & b^{n+1}_n \end{vmatrix} b^1_j + \dots + (-1)^n \begin{vmatrix} b^1_1 & \dots & b^1_n \\ \vdots & \ddots & \vdots \\ b^n_1 & \dots & b^n_n \end{vmatrix} b^{n+1}_j = 0$$

for $j = 1, \ldots, n$, where recall from the previous section that

$$\begin{vmatrix} b_1^2 & \ldots & b_n^2 \\ \vdots & \ddots & \vdots \\ b_1^{n+1} & \ldots & b_n^{n+1} \end{vmatrix}, \ldots, \begin{vmatrix} b_1^1 & \ldots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \ldots & b_n^n \end{vmatrix}$$

are respectively the local expressions of the sections $\omega_1, \ldots, \omega_{n+1}$. Now let d_U be a local equation of $d \in H^0(X, \mathcal{O}_X(D_W))$. By (153) we have

$$\begin{vmatrix} b_1^2 & \ldots & b_n^2 \\ \vdots & \ddots & \vdots \\ b_1^{n+1} & \ldots & b_n^{n+1} \end{vmatrix} = d_U \cdot f_1,$$

$$\vdots$$

$$\begin{vmatrix} b_1^1 & \ldots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \ldots & b_n^n \end{vmatrix} = d_U \cdot f_{n+1}$$

where the functions f_i are local expressions of $\tilde{\omega}_i$. Then we have

$$d_U \cdot (f_1 \cdot c^1 + \cdots + (-1)^n f_{n+1} \cdot c^{n+1}) = 0$$

and

$$d_U \cdot (f_1 \cdot b_j^1 + \cdots + (-1)^n f_{n+1} \cdot b_j^{n+1}) = 0$$

for $j = 1, \ldots, n$. Since by definition d_U vanishes on $D_W \cap U$, then on U

$$f_1 \cdot c^1 + \cdots + (-1)^n f_{n+1} \cdot c^{n+1} = 0$$

and

$$f_1 \cdot b_j^1 + \cdots + (-1)^n f_{n+1} \cdot b_j^{n+1} = 0$$

for $j = 1, \ldots, n$. We immediately obtain

$$f_1 \cdot s_1 + \cdots + (-1)^n f_{n+1} \cdot s_{n+1} = 0. \tag{156}$$

By definition the scheme $Z_W \cap U$ is given by $(f_1 = 0, \ldots, f_{n+1} = 0)$. Let $P \in U$ be a point not in $\mathrm{supp}(Z_W)$. At least one of the functions f_i can be inverted in a neighbourhood of P, for example let the germ of f_1 be nonzero in P. We have then a relation

$$s_1 = g_2 \cdot s_2 + \cdots + g_{n+1} \cdot s_{n+1}$$

and so we can easily find holomorphic functions h_i such that

$$\Omega_i = h_i \cdot \Omega_1$$

for $i = 2, \dots, n+1$. Since

$$\mathcal{E}^{(n)} \xrightarrow{\psi} \bigwedge^n \mathcal{E}$$

is injective, then we have

$$\tilde{\Omega}_i = h_i \cdot \tilde{\Omega}_1$$

for $i = 2, \dots, n+1$. The section $\tilde{\Omega}_1$ is nonzero, otherwise $\Omega_i = 0$ for $i = 1, \dots, n+1$ and then also $\omega_i = 0$ for $i = 1, \dots, n+1$, but this fact contradicts our hypothesis that $\lambda^n W$ is not trivial (see Definition 3.1.8). In particular we have proved that the sheaf $\operatorname{Im}\tilde{\mu}$ has rank one outside Z_W. Furthermore, since it is a subsheaf of the locally free sheaf $\mathcal{E}^{(n)}$, then $\operatorname{Im}\tilde{\mu}$ is torsion free. Denote $\operatorname{Im}\tilde{\mu}$ by $\mathcal{L}$.

By definition

$$\operatorname{Im}\mu := \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}.$$

The morphism

$$\alpha \colon \mathcal{E}^{(n)} \to \det \mathcal{F}(-D_W)$$

restricts to a surjective morphism, that we continue to call α,

$$\mathcal{L} \xrightarrow{\alpha} \operatorname{Im}\mu,$$

between two sheaves that are locally free of rank one outside Z_W. The kernel of α is then a torsion subsheaf of $\mathcal{L}$, which is torsion free, hence α gives an isomorphism $\mathcal{L} \cong \operatorname{Im}\mu$.

Since X is normal (actually it is smooth) and Z_W has codimension at least 2, we have that the inclusion

$$\mathcal{E}^{(n)} \supset (\mathcal{L}^\vee)^\vee \cong \det \mathcal{F}(-D_W)$$

gives the splitting

$$0 \longrightarrow \bigwedge^{n-1} \mathcal{F} \longrightarrow \mathcal{E}^{(n)} \longrightarrow \det \mathcal{F}(-D_W) \longrightarrow 0.$$

Since ξ_{D_W} is the element of $H^1(X, \mathcal{F}^\vee(D_W))$ associated to this extension, we conclude that $\xi_{D_W} = 0$.

□

Comparing this theorem to the Adjoint Theorem for smooth curves 2.2.10 we see that in the case of curves there is an equivalence between the conditions $\xi_{D_W} = 0$ and $[\omega = 0]$, while in this case we have only one implication. We prove now a theorem which gives an inverse of this result.

3.3.1 *An inverse of the Adjoint Theorem*

Let X be a smooth compact complex variety of dimension m and $\mathcal{F}$ a locally free sheaf of rank n. Let $\xi \in H^1(X, \mathcal{F}^\vee)$ be the class associated to the extension (141). Fix $W = \langle \eta_1, \dots, \eta_{n+1} \rangle \subset \operatorname{Ker}(\delta_\xi) \subset H^0(X, \mathcal{F})$ a subspace of dimension $n+1$ and take an adjoint form $\omega = \omega_{\xi, W, \mathcal{B}}$ for chosen $s_1, \dots, s_{n+1} \in H^0(X, \mathcal{E})$ liftings of respectively $\eta_1, \dots, \eta_{n+1}$. As above set $|\lambda^n W| = D_W + |M_W|$. Note that by construction the adjoint form ω can be lifted to $H^0(X, \det \mathcal{F}(-D_W))$.

Theorem 3.3.2. *Assume that* $h^0(X, \mathcal{O}_X(D_W)) = 1$. *If* $\xi_{D_W} = 0$ *then* $[\omega] = 0$.

Remark 3.3.3. *If* $n = m = 1$ *an argument identical to the one of Theorem 2.2.10 gives our thesis without the additional hypothesis* $h^0(X, \mathcal{O}_X(D_W)) = 1$. *Hence we only need to deal with the remaining cases.*

We start with a couple of technical Lemmas

Lemma 3.3.4. *If* $\xi_{D_W} = 0$, *there exists a lifting* $\Omega \in H^0(X, \bigwedge^n \mathcal{E})$ *of* ω.

Proof. Let $(\omega) = D_W + C$ be the decomposition of the adjoint divisor in its fixed component and in its moving part.

By (152), that is

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{n-1}\mathcal{F} & \to & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det\mathcal{F}(-D_W) & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \cdot D_W} & & \\
0 & \to & \bigwedge^{n-1}\mathcal{F} & \to & \bigwedge^{n}\mathcal{E} & \xrightarrow{\rho_n} & \det\mathcal{F} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & = & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

take a global lifting $c \in H^0(X, \det \mathcal{F}(-D_W))$ of ω. Since $\xi_{D_W} = 0$, c can be lifted to a section $e \in H^0(X, \mathcal{E}^{(n)})$. Define $\Omega := \psi(e)$. By commutativity the claim follows. □

Lemma 3.3.5. *Under the hypothesis of Lemma 3.3.4, there exists a global section*

$$\mu \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_C)$$

such that $\delta(\mu) = \xi_{D_W}$ *where* δ *is the coboundary homomorphism of*

$$0 \to \bigwedge^{n-1} \mathcal{F}(-C) \to \bigwedge^{n-1} \mathcal{F} \to \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_C \to 0. \tag{157}$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F}(-C) & \longrightarrow & \bigwedge^{n} \mathcal{E}(-C) & \xrightarrow{G_2} & \det \mathcal{F}(-C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle G_1} & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F} & \xrightarrow{d\epsilon} & \bigwedge^{n} \mathcal{E} & \xrightarrow{\rho_n} & \det \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle H_1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C & \xrightarrow{H_2} & \bigwedge^{n} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_C & \xrightarrow{H_3} & \det \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 3.3.4 we have the global section Ω.

By commutativity, $H_3(\Omega|_C) = 0$. Hence there exists

$$\mu \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_C)$$

such that $H_2(\mu) = \Omega|_C$. It remains to show that $\delta(\mu) = \xi_{D_W}$.

To do this we consider the cocycle $\{\xi_{\alpha\beta}\}$ which gives ξ with respect to an open cover $\{U_\alpha\}$ of X. For any α there exists $\gamma_\alpha \in \bigwedge^{n-1} \mathcal{F}(U_\alpha)$ such that:

$$\Omega|_{U_\alpha} = \omega_\alpha + \gamma_\alpha \wedge d\epsilon \tag{158}$$

where ω_α is obtained by the analogous expression to the one in equation (149) but on $\bigwedge^n \mathcal{E}$. A local computation on $U_\alpha \cap U_\beta$ shows that $\omega_\beta - \omega_\alpha = \xi_{\alpha\beta}(\omega) \wedge d\epsilon$, where the notation $\xi_{\alpha\beta}(\omega)$ indicates the natural contraction. Since we have a global lifting of ω, this cocycle must be a coboundary. Hence by (158)

$$(\gamma_\beta - \gamma_\alpha) \wedge d\epsilon = \omega_\beta - \omega_\alpha = \xi_{\alpha\beta}(\omega) \wedge d\epsilon,$$

that is $\gamma_\beta - \gamma_\alpha = \xi_{\alpha\beta}(\omega)$. Note that the sections γ_α are local liftings of μ. Indeed by the injectivity of H_2, it is enough to show that $(\gamma_\alpha \wedge d\epsilon)|_C = \Omega|_C$, and this is obvious by (158) and by the fact that $\omega_\alpha|_C = 0$. Since the morphism

$$\bigwedge^{n-1} \mathcal{F}(-C)(U_\alpha \cap U_\beta) \to \bigwedge^{n-1} \mathcal{F}(U_\alpha \cap U_\beta)$$

is given by multiplication by $c_{\alpha\beta}$, where $c_{\alpha\beta}$ is a local equation of C on $U_\alpha \cap U_\beta$, then the desired expression of the coboundary $\delta(\mu)$ is given by the cocycle

$$\frac{\gamma_\beta - \gamma_\alpha}{c_{\alpha\beta}} = \frac{\xi_{\alpha\beta}(\omega)}{c_{\alpha\beta}}.$$

By the above local computation, we see that this cocycle gives ξ_{D_W} via the isomorphism

$$H^1(X, \bigwedge^{n-1} \mathcal{F}(-C)) \cong H^1(X, \mathcal{F}^\vee(D_W)). \tag{159}$$

This is easy to prove since ξ_{D_W} is locally given by $\xi_{\alpha\beta} \cdot d_{\alpha\beta}$, where $d_{\alpha\beta}$ is the equation of D_W on $U_\alpha \cap U_\beta$, so the image of ξ_{D_W} through the isomorphism (159) is $\frac{\xi_{\alpha\beta} \cdot d_{\alpha\beta} \cdot c_{\alpha\beta}}{c_{\alpha\beta}} = \frac{\xi_{\alpha\beta}(\omega)}{c_{\alpha\beta}}$, which is exactly $\delta(\mu)$. □

We can now prove Theorem 3.3.2.

Proof. We split the proof into two parts.

First Part. In the first step of the proof we construct a global section

$$\Omega' \in H^0(X, \bigwedge^{n} \mathcal{E}(-C))$$

which restricts, through the natural map $\bigwedge^n \mathcal{E}(-C) \to \det \mathcal{F}(-C)$, to $d \in H^0(X, \det \mathcal{F}(-C))$, where $(d) = D_W$. By Lemma 3.3.5 there exists $\mu \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_C)$ such that $\delta(\mu) = \xi_{D_W}$.

Now we use our hypothesis $\xi_{D_W} = 0$ to write $\delta(\mu) = 0$. Then there exists a global section $\bar{\mu} \in H^0(X, \bigwedge^{n-1} \mathcal{F})$ which is a lifting of μ. Define

$$\tilde{\Omega} := \Omega - \bar{\mu} \wedge d\epsilon \in H^0(X, \bigwedge^{n} \mathcal{E}).$$

By construction $\tilde{\Omega}$ is a new lifting of ω which now vanishes once restricted to C:

$$\tilde{\Omega}|_C = \Omega|_C - \bar{\mu} \wedge d\epsilon|_C = \Omega|_C - H_2(\mu) = 0$$

(recall by Lemma 3.3.5 that $H_2(\mu) = \Omega|_C$). The wanted section $\Omega' \in H^0(X, \bigwedge^n \mathcal{E}(-C))$ is the global section which lifts $\tilde{\Omega}$ and by construction satisfies $\rho_n(G_1(\Omega')) = \omega$ and $G_2(\Omega') = d$.

Second Part. In the second part we show that $[\omega] = 0$ using the section Ω' from the first part: the existence of Ω' together with the fact that $h^0(X, \mathcal{O}_X(D_W)) = 1$ will force the adjoint ω to be a linear combination of $\eta_1, \dots, \eta_{n+1}$.

The global sections $\omega_i := \lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$ generate $\lambda^n W$ and by definition they vanish on D_W, that is there exist global sections $\tilde{\omega}_i \in H^0(X, \det \mathcal{F}(-D_W))$ such that $\omega_i = \tilde{\omega}_i \cdot d$. We consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-C) & \xrightarrow{\alpha} & W \otimes \mathcal{O}_X & \xrightarrow{\gamma} & \bar{\mathcal{F}} & \longrightarrow & 0 \\ & & \downarrow{\cdot C} & & \downarrow{\beta} & & \downarrow{\iota} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{d\epsilon} & \mathcal{E} & \xrightarrow{\rho_1} & \mathcal{F} & \longrightarrow & 0. \end{array} \tag{160}$$

The homomorphism β is locally defined by

$$(f_1, \dots, f_{n+1}) \mapsto (-1)^n f_1 \cdot s_1 + \dots + f_{n+1} \cdot s_{n+1}$$

and similarly $\rho_1 \circ \beta$ is given by

$$(f_1, \dots, f_{n+1}) \mapsto (-1)^n f_1 \cdot \eta_1 + \dots + f_{n+1} \cdot \eta_{n+1}.$$

The homomorphism $\iota \circ \gamma$ is the factorization of this map and it defines $\bar{\mathcal{F}}$. The homomorphism α is defined in the following way: if $f \in \mathcal{O}_X(-C)(U)$ is a local section, then $\tilde{\omega}_i$ are sections of the dual sheaf $\mathcal{O}_X(C)$ and, locally on U, α is given by

$$f \mapsto (\tilde{\omega}_1(f), \dots, \tilde{\omega}_{n+1}(f)).$$

We want to verify that the first square is commutative. If we write locally $\eta_i = \sum_{j=1}^n b_j^i \cdot \sigma_j$, and

$$s_i = \sum_{j=1}^n b_j^i \cdot \sigma_j + c^i \cdot d\epsilon,$$

then

$$\beta(\alpha(f)) = \beta\left(f_U \begin{vmatrix} b_1^2 & \dots & b_n^2 \\ \vdots & \ddots & \vdots \\ b_1^{n+1} & \dots & b_n^{n+1} \end{vmatrix} \frac{1}{d_U}, \dots, f_U \begin{vmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \dots & b_n^n \end{vmatrix} \frac{1}{d_U} \right) =$$

$$= (-1)^n f_U \begin{vmatrix} b_1^2 & \dots & b_n^2 \\ \vdots & \ddots & \vdots \\ b_1^{n+1} & \dots & b_n^{n+1} \end{vmatrix} \frac{1}{d_U} s_1 + \dots + f_U \begin{vmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \dots & b_n^n \end{vmatrix} \frac{1}{d_U} s_{n+1} =$$

$$= f_U \begin{vmatrix} b_1^1 & \dots & b_n^1 & c^1 \\ b_1^2 & \dots & b_n^2 & c^2 \\ \vdots & \ddots & \vdots & \vdots \\ b_1^{n+1} & \dots & b_n^{n+1} & c^{n+1} \end{vmatrix} \frac{1}{d_U} d\epsilon = (f_U \cdot c_U) d\epsilon,$$

where f_U and d_U are local holomorphic functions which represent the sections f and d respectively. The first equality uses the fact that the determinants appearing in the first line are the local equations of the sections ω_i (see also the proof of Theorem 3.3.1); the last equality comes from the fact that the determinant in the second to last line is the local equation of ω, and $\omega = d \cdot c$. To dualize diagram (160) we recall the sheaves isomorphisms $\mathcal{F}^\vee \cong \bigwedge^{n-1} \mathcal{F}(-C - D_W)$ and $\mathcal{E}^\vee \cong \bigwedge^n \mathcal{E}(-C - D_W)$. Moreover we also recall the isomorphism $W^\vee \cong \bigwedge^n W$, given by

$$\eta^i \mapsto \eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}$$

where $\eta^1, \ldots, \eta^{n+1}$ is the basis of $W^\vee$ dual to the basis $\eta_1, \ldots, \eta_{n+1}$ of W. Define

$$e_i := \eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}.$$

Now we dualize (160):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^\vee & \xrightarrow{\gamma^\vee} & \bigwedge^n W \otimes \mathcal{O}_X & \xrightarrow{\alpha^\vee} & \det \mathcal{F}(-D_W) & & \\ & & \uparrow & & \uparrow{\scriptstyle \beta^\vee} & & \uparrow{\scriptstyle \cdot c} & & \\ 0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F}(-C - D_W) & \longrightarrow & \bigwedge^n \mathcal{E}(-C - D_W) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0. \end{array} \tag{161}$$

Here $\alpha^\vee$ is the evaluation map given by the global sections $\tilde{\omega}_i$, not necessarily surjective. Nevertheless we tensor by $\mathcal{O}_X(D_W)$ and we obtain

$$\begin{array}{ccc} \bigwedge^n W \otimes H^0(X, \mathcal{O}_X(D_W)) & \xrightarrow{\overline{\alpha^\vee}} & H^0(X, \det \mathcal{F}) \\ \uparrow{\scriptstyle \overline{\beta^\vee}} & & \uparrow{\scriptstyle \cdot c} \\ H^0(X, \bigwedge^n \mathcal{E}(-C)) & \longrightarrow & H^0(X, \mathcal{O}_X(D_W)). \end{array}$$

The section $\Omega' \in H^0(X, \bigwedge^n \mathcal{E}(-C))$ constructed in the first part of the proof gives in $H^0(X, \det \mathcal{F})$ the adjoint ω. By commutativity

$$\omega = \overline{\alpha^\vee}(\overline{\beta^\vee}(\Omega')).$$

By our hypothesis $h^0(X, \mathcal{O}_X(D_W)) = 1$, the section d is a basis of $H^0(X, \mathcal{O}_X(D_W))$, so $(e_1 \otimes d, \ldots, e_{n+1} \otimes d)$ is a basis of the vector space $\bigwedge^n W \otimes H^0(X, \mathcal{O}_X(D_W))$. We have then

$$\overline{\beta^\vee}(\Omega') = \sum_{i=1}^{n+1} c_i \cdot e_i \otimes d$$

where $c_i \in \mathbb{C}$ and

$$\omega = \overline{\alpha^\vee}(\overline{\beta^\vee}(\Omega')) = \overline{\alpha^\vee}(\sum_{i=1}^{n+1} c_i \cdot e_i \otimes d) = \sum_{i=1}^{n+1} c_i \cdot \tilde{\omega}_i \cdot d = \sum_{i=1}^{n+1} c_i \cdot \omega_i,$$

and hence $[\omega] = 0$. □

We remark that in the case $h^0(X, \mathcal{O}_X(D_W)) = 1$, Theorem 3.3.1 and Theorem 3.3.2 give a full characterization for the condition $\xi_{D_W} = 0$, that is

$$\xi_{D_W} = 0 \quad \text{iff} \quad [\omega = 0].$$

In the particular case $D_W = 0$, the hypothesis $h^0(X, \mathcal{O}_X(D_W)) = 1$ is trivially true and this characterization simply is

$$\xi = 0 \quad \text{iff} \quad [\omega = 0].$$

3.4 AN INFINITESIMAL TORELLI THEOREM

As an easy consequence of the Adjoint Theorem we have the infinitesimal Torelli theorem for primitive varieties with $p_g = q = n + 1$.

We recall that any irregular variety X comes equipped with its Albanese variety $\text{Alb}(X)$ and its Albanese morphism

$$\text{alb} : X \to \text{Alb}(X).$$

Definition 3.4.1. *A* fibration *is a surjective proper flat morphism* $f\colon X \to Z$ *with connected fibers between the smooth varieties* X *and* Z*. A fibration* $f\colon X \to Z$ *is* irregular *if* Z *is an irregular variety.*

A smooth irregular variety X *is said to be of* maximal Albanese dimension *if* $\dim \text{alb}(X) = \dim X$*. If* $\dim \text{alb}(X) = \dim X$ *and* $\text{alb}\colon X \to \text{Alb}(X)$ *is not surjective, that is* $q(X) > \dim X$*,* X *is said to be of* Albanese general type*.*

A fibration $f\colon X \to Z$ *is called a* higher irrational pencil *if* Z *is of Albanese general type. An irregular variety* X *is said to be* primitive *if it does not admit any higher irrational pencil; see: [12, Definition 1.20], [25, Definition 1.2.4].*

Note that irregular fibrations (resp. higher irrational pencils) are higher-dimensional analogous to fibrations over non-rational curves (resp. curves of genus $g \geqslant 2$).

If X is primitive, the Generalized Castelnuovo-de Franchis Theorem (see [12, Theorem 1.14]), implies that the maps

$$\bigwedge^k H^0(X, \Omega^1_X) \to H^0(X, \Omega^k_X)$$

do not map to 0 decomposable elements.

Corollary 3.4.2. *Let X be an n-dimensional primitive variety of general type such that $p_g = q = n+1$. If Ω^1_X is globally generated by global sections then the infinitesimal Torelli theorem holds for X.*

Proof. Take $W = H^0(X, \Omega^1_X)$. Since X is primitive,

$$\lambda^n W = \langle \omega_1, \dots, \omega_{n+1} \rangle = H^0(X, \omega_X).$$

By the Torelli hypothesis $W = \operatorname{Ker} \delta_\xi$, we can construct an adjoint form $\omega_{\xi,W,\mathcal{B}}$ and obviously $\omega_{\xi,W,\mathcal{B}} \in \lambda^n W$. Hence, by Theorem 3.3.1, $\xi_{D_W} = 0$. If Ω^1_X is generated by global sections then $D_W = 0$, which implies that $\xi = 0$. □

In a similar way we have

Corollary 3.4.3. *Let X be a surface of general type such that $p_g = q = 3$, $K^2_X = 6$ and the canonical system $|K_X|$ is base point free. Then the infinitesimal Torelli theorem holds for X.*

Proof. Indeed a surface of general type such that $p_g = q = 3$ and $K^2_X = 6$ is the symmetric product of a non-hyperelliptic curve of genus 3 by [13, Proposition 3.17, (i)]. Hence the proof is similar to the one of Corollary 3.4.2. □

Remark 3.4.4. *Note that for surfaces of general type which are the symmetric product of a hyperelliptic curve of genus 3, the infinitesimal Torelli theorem does not hold by [51]. Indeed $D_{\Omega^1_X}$ exists and it is a rational -2 curve; see: [13, Proposition 3.17, (ii)]. Hence an infinitesimal Torelli deformation is supported on $D_{\Omega^1_X}$.*

3.5 ADJOINT QUADRICS

Consider a locally free sheaf $\mathcal{F}$ of rank n over an m-dimensional smooth variety X. Naturally associated to $\mathcal{F}$ there is the invertible sheaf $\det \mathcal{F}$ and the natural homomorphism:

$$\lambda^n \colon \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F}). \tag{162}$$

We denote by $\lambda^n H^0(X, \mathcal{F})$ its image. Consider the linear system

$$\mathbb{P}(\lambda^n H^0(X, \mathcal{F}))$$

and call $D_\mathcal{F}$ its fixed component and $|M_\mathcal{F}|$ its associated mobile linear system. Moreover we denote by $|\det \mathcal{F}|$ the linear system associated to $\det \mathcal{F}$ and by $D_{\det \mathcal{F}}$, $M_{\det \mathcal{F}}$ respectively its fixed and its movable part; that is: $|\det \mathcal{F}| = D_{\det \mathcal{F}} + |M_{\det \mathcal{F}}|$. Finally note that $D_{\det \mathcal{F}}$ is a sub-divisor of $D_\mathcal{F}$.

Definition 3.5.1. *An* adjoint quadric *for ω, or simply ω-adjoint quadric, is a quadric in the projective space $\mathbb{P}(H^0(X, \det \mathcal{F})^\vee)$ which can be written in the form*

$$\omega^2 = \sum L_i \cdot \omega_i,$$

where ω is an adjoint form of ξ and $W \subset H^0(X, \mathcal{F})$, $\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$ and $L_i \in H^0(X, \det \mathcal{F})$.

Obviously an adjoint quadric has rank less than or equal to $2n+3$, and, if it exists, it is constructed by the extension class ξ.

The following theorem relates adjoint quadrics and adjoint forms. It turns out that the absence of adjoint quadrics is strictly related to the vanishing of the adjoint image.

Theorem 3.5.2. *Let X be an m-dimensional smooth variety. Let $\mathcal{F}$ be a locally free sheaf of rank n such that $h^0(X, \mathcal{F}) \geqslant n+1$, let $\xi \in H^1(X, \mathcal{F}^\vee)$ and let Y be the schematic image of*

$$\phi_{|M_{\det \mathcal{F}}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \det \mathcal{F})^\vee).$$

If ξ is such that $\delta_\xi^{0,n}(\omega) = 0$, where ω is an adjoint form associated to an $n+1$-dimensional subspace $W \subset \operatorname{Ker} \delta_\xi \subset H^0(X, \mathcal{F})$, then $[\omega] = 0$, provided that there are no ω-adjoint quadrics vanishing on Y.

Proof. Let $\mathcal{B} = \{\eta_1, \ldots, \eta_{n+1}\}$ be a basis of W. Set ω_i for $i = 1, \ldots, n+1$ as above and denote by $\tilde{\omega}_i \in H^0(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$ the corresponding sections via

$$0 \to H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^0(X, \det \mathcal{F}).$$

Recall that $\lambda^n W := \langle \omega_1, \ldots, \omega_{n+1} \rangle \subset H^0(X, \det \mathcal{F})$ is the vector space generated by the sections ω_i. The standard evaluation map

$$\bigwedge^n W \otimes \mathcal{O}_X \to \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}$$

given by $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n+1}$ results in the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigwedge^n W \otimes \mathcal{O}_X \longrightarrow \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \longrightarrow 0 \tag{163}$$

which is associated to the class $\xi' \in \operatorname{Ext}^1(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}, \mathcal{K})$. This sequence fits into the following commutative diagram (cf. (161))

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \longrightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0,
\end{array}$$

where f is the map given by the contraction by the sections $(-1)^{n+1-i}s_i$, for $i = 1, \dots, n+1$, and g is given by the global section

$$\sigma \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$$

corresponding to the adjoint ω. We have the standard factorization

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \longrightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

where the sequence in the middle is associated to the class $\xi'' \in H^1(X, \mathcal{K})$ which is the image of $\xi \in H^1(X, \mathcal{F}^\vee)$ through the map $H^1(X, \mathcal{F}^\vee) \to H^1(X, \mathcal{K})$. In particular we obtain the commutative square

$$\begin{array}{ccc}
H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) & \longrightarrow & H^1(X, \mathcal{K}) \\
\uparrow & & \| \\
H^0(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{K}).
\end{array}$$

By commutativity we immediately have that the image of σ through the coboundary map

$$H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^1(X, \mathcal{K})$$

is ξ''. Tensoring by $\det \mathcal{F}$, the map $\mathcal{F}^\vee \to \mathcal{K}$ gives

$$\begin{array}{ccc}
\mathcal{F}^\vee \otimes \det \mathcal{F} & \longrightarrow & \mathcal{K} \otimes \det \mathcal{F} \\
\| & \nearrow{\scriptstyle \Gamma} & \\
\bigwedge^{n-1} \mathcal{F} & &
\end{array}$$

and, since $\xi \cdot \omega \in H^1(X, \mathcal{F}^\vee \otimes \det \mathcal{F})$ is sent to $\xi'' \cdot \omega \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, we have that

$$H^1(\Gamma)(\xi \cup \omega) = \xi'' \cdot \omega,$$

where $\xi \cup \omega$ is the cup product.

By hypothesis $\delta_\xi^{0,n}(\omega) = \xi \cup \omega = 0 \in H^1(X, \bigwedge^{n-1} \mathcal{F})$, so also $\xi'' \cdot \omega = 0 \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, hence the global section

$$\sigma \cdot \omega \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F})$$

is in the kernel of the coboundary map

$$H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F}) \to H^1(X, \mathcal{K} \otimes \det \mathcal{F})$$

associated to the sequence

$$0 \longrightarrow \mathcal{K} \otimes \det \mathcal{F} \longrightarrow \bigwedge^n W \otimes \det \mathcal{F} \longrightarrow \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F} \longrightarrow 0. \tag{164}$$

This occurs iff there exist $L_i^\sigma \in H^0(X, \det \mathcal{F})$, $i = 1, \dots, n+1$ such that

$$\sigma \cdot \omega = \sum_{i=1}^{n+1} L_i^\sigma \cdot \tilde{\omega}_i. \tag{165}$$

This relation gives the following relation in $H^0(X, \det \mathcal{F}^{\otimes 2})$:

$$\omega \cdot \omega = \sum_{i=1}^{n+1} L_i^\sigma \cdot \omega_i. \tag{166}$$

Assume now that the adjoint form ω is not in the vector space $\lambda^n W$. Then the equation (166) gives an adjoint quadric. By contradiction the claim follows. □

Corollary 3.5.3. *Under the above hypotheses, ξ is a supported deformation on D_W; that is, ξ_{D_W} is trivial.*

Proof. The claim follows by Theorem 3.5.2 and by Theorem 3.3.1. □

Remark 3.5.4. *Consider a generically globally generated sheaf $\mathcal{F}$ of rank $n = \dim X$. By [55, Proposition 3.1.6], if W is a generic $n+1$-dimensional subspace of $H^0(X, \mathcal{F})$, then $D_\mathcal{F} = D_W$.*

This allows us to prove the following

Corollary 3.5.5. *Under the hypotheses of Theorem 3.5.2, if we further assume that $\mathcal{F}$ has rank $n = \dim X$ and W is generic in $H^0(X, \mathcal{F})$, it follows that ξ is a deformation supported on $D_\mathcal{F}$, that is $\xi_{D_\mathcal{F}}$ is trivial.*

Proof. If W is a generic $n+1$-dimensional subspace of $H^0(X,\mathcal{F})$, we have that $D_{\mathcal{F}} = D_W$. Then the claim follows by Corollary 3.5.3. □

Note that there are cases where these hypotheses easily apply:

Corollary 3.5.6. *Let X be an n-dimensional smooth variety. Let $\mathcal{F}$ be a locally free sheaf of rank n. Assume that*

$$\phi_{|M_{\det\mathcal{F}}|}\colon X \dashrightarrow \mathbb{P}(H^0(X,\det\mathcal{F})^\vee)$$

is a non trivial rational map such that its schematic image is a complete intersection of hypersurfaces of degree > 2. Let $\xi \in H^1(X,\mathcal{F}^\vee)$. If $\delta_\xi = 0$ and $\delta_\xi^{0,n}(\omega) = 0$, where ω is an adjoint form associated to a generic $n+1$-dimensional subspace $W \subset H^0(X,\mathcal{F})$, then ξ is a deformation supported on $D_{\mathcal{F}}$. In particular if $D_{\mathcal{F}} = 0$ then $\xi = 0$.

Proof. The claim follows directly by Corollary 3.5.5. □

Corollary 3.5.7. *Let X be an n-dimensional variety of general type with irregularity $\geqslant n+1$ and such that its cotangent sheaf is generated by global sections. Suppose also that there are no adjoint quadrics containing the canonical image of X. Then the infinitesimal Torelli theorem holds for X. In particular it holds if there are no quadrics of rank less than or equal to $2n+3$ passing through the canonical image of X.*

Proof. By Corollary 3.5.5, any $\xi \in H^1(X,T_X)$ such that $\delta_\xi = 0$ and $\delta_\xi^{0,n}(\omega) = 0$, where ω is an adjoint form associated to a generic $n+1$-dimensional subspace $W \subset H^0(X,\Omega^1_X)$, is supported on the branch locus of the Albanese morphism and, since we have assumed it to be trivial, then the trivial infinitesimal deformation is the only possible case. □

Remark 3.5.8. *As a typical application we obtain infinitesimal Torelli for a smooth divisor X of an $n+1$-dimensional Abelian variety A such that $p_g(X) = n+2$; for explicit examples consider the case of a smooth surface X in a polarization of type $(1,1,2)$ in an abelian 3-fold A. The invariants of X are $p_g(X) = 4$, $q(X) = 3$ and $K^2 = 12$. The canonical map is in general a birational morphism onto a surface of degree 12. See [14, Theorem 6.4].*

3.6 FAMILIES WITH BIRATIONAL FIBERS

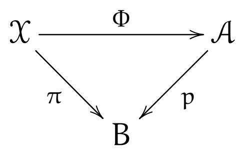
3.6.1 *Albanese type families*

Theorem 3.5.2 provides a criterion to understand which families of irregular varieties of general type have birational fibers.

To study a family $\pi\colon \mathcal{X} \to B$ of irregular varieties it is natural to consider the case where it comes equipped with a family $p\colon \mathcal{A} \to B$ of Abelian varieties; that is, the fiber $A_b := p^{-1}(b)$ is an Abelian variety of dimension $a > 0$.

Definition 3.6.1. *Let* $\pi\colon \mathcal{X} \to B$ *be a family of irregular varieties of general type and* $p : \mathcal{A} \to B$ *a family of Abelian varieties. A morphism* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *will be called a* family of Albanese type *over* B *if:*

- Φ *fits into the commutative diagram*



- *The induced map* $\phi_b\colon X_b \to A_b$ *of* Φ *on* X_b *is birational onto its image* Y_b.

- *The cycle* Y_b *generates the fiber* A_b *as a group.*

See: [55, Definition 1.1.1.]. We remark that $a > n$ (cf. see [41, p.311 and Corollary to Theorem 10.12,(i)]). We shall say that two families over B, $\Phi : \mathcal{X} \to \mathcal{A}$ and $\Psi : \mathcal{Y} \to \mathcal{A}$, of Albanese type have the same image if it is true fiberwise, that is $\phi_b(X_b) = \psi_b(Y_b)$ for every $b \in B$.

Remark 3.6.2. *Albanese type families have a good behaviour under base change. In fact let* $\mu : B' \to B$ *be a base change, then* $\mu^*(\Phi) = \Phi \times \mathrm{id} : \mathcal{X} \times_B B' \to \mathcal{A} \times_B B'$ *is an Albanese type family over* B'. *In particular for a connected subvariety* $C \hookrightarrow B$ *the base change of* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *to* C *is well defined and it will be denoted by* $\Phi_C\colon \mathcal{X}_C \to \mathcal{A}_C$.

If $s\colon B \to \mathcal{A}$ is a section of $p : \mathcal{A} \to B$, we define the translated family $\Phi_s : \mathcal{X} \to \mathcal{A}$ of Φ by the formula:

$$\Phi_s(x) = \Phi(x) + s(\pi(x)).$$

Notice that $\Phi_s : \mathcal{X} \to \mathcal{A}$ is a family of Albanese type. Two families Φ and Ψ over B are said to be *translation equivalent* if there exists a section σ of p such that the images of Φ_σ and Ψ (fiberwise) coincide.

We are interested in a condition that guarantees that, up to restriction, the fibers of the restricted family are birationally equivalent. For the reader's convenience we recall the following definition given in [55, Definition 1.1.2]:

Definition 3.6.3. *Two families of Albanese type* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *and* $\Phi'\colon \mathcal{X}' \to \mathcal{A}'$ *over, respectively,* B *and* B' *will be said* locally equivalent, *if there exist an open set* $U \subset B$ *an open set* $U' \subset B'$ *and a biregular map* $\mu\colon U' \to U := \mu(U') \subset B$ *such that the pull-back families* $\mu^*(\Phi_U)$ *and* $\Phi'_{U'}$ *are translation equivalent. We will say that* Φ *is* trivial *if* $\mathcal{X} = X \times B$, $\mathcal{A} = A \times B$ *and* $\pi_A(\Phi(X_b)) = \pi_A(\Phi(X_{b_0}))$ *for all* b *where* $\pi_A\colon A \times B \to A$ *is the natural projection.*

Example 3.6.4. *The standard example of Albanese type family is given by a family* $\pi : \mathcal{X} \to B$ *with a section* $s\colon B \to \mathcal{X}$*. Indeed by* $s\colon B \to \mathcal{X}$ *we have a family* $p\colon \mathrm{Alb}(\mathcal{X}) \to B$ *whose fiber is* $p^{-1}(b) = \mathrm{Alb}(X_b)$*; the section gives a family* $\Phi\colon \mathcal{X} \to \mathrm{Alb}(\mathcal{X})$ *with fiber:*

$$\mathrm{alb}(X_b) : X_b \to \mathrm{Alb}(X_b).$$

If we also assume that $\phi_b = \mathrm{alb}(X_b)$ *has degree* 1 *onto the image then* $\Phi\colon \mathcal{X} \to \mathrm{Alb}(\mathcal{X})$ *is an Albanese type family. We will call* $\mathcal{X} \stackrel{\Phi}{\to} \mathrm{Alb}(\mathcal{X})$ *an* Albanese family.

We will use the following:

Proposition 3.6.5. *An Albanese type family* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *is locally equivalent to a trivial family if and only if the fibers* X_b *are birationally equivalent.*

Proof. See [55, Proposition 1.1.3]. □

3.6.2 *Families with liftability conditions*

To find conditions which ensure that the fibers of a family $\pi\colon \mathcal{X} \to B$ are birationally equivalent, it is natural to understand conditions on Albanese type families whose associated family of Abelian varieties is trivial. The easiest condition to think of is given by the condition of liftability for any 1-form.

Proposition 3.6.6. *Let* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *be an Albanese type family such that for every* $b \in B$ *the map* $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$ *is surjective. Then up to shrinking* B *the fibers of* $p\colon \mathcal{A} \to B$ *are isomorphic.*

Proof. Let $\mu_b \in \mathrm{Ext}^1(\Omega^1_{A_b}, \mathcal{O}_{A_b})$ be the class given by the family $p\colon \mathcal{A} \to B$, that is the class of the following extension:

$$0 \to \mathcal{O}_{A_b} \to \Omega^1_{\mathcal{A}|A_b} \to \Omega^1_{A_b} \to 0.$$

Now $\phi_b^*\mathcal{O}_{A_b} = \mathcal{O}_{X_b}$ and the map $\phi_b^*\mathcal{O}_{A_b} \to \phi_b^*\Omega^1_{\mathcal{A}|A_b}$ is generically injective, hence it is injective because otherwise the kernel would be a torsion subsheaf of $\mathcal{O}_{X_b}$. Thus we have the following exact sequence

$$0 \to \phi_b^*\mathcal{O}_{A_b} \to \phi_b^*\Omega^1_{\mathcal{A}|A_b} \to \phi_b^*\Omega^1_{A_b} \to 0$$

which fits into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \phi_b^*\mathcal{O}_{A_b} & \longrightarrow & \phi_b^*\Omega^1_{\mathcal{A}|A_b} & \longrightarrow & \phi_b^*\Omega^1_{A_b} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X_b} & \longrightarrow & \Omega^1_{\mathcal{X}|X_b} & \longrightarrow & \Omega^1_{X_b} & \longrightarrow & 0. \end{array}$$

In cohomology we have

$$\begin{array}{ccc} H^0(X_b, \phi_b^*\Omega^1_{A_b}) & \longrightarrow & H^1(X_b, \mathcal{O}_{X_b}) \\ \downarrow & & \| \\ H^0(X_b, \Omega^1_{X_b}) & \longrightarrow & H^1(X_b, \mathcal{O}_{X_b}) \end{array}$$

so, by commutativity and by the surjectivity hypothesis

$$H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b}),$$

we immediately obtain

$$H^0(X_b, \phi_b^*\Omega^1_{\mathcal{A}|A_b}) \twoheadrightarrow H^0(X_b, \phi_b^*\Omega^1_{A_b})$$

and hence the coboundary

$$\delta_{\mu_b}\colon H^0(A_b, \Omega^1_{A_b}) \to H^1(A_b, \mathcal{O}_{A_b})$$

is trivial.

Then by [18, Page 78] we conclude. □

For an Albanese type family $\Phi\colon \mathcal{X} \to \mathcal{A}$ such that $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$ we can say more. Actually up to shrinking B, it is trivial to show that for every $b \in B$ it holds that $\mathrm{Alb}(X_b) = A$ where A is a fixed Abelian variety. For later reference we state:

Corollary 3.6.7. *Let* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *be an Albanese type family such that* $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$ *where* $b \in B$. *Then for every* $b \in B$ *there exists an open neighbourhood* U *such that the restricted family* $\Phi_U\colon \mathcal{X}_U \to \mathcal{A}_U$ *is locally equivalent to* $\Psi\colon \mathcal{X}_U \to \widehat{A} \times U$, *where* $\widehat{A}$ *is an Abelian variety. Moreover there exists another Abelian variety* A *such that for every* $b \in U$ *it holds that* $A = \mathrm{Alb}(X_b)$ *and the natural morphism* $A \to \widehat{A}$ *gives a factorization of* $\Psi\colon \mathcal{X}_U \to \widehat{A} \times U$ *via* $\mathrm{alb}(\mathcal{X}_U)\colon \mathcal{X}_U \to A \times U$.

Definition 3.6.8. *We say that a family* $f\colon \mathcal{X} \to B$ *of relative dimension* n *satisfies* extremal liftability conditions *over* B *if*

- $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$;
- $H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^n_{X_b})$.

This definition means that all the 1-forms and all the n-forms of the fibers are obtained by restriction from the family $\mathcal{X}$. Comparing the two conditions with the hypotheses of Theorem 3.5.2, they ensure that $\delta_\xi = 0$ and $\delta^{0,n}_\xi = 0$.

Remark 3.6.9. *Let* X *be a smooth variety such that* $\mathrm{alb}(X)\colon X \to \mathrm{Alb}(X)$ *has degree* 1. *Let* $f\colon X \longrightarrow Z$ *be a fibration of relative dimension* n *such that the general fiber* $f^{-1}(y) = X_y$ *is smooth of general type and with irregularity* $q \geqslant n+1$. *By rigidity of Abelian subvarieties, the image of the map* $\mathrm{Alb}(X_y) \longrightarrow \mathrm{Alb}(X)$ *is a (translate of a) fixed abelian variety* A. *If, moreover, we have surjections* $H^0(X, \Omega^1_X) \twoheadrightarrow H^0(X_y, \Omega^1_{X_y})$ *and* $H^0(X, \Omega^n_X) \twoheadrightarrow H^0(X, \Omega^n_{X_y})$, *then* $A = \mathrm{Alb}(X_y)$ *and taking a sufficiently small polydisk around any smooth fibre of* f *we obtain a family which satisfies extremal liftability conditions.*

The main theorem of this section is based on the Volumetric Theorem; see [55, Theorem 1.5.3]:

Theorem 3.6.10. *Let* $\Phi\colon \mathcal{X} \to \mathcal{A}$ *be an Albanese type family such that* $p\colon \mathcal{A} \to B$ *has fibers isomorphic to a fixed Abelian variety* A. *Let* $W \subset H^0(A, \Omega^1_A)$ *be a generic* $n+1$*-dimensional subspace and* $W_b \subset H^0(X_b, \Omega^1_{X_b})$ *its pull-back over the fiber* X_b. *Assume that for every point* $b \in B$ *it holds that* $\omega_{\xi_b, W_b, \mathcal{B}_b} \in \lambda^n W_b$ *where* $\xi_b \in H^1(X_b, T_{X_b})$ *is the class given on* X_b *by* $\pi\colon \mathcal{X} \to B$, *then the fibers of* $\pi\colon \mathcal{X} \to B$ *are birational.*

Proof. See [55, Theorem 1.5.3]. □

Theorem 3.6.11. *Let $\Phi\colon \mathcal{X} \to \mathcal{A}$ be a family of Albanese type whose associated family of n-dimensional irregular varieties $\pi\colon \mathcal{X} \to B$ satisfies extremal liftability conditions. Assume that every fiber X of $\pi\colon \mathcal{X} \to B$ has irregularity $\geqslant n+1$ and that there are no adjoint quadrics through the canonical image of any fiber X. Then the fibers of $\pi\colon \mathcal{X} \to B$ are birational.*

Proof. Since our claim is local in the analytic category, up to base change, we can assume that B is a 1-dimensional disk and that $\pi\colon \mathcal{X} \to B$ has a section. More precisely we take two points in B and a curve C connecting them, then make a base change from B to C. Therefore in the rest of this proof B is a curve. By proposition 3.6.6 $p\colon \mathcal{A} \to B$ is trivial. Moreover the Albanese family $\mathrm{alb}(\mathcal{X})\colon \mathcal{X} \to \mathrm{Alb}(\mathcal{X})$ exists and by Corollary 3.6.7 we can also assume that $\mathrm{Alb}(\mathcal{X}) = A \times B$.

We denote by $\xi_b \in H^1(X_b, T_{X_b})$ the class associated to the infinitesimal deformation of X_b induced by $\pi\colon \mathcal{X} \to B$.

Let $\mathcal{B} := \{dz_1, \dots, dz_{n+1}\}$ be a basis of an $n+1$-dimensional generic subspace W of $H^0(A, \Omega^1_A)$. For every $b \in B$ let $\eta_i(b) := \mathrm{alb}(X_b)^* dz_i$, $i = 1, \dots, n+1$. By standard theory of the Albanese morphism it holds that $\mathcal{B}_b := \{\eta_1(b), \dots, \eta_{n+1}(b)\}$ is a basis of the pull-back W_b of W inside $H^0(X_b, \Omega^1_{X_b})$. Let $\omega_i(b) := \lambda^n(\eta_1(b) \wedge \dots \wedge \eta_{i-1}(b) \wedge \widehat{\eta_i(b)} \wedge \dots \wedge \eta_{n+1}(b))$ for $i = 1, \dots, n+1$. Since $\Phi\colon \mathcal{X} \to \mathcal{A}$ is a family of Albanese type, $\dim \lambda^n W_b \geqslant 1$, actually by [55, Theorem 1.3.3] it follows that $\lambda^n W_b$ has dimension $n+1$, and we can write:

$$\lambda^n W_b = \langle \omega_1(b), \dots, \omega_{n+1}(b) \rangle.$$

Let $\omega_b := \omega_{\xi_b, W_b, \mathcal{B}_b}$ be an adjoint image of W_b.

By Theorem 3.5.2 it follows that $\omega_b \in \lambda^n W_b$. By Theorem 3.6.10 we conclude. □

The above theorem gives, in particular, an answer to the generic Torelli problem if the fibers of $\pi\colon \mathcal{X} \to B$ are smooth minimal with unique minimal model. Indeed we have:

Corollary 3.6.12. *Let $\pi\colon \mathcal{X} \to B$ be a family of n-dimensional irregular varieties which satisfies extremal liftability conditions. Assume that every fiber X is minimal, it has a unique minimal model and its Albanese morphism has degree 1. If there are no adjoint quadrics containing the canonical image of X, then the generic Torelli theorem holds for $\pi\colon \mathcal{X} \to B$. In particular the claim holds if no quadric of rank $\leqslant 2n+3$ contains the canonical image of X.*

Proof. Since the fiber X has a unique minimal model the claim follows directly by Theorem 3.6.11. □

3.7 EXAMPLES

We discuss a couple of examples of varieties whose canonical image is not contained in any adjoint quadric. Consider the product $X = C_1 \times C_2$ of two canonical curves of genus 3. The space of first order deformations of X, $H^1(X, T_X)$, is isomorphic to the direct sum $H^1(C_1, T_{C_1}) \oplus H^1(C_2, T_{C_2})$, hence every deformation ξ of X is uniquely associated to a pair ξ_1, ξ_2. We consider a first order deformation ξ satisfying the Torelli hypothesis $\delta_\xi = 0$. The canonical map of X is given by the Segre embedding

$$X = C_1 \times C_2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8.$$

We call X_0, X_1, X_2 the coordinates on $\mathbb{P}^2 = \mathbb{P}(H^0(C_1, \omega_{C_1})^\vee)$ and Y_0, Y_1, Y_2 those on $\mathbb{P}^2 = \mathbb{P}(H^0(C_2, \omega_{C_2})^\vee)$. On X we take the 1-forms η_1, η_2, η_3 corresponding to $X_1 + Y_1$, $X_2 + Y_2$ and $X_1 + Y_2$. Since they are linearly independent we can construct the adjoint form ω and the sections ω_i as described above. A simple computation shows that if both the adjoint image of X_1, X_2 and the adjoint image of Y_1, Y_2 are zero (on C_1 and C_2 respectively), then also the class of ω is zero. On the other hand, assume that at least one of the adjoint images on the curves, for example the adjoint on C_1, is not zero. Then, changing coordinates if necessary, we can assume that it is X_0. In the usual coordinates $U_{i,j}$ on $\mathbb{P}^8$, we have that $\omega_1 = U_{2,2} - U_{1,2}$, $\omega_2 = U_{1,2} - U_{1,1}$, $\omega_3 = U_{1,2} - U_{2,1}$ and the expression of the adjoint form ω contains, for example, the term $U_{0,2}$. We want now to show that there are no adjoint quadrics of ω containing the image of $X \subset \mathbb{P}^8$. Cohomological computations on the Segre embedding show that, since the image of C_i in $\mathbb{P}^2$ is not contained in any quadric (it is in fact a quartic), the quadrics of $\mathbb{P}^8$ vanishing on X are exactly those vanishing on the image of $\mathbb{P}^2 \times \mathbb{P}^2$. Now, an adjoint quadric of ω, that is a quadric in the form $\omega^2 - \sum L_i \cdot \omega_i$, is homogeneous of degree 2 and contains the term $U_{0,2}^2$ which comes from ω^2 and is not simplified by the sum $\sum L_i \cdot \omega_i$ (this is obvious looking at the expressions of the ω_i). It is easy to see that such a quadric is not contained in the ideal of $\mathbb{P}^2 \times \mathbb{P}^2$, which is generated by the quadrics $U_{i,j}U_{k,l} - U_{i,l}U_{k,j}$. Hence, with our choice of the sections η_i, we have proved that there are no adjoint quadrics of ω containing the canonical image of the product of two canonical curves of genus 3.

A similar computation can be done for the product of two canonical curves C_1 and C_2 of genus 4. The main difference is that the canonical image of C_i in $\mathbb{P}^3$ is the complete intersection of a quadric and a cubic. Assume that these quadrics are of rank 4 and choose coordinates X_0, X_1, X_2, X_3 on $\mathbb{P}^3 = \mathbb{P}(H^0(C_1, \omega_{C_1})^\vee)$ and Y_0, Y_1, Y_2, Y_3 on $\mathbb{P}^3 = \mathbb{P}(H^0(C_2, \omega_{C_2})^\vee)$ such that the quadric containing C_1 is $X_0 \cdot X_1 - X_2 \cdot X_3 = 0$ and the quadric containing C_2 is $Y_0 \cdot Y_1 - Y_2 \cdot Y_3 = 0$. Taking η_1, η_2, η_3 corresponding to $X_1 + Y_1$, $X_3 + Y_3$ and $X_1 + Y_3$ we have $\omega_1 = U_{3,3} - U_{1,3}$, $\omega_2 = U_{1,3} - U_{1,1}$ and $\omega_3 = U_{1,3} - U_{3,1}$. As before if the adjoint classes on the curves are zero, then also the adjoint class on the product is zero. In the other case in the expression of ω there are always terms (like for example $U_{0,3}$ or $U_{2,3}$) that prevent an adjoint quadric $\omega^2 - \sum L_i \cdot \omega_i$ to be in the ideal of $C_1 \times C_2$ inside $\mathbb{P}^{15}$. The difference with the case of genus 3 is that one has to consider also quadrics vanishing on $C_1 \times C_2$, but not on $\mathbb{P}^3 \times \mathbb{P}^3$, that is the quadrics of the form $U_{0,i} \cdot U_{1,j} - U_{2,i} \cdot U_{3,j} = 0$ and $U_{i,0} \cdot U_{j,1} - U_{i,2} \cdot U_{j,3} = 0$ which come from $X_0 \cdot X_1 - X_2 \cdot X_3 = 0$ and $Y_0 \cdot Y_1 - Y_2 \cdot Y_3 = 0$ respectively. The case where at least one of the quadrics containing the canonical image of the curves is of rank 3 is similar. If for example the canonical image of C_1 is contained in a quadric of rank 3, we take coordinates such that this quadric is $X_0 \cdot X_1 - X_0 \cdot X_2 - X_1 \cdot X_2 = 0$, then the computations are the same as in the rank 4 case.

Theorem 3.6.11, and consequently Corollary 3.6.12, applies directly to the families whose fiber X has canonical map which is not an isomorphism and X is an irregular variety such that its canonical image Y is a (possibly very singular) hypersurface of degree > 2 or a (possibly very singular) complete intersection of hypersurfaces of degree > 2: see also Remark 3.5.8. Many of these examples are not well studied in the literature.

As far as other possible examples are concerned, the class of minimal irregular surfaces with very ample and primitive canonical bundle should be worthy of study. Indeed we remind the reader that the space $\mathcal{Q}_{k,n}$ of quadrics of the projective space $\mathbb{P}^n$ of rank $\leqslant k$ has dimension $k(n-k+1) + \binom{k+1}{2} - 1$, and letting $n = p_g(S) - 1$, by Riemann-Roch theorem the dimension of the vector space of quadrics containing $S = Y$ is $h^0(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(2)) = \binom{n+2}{2} - K_S^2 - \chi(\mathcal{O}_S)$. Now to apply our theory in the case $n = 2$ we have to take $k = 2n + 3 = 7$; hence, imposing the natural condition $K_S^2 + \chi(\mathcal{O}_S) \geqslant 7n - 15$, it is at least reasonable, by the obvious dimensional computation, to expect that, for such a generic irregular surface, $\mathcal{Q}_{7,n}$ is a direct summand of $H^0(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(2))$. Then Corollary 3.6.12 applies to give generic Torelli theorem for such irregular surfaces of general type (with Albanese map of degree 1 and irregularity $\geqslant 4$).

Clearly if X is a variety such that there exists a divisor D with $h^0(X, \mathcal{O}_X(D)) \geqslant 2$ and $h^0(X, \mathcal{O}_X(K_X - D)) \geqslant 2$ then the usual argument of the Petri map for curves gives a quadric of rank 3 or 4 containing the canonical image. It is also easy to construct varieties such that this D exists. Nevertheless our theorem requires that no *adjoint quadrics* exist, so it can happen that those low rank quadrics obtained by Petri-like arguments are not adjoint quadrics. The geometry of 1-forms and of the quadrics of low rank through the canonical image is an interesting question as the problem of finding conditions such that $\mathcal{Q}_{7,n} \cap H^0(\mathbb{P}^n, \mathcal{I}_{S/\mathbb{P}^n}(2)) = \{0\}$.

4 GENERALIZED ADJOINT FORM

The adjoint theory developed in Chapter 3 deals with exact sequences of the form

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

and, as we have seen, it is useful for the study of Torelli type problems for irregular varieties. In this chapter on the other hand we want to develop the tools for the study of a broader class of varieties. In particular we will generalize the adjoint theory of Chapter 3 for sequences of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is a locally free sheaf of rank 1 possibly different from the structure sheaf $\mathcal{O}_X$.

4.1 THE THEORY OF GENERALIZED ADJOINT FORMS

Let X be a smooth m-dimensional compact complex smooth variety and

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0 \tag{167}$$

a short exact sequence of locally free sheaves on X with $\mathcal{F}$ of rank n. In the previous chapters we have introduced the theory of adjoint forms which is suitable for the study of the extension class associated to such a sequence. However, one condition is essential for the theory to work:

- $h^0(X, \mathcal{F}) \geqslant n+1$.

Sometimes this can be quite restrictive, hence we want to bypass this requirement. The first idea is to take the tensor product of sequence (167) with a sufficiently ample line bundle $\mathcal{L}$ and obtain the exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \otimes \mathcal{L} \to \mathcal{F} \otimes \mathcal{L} \to 0. \tag{168}$$

We can choose $\mathcal{L}$ in such a way that $h^0(X, \mathcal{F} \otimes \mathcal{L}) \geqslant n+1$. This new sequence is still associated to the same extension class as before, since there is a natural isomorphism $\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong$

$\text{Ext}^1(\mathcal{F} \otimes \mathcal{L}, \mathcal{L})$. Therefore the study of (167) is equivalent to the study of (168) under this point of view.

In this way we have bypassed the problem on the number of global sections, but another obstruction to the application of the adjoint theory arises: the leftmost term of (168) is not the structure sheaf $\mathcal{O}_X$, but an invertible sheaf $\mathcal{L}$. In this chapter we want to develop a generalized version of the adjoint theory that will be able to deal with sequences of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{169}$$

and hence solve the aforementioned problems.

As we have seen in the previous chapters the adjoint theory is well-suited for the study of Torelli-type problems for irregular varieties with $q \geqslant n+1$. In fact in this cases the exact sequence associated to an infinitesimal deformation is

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0$$

and by hypothesis $h^0(X, \Omega^1_X) = q \geqslant n+1$. The generalized version we present in this chapter can deal not only with irregular varieties with $q \leqslant n$, but also with regular varieties. In particular we analyze the case of smooth projective hypersurfaces.

4.1.1 *Definition of generalized adjoint form*

Let X be a smooth compact complex variety of dimension m and let $\mathcal{F}$ and $\mathcal{L}$ be two locally free sheaves on X of rank n and 1 respectively. Consider the exact sequence of locally free sheaves

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{170}$$

associated to an element $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{L}) \cong H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$. Recall that the invertible sheaf $\det \mathcal{F} := \bigwedge^n \mathcal{F}$ fits into the exact sequence

$$0 \to \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L} \to \bigwedge^{n} \mathcal{E} \to \det \mathcal{F} \to 0, \tag{171}$$

which still corresponds to ξ under the isomorphism $\text{Ext}^1(\mathcal{F}, \mathcal{L}) \cong \text{Ext}^1(\det \mathcal{F}, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}) \cong H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$. Furthermore $\det \mathcal{F}$ satisfies

$$\det \mathcal{F} \otimes \mathcal{L} \cong \det \mathcal{E}. \tag{172}$$

Note that sequence (171) corresponds to sequence (144) of the previous chapter while isomorphism (172) corresponds to (145).

Let $\delta_\xi \colon H^0(X, \mathcal{F}) \to H^1(X, \mathcal{L})$ be the connecting homomorphism related to (170), and let $W \subset \operatorname{Ker}(\delta_\xi)$ be a vector subspace of dimension $n+1$. Choose a basis $\mathcal{B} := \{\eta_1, \dots, \eta_{n+1}\}$ of W. Take liftings $s_1, \dots, s_{n+1} \in H^0(X, \mathcal{E})$ of the sections $\eta_1, \dots, \eta_{n+1}$. If we consider the natural map

$$\Lambda^n \colon \bigwedge^n H^0(X, \mathcal{E}) \to H^0(X, \bigwedge^n \mathcal{E})$$

we can define the sections

$$\Omega_i := \Lambda^n(s_1 \wedge \ldots \wedge \widehat{s_i} \wedge \ldots \wedge s_{n+1}) \qquad (173)$$

for $i = 1, \dots, n+1$. As in the previous chapter, denote by ω_i, for $i = 1, \dots, n+1$, the section $\lambda^n(\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$. We have that ω_i is the image of Ω_i via the morphism

$$H^0(X, \bigwedge^n \mathcal{E}) \to H^0(X, \det \mathcal{F}).$$

The vector subspace of $H^0(X, \det \mathcal{F})$ generated by $\omega_1, \dots, \omega_{n+1}$ is denoted again by $\lambda^n W$.

Definition 4.1.1. *If $\lambda^n W$ is nontrivial, it induces a sublinear system $|\lambda^n W| \subset \mathbb{P}(H^0(X, \det \mathcal{F}))$ that we will call* adjoint sublinear system*. We call D_W its fixed divisor and Z_W the base locus of its moving part $|M_W| \subset \mathbb{P}(H^0(X, \det \mathcal{F}(-D_W)))$.*

Definition 4.1.2. *As in the previous chapters, we say that an extension $\xi \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{L})$ is* supported on a divisor D *if*

$$\xi \in \operatorname{Ker} \operatorname{Ext}^1(\mathcal{F}, \mathcal{L}) \to \operatorname{Ext}^1(\mathcal{F}(-D), \mathcal{L}). \qquad (174)$$

Definition 4.1.3. *The section $\Omega \in H^0(X, \det \mathcal{E})$ corresponding to $s_1 \wedge \ldots \wedge s_{n+1}$ via*

$$\Lambda^{n+1} \colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{E}) \qquad (175)$$

is called generalized adjoint form.

Remark 4.1.4. *It is easy to see by local computation that this section is in the image of the natural injection $\det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} \to \det \mathcal{E}$.*

We want to find a condition corresponding to the one in Problem 3.1.4. Note that in that case the condition $\omega_{\xi, W, \mathcal{B}} \in \lambda^n W$ can be written equivalently as

$$\omega_{\xi, W, \mathcal{B}} \in \operatorname{Im}(H^0(X, \mathcal{O}_X) \otimes \lambda^n W \to H^0(X, \det \mathcal{F}))$$

by the obvious reason that $H^0(X, \mathcal{O}_X) = \mathbb{C}$. Hence in the general case this condition can be replaced by

$$\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E})\right). \tag{176}$$

Note that since $\det \mathcal{E}$ is no longer isomorphic to $\det \mathcal{F}$, we have a difference in the codomain. Using the commutativity of

$$\begin{array}{ccc} H^0(X, \mathcal{L}) \otimes \bigwedge^n \mathcal{E} & \longrightarrow & H^0(X, \mathcal{L}) \otimes \det \mathcal{F} \\ & \searrow & \downarrow \\ & & \det \mathcal{E}, \end{array}$$

(176) can be equivalently written as

$$\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E})\right). \tag{177}$$

Note that the map in (177) is given by the wedge product, while the map in (176) by the isomorphism (172).

Remark 4.1.5. *If $H^0(X, \mathcal{L}) = 0$, then these conditions are equivalent to $\Omega = 0$.*

Remark 4.1.6. *As in the previous chapter, the choice of the liftings is not relevant for our purposes. Take different liftings $s_1', \dots, s_{n+1}' \in H^0(X, \mathcal{E})$ of $\eta_1, \dots, \eta_{n+1}$ and call $\Omega_i' \in H^0(X, \bigwedge^n \mathcal{E})$ and $\Omega' \in H^0(X, \det \mathcal{E})$ the corresponding sections constructed as above. Obviously*

$$\operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E})\right) = \\ = \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \left\langle \Omega_i' \right\rangle \to H^0(X, \det \mathcal{E})\right),$$

since they are both equal to $\operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E})\right)$. It is also easy to see that $\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E})\right)$ iff $\Omega' \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \left\langle \Omega_i' \right\rangle \to H^0(X, \det \mathcal{E})\right)$.

Remark 4.1.7. *Consider another basis $\mathcal{B}' := \{\eta_1', \dots, \eta_{n+1}'\}$ of W and let A be the matrix of the basis change. The sections $s_1', \dots, s_{n+1}'$ obtained from $s_1, \dots, s_{n+1}$ through the matrix A are liftings of the new basis $\eta_1', \dots, \eta_{n+1}'$. The section $\Omega' := \Lambda^{n+1}(s_1' \wedge \ldots \wedge s_{n+1}')$ satisfies $\Omega' = \det A \cdot \Omega$. Moreover $\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E})\right)$ iff $\Omega' \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \left\langle \Omega_i' \right\rangle \to H^0(X, \det \mathcal{E})\right)$.*

Lemma 4.1.8. *If $\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E})\right)$, then we can find liftings $t_i \in H^0(X, \mathcal{E})$, $i = 1, \dots, n+1$, such that $\tilde{\Omega} := \Lambda^{n+1}(t_1 \wedge \ldots \wedge t_{n+1}) = 0$.*

Proof. By Remark 4.1.5, the proof is trivial if $H^0(X, \mathcal{L}) = 0$. Hence we assume $h^0(X, \mathcal{L}) > 0$. By hypothesis there exist $\sigma_i \in H^0(X, \mathcal{L})$ such that

$$\Omega = \sum_{i=1}^{n+1} \sigma_i \wedge \Omega_i, \tag{178}$$

that is

$$s_1 \wedge \ldots \wedge s_{n+1} = \sum_{i=1}^{n+1} s_1 \wedge \ldots \wedge \hat{s_i} \wedge \ldots \wedge s_{n+1} \wedge \sigma_i$$

for a choice of liftings $s_1, \ldots, s_{n+1}$ of $\eta_1, \ldots, \eta_{n+1}$. We can define new liftings:

$$t_i := s_i + (-1)^{n-i} \sigma_i.$$

Now, since

$$t_1 \wedge \ldots \wedge t_{n+1} = s_1 \wedge \ldots \wedge s_{n+1} - \sum_{i=1}^{n+1} s_1 \wedge \ldots \wedge \hat{s_i} \wedge \ldots \wedge s_{n+1} \wedge \sigma_i,$$

we immediately deduce $\tilde{\Omega} = 0$. □

Notation 4.1.9. *From now on for simplicity call $\Psi_{\mathcal{B}}$ the map*

$$\Psi_{\mathcal{B}} \colon H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E}). \tag{179}$$

Then conditions (176) and (177) read $\Omega \in \operatorname{Im} \Psi_{\mathcal{B}}$.

From the natural map

$$\mathcal{F}^\vee \otimes \mathcal{L} \to \mathcal{F}^\vee \otimes \mathcal{L}(D_W)$$

we have a homomorphism

$$H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \xrightarrow{\rho} H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W));$$

we call $\xi_{D_W} = \rho(\xi)$.

4.1.2 *Castelnuovo's free pencil trick*

The case where both $\mathcal{L}$ and $\mathcal{F}$ are of rank one, while X has dimension m, is peculiar. In this case $W = \langle \eta_1, \eta_2 \rangle \subset H^0(X, \mathcal{F})$ has dimension two; as usual we choose liftings $s_1, s_2 \in H^0(X, \mathcal{E})$ of η_1, η_2. Note also that $\omega_1 = \eta_2$ and $\omega_2 = \eta_1$, in particular $W = \lambda^1 W$ so D_W is the fixed part of W and Z_W is the base locus of its moving part. Call $\tilde{\eta}_i \in H^0(X, \mathcal{F}(-D_W))$ the sections corresponding to the η_i's via $H^0(X, \mathcal{F}(-D_W)) \to H^0(X, \mathcal{F})$. The following lemma is well known and it is the core of the Castelnuovo base point free pencil trick (see Lemma 2.4.2).

Lemma 4.1.10. *We have an exact sequence*

$$0 \to \mathcal{F}^\vee(D_W) \xrightarrow{i} \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{\nu} \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \to 0 \qquad (180)$$

where the morphism i is given by contraction with $-\tilde{\eta}_1$ and $\tilde{\eta}_2$, while ν is given by evaluation with $\tilde{\eta}_2$ on the first component and $\tilde{\eta}_1$ on the second one.

It is easy to see by local computation that sequence (180) fits into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{L}^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot D_W} & & \downarrow{\scriptstyle (-s_1, s_2)} & & \downarrow{\scriptstyle \Omega} & & \\
0 & \longrightarrow & \mathcal{F}^\vee(D_W) & \xrightarrow{i} & \mathcal{O}_X \oplus \mathcal{O}_X & \xrightarrow{\nu} & \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0.
\end{array} \qquad (181)$$

The morphism $\mathcal{E}^\vee \to \mathcal{O}_X \oplus \mathcal{O}_X$ is given by contraction with the sections $-s_1$ and s_2, the morphism $\mathcal{L}^\vee \to \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}$ by contraction with the adjoint Ω. We can prove now the following

Theorem 4.1.11. *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$, $\mathcal{L}$ be invertible sheaves on X. Consider $\xi \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$ associated to the extension (170). Define $W = \langle \eta_1, \eta_2 \rangle \subset \operatorname{Ker}(\delta_\xi) \subset H^0(X, \mathcal{F})$ and Ω as above. We have that $\Omega \in \operatorname{Im} \Psi_{\mathcal{B}}$ if and only if $\xi_{D_W} = 0$.*

Proof. Tensoring (181) by $\mathcal{L}$ and passing to cohomology we have the following diagram

$$\begin{array}{ccccc}
H^0(\mathcal{E}^\vee \otimes \mathcal{L}) & \longrightarrow & \mathbb{C} & \longrightarrow & H^1(\mathcal{F}^\vee \otimes \mathcal{L}) \\
\downarrow{\scriptstyle (s_1, -s_2)} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \rho} \\
H^0(\mathcal{L} \oplus \mathcal{L}) & \xrightarrow{\nu} & H^0(\mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \mathcal{L}) & \xrightarrow{\delta} & H^1(\mathcal{F}^\vee(D_W) \otimes \mathcal{L}).
\end{array} \qquad (182)$$

Obviously $\beta(1) = \Omega$ and, by commutativity, $\delta(\beta(1)) = \xi_{D_W}$. We have then $\xi_{D_W} = 0$ if and only if $\Omega \in \operatorname{Im}(H^0(\mathcal{L} \oplus \mathcal{L}) \xrightarrow{\nu} H^0(\mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \mathcal{L}))$. Since ν is given by the sections $\tilde{\eta}_2$ and $\tilde{\eta}_1$, this condition is equivalent to $\Omega \in \operatorname{Im}(H^0(X, \mathcal{L}) \otimes W \to H^0(X, \det \mathcal{E}))$, since $\det \mathcal{E} = \mathcal{F} \otimes \mathcal{L}$. □

4.1.3 *The Adjoint Theorem*

We go back now to the general case with $\mathcal{F}$ locally free of rank n. By obvious identifications the natural map

$$\operatorname{Ext}^1(\det \mathcal{F}, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}) \to \operatorname{Ext}^1(\det \mathcal{F}(-D_W), \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L})$$

gives an extension $\mathcal{E}^{(n)}$ and a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} & \longrightarrow & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det\mathcal{F}(-D_W) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} & \longrightarrow & \bigwedge^{n}\mathcal{E} & \longrightarrow & \det\mathcal{F} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & = & \det\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{D_W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{183}$$

Theorem 4.1.12 (Generalized Adjoint Theorem)**.** *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\mathcal{L}$ an invertible sheaf with $H^0(X,\mathcal{L}) \neq 0$. Consider an extension*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

corresponding to $\xi \in \text{Ext}^1(\mathcal{F},\mathcal{L}) \cong H^1(X,\mathcal{F}^\vee \otimes \mathcal{L})$. Define $W = \langle \eta_1, \dots, \eta_{n+1}\rangle \subset \text{Ker}\,(\delta_\xi) \subset H^0(X,\mathcal{F})$ and Ω as above.

If $\Omega \in \text{Im}\,\Psi_{\mathcal{B}}$ then $\xi \in \text{Ker}\,(H^1(X,\mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X,\mathcal{F}^\vee \otimes \mathcal{L}(D_W)))$.

Proof. By the hypothesis $\Omega \in \text{Im}\,\Psi_{\mathcal{B}}$ and by lemma (4.1.8), we can choose liftings $s_i \in H^0(X,\mathcal{E})$ of η_i with $\Omega = 0$.

Since D_W is the fixed divisor of the linear system $|\lambda^n W|$ and the sections ω_i generate this linear system, then the ω_i are in the image of

$$\det\mathcal{F}(-D_W) \to \det\mathcal{F},$$

so we can find sections $\tilde{\omega}_i \in H^0(X, \det\mathcal{F}(-D_W))$ such that

$$\tilde{\omega}_i \cdot d = \omega_i, \tag{184}$$

where d is a global section of $\mathcal{O}_X(D_W)$ with $(d) = D_W$. Hence, using the commutativity of (183), we can find liftings $\tilde{\Omega}_i \in H^0(X,\mathcal{E}^{(n)})$ of the sections Ω_i. The evaluation map

$$\bigoplus_{i=1}^{n+1} \mathcal{O}_X \xrightarrow{\tilde{\mu}} \mathcal{E}^{(n)}$$

given by the global sections $\tilde{\Omega}_i$, composed with the map α of (183), induces a map μ which fits into the following diagram

$$\begin{array}{ccccccccc} & & & & \bigoplus_{i=1}^{n+1}\mathcal{O}_X & = & \bigoplus_{i=1}^{n+1}\mathcal{O}_X & & \\ & & & & \downarrow{\scriptstyle\tilde{\mu}} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \longrightarrow & \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} & \longrightarrow & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det\mathcal{F}(-D_W) & \longrightarrow & 0. \end{array}$$

We point out that the morphism μ is given by multiplication by $\tilde{\omega}_i$ on the i-th component. The sheaf $\operatorname{Im}\tilde{\mu}$ is torsion free since it is a subsheaf of the locally free sheaf $\mathcal{E}^{(n)}$. Moreover, since $\Omega = 0$, a local computation shows that $\operatorname{Im}\tilde{\mu}$ has rank one outside Z_W. On the other hand the sheaf $\operatorname{Im}\mu$ is by definition

$$\operatorname{Im}\mu = \det\mathcal{F}(-D_W)\otimes\mathcal{I}_{Z_W}.$$

The morphism

$$\alpha\colon \mathcal{E}^{(n)} \to \det\mathcal{F}(-D_W)$$

induces a surjective morphism, that we continue to call α,

$$\operatorname{Im}\tilde{\mu} \xrightarrow{\alpha} \operatorname{Im}\mu$$

between two sheaves that are locally free of rank one outside Z_W. This morphism is also injective, because its kernel is a torsion subsheaf of the torsion free sheaf $\operatorname{Im}\tilde{\mu}$, hence it is trivial.

We have proved that

$$\operatorname{Im}\tilde{\mu} \cong \det\mathcal{F}(-D_W)\otimes\mathcal{I}_{Z_W},$$

so

$$\mathcal{E}^{(n)} \supset (\operatorname{Im}\tilde{\mu})^{\vee\vee} \cong \det\mathcal{F}(-D_W).$$

This isomorphism gives the splitting

$$0 \longrightarrow \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} \longrightarrow \mathcal{E}^{(n)} \curvearrowleft \det\mathcal{F}(-D_W) \longrightarrow 0.$$

Since ξ_{D_W} is the element of $H^1(X, \mathcal{F}^\vee\otimes\mathcal{L}(D_W))$ associated to this extension, we conclude that $\xi_{D_W} = 0$.

We have proved the Generalized Adjoint Theorem. □

4.1.3.1 *An inverse of the Generalized Adjoint Theorem*

As in the previous chapter, we have an inverse of the Generalized Adjoint Theorem.

Theorem 4.1.13. *Let* X *be an* m*-dimensional complex compact smooth variety. Let* $\mathcal{F}$ *be a rank* n *locally free sheaf on* X *and* $\mathcal{L}$ *an invertible sheaf. Consider an extension* $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ *corresponding to* $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})$*. Let* $W = \langle \eta_1, \dots, \eta_{n+1} \rangle$ *be a* $n+1$*-dimensional sublinear system of* $\mathrm{Ker}\,(\delta_\xi) \subset H^0(X, \mathcal{F})$*. Let* $\Omega \in H^0(X, \det \mathcal{E})$ *be an adjoint form associated to* W *as above. Assume that* $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$*. If* $\xi \in \mathrm{Ker}\,(H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W)))$*, then* $\Omega \in \mathrm{Im}\,\Psi_{\mathcal{B}}$*.*

Proof. If $\mathcal{F}$ is a rank one sheaf, then Theorem 4.1.11 gives the thesis without the extra assumption $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$. We assume then rank $\mathcal{F} \geqslant 2$.

In the remaining cases the proof is very similar to the one of Theorem 3.3.2. The hypothesis $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$ replaces $H^0(X, \mathcal{O}_X(D_W)) = \mathbb{C}$. □

By the Generalized Adjoint Theorem and Theorem 4.1.13 we deduce the following

Corollary 4.1.14. *If* $D_W = 0$*, then* $\xi = 0$ *iff* $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$*.*

4.2 GENERALIZED ADJOINT QUADRICS

The notion of adjoint quadric introduced in the previous chapter can be generalized also in this context. With the same notations introduced in Section 3.5, we give the following

Definition 4.2.1. *A* generalized adjoint quadric *for* Ω *is a quadric in* $\mathbb{P}(H^0(X, \det \mathcal{E})^\vee)$ *of the form*

$$\Omega^2 = \sum E_i \cdot \sigma_i,$$

where Ω *is a generalized adjoint of* $W \subset H^0(X, \mathcal{F})$ *and* σ_i*,* E_i *are elements of* $H^0(X, \det \mathcal{E})$ *for* $i = 1, \dots, n+1$ *with* $\sigma_i \in \mathrm{Im}\, H^0(X, \mathcal{L}) \otimes \lambda^n W$*.*

Theorem 4.2.2. *Let* X *be a* m*-dimensional smooth compact variety where* $m \geqslant 2$*. Let* $\mathcal{F}$ *be a generically globally generated locally free sheaf of rank* n *such that* $h^0(X, \mathcal{F}) \geqslant n+1$*. Let* Y *be the schematic image of* $\phi_{|M_{\det \mathcal{E}}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \det \mathcal{E})^\vee)$ *and assume that* $H^0(Y, \mathcal{I}_Y(2))$ *contains no adjoint quadrics, where* $\mathcal{I}_Y$ *is the ideal sheaf of* Y*. Assume also that the map* $H^0(X, \det \mathcal{E}) \otimes H^0(X, \mathcal{L}) \to H^0(X, \det \mathcal{E} \otimes \mathcal{L})$ *is surjective. If* $W \subset \mathrm{Ker}\,\delta_\xi$ *is an* $n+1$*-dimensional subspace and the associated adjoint* Ω *is sent to zero by the cup product with* ξ*, then* $\Omega \in \mathrm{Im}\,\Psi_{\mathcal{B}}$*.*

Proof. Let $\mathcal{B} = \{\eta_1, \dots, \eta_{n+1}\}$ be a basis of W. Set $\omega_i := \lambda^n(\eta_1 \wedge \dots \wedge \widehat{\eta_i} \wedge \dots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$ where $i = 1, \dots, n+1$, and denote by $\tilde{\omega}_i \in H^0(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W})$ the corresponding sections via $0 \to H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^0(X, \det \mathcal{F})$. The standard evaluation map $\bigwedge^n W \otimes \mathcal{O}_X \to \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}$ given by $\tilde{\omega}_1, \dots, \tilde{\omega}_{n+1}$ results in the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigwedge^n W \otimes \mathcal{O}_X \longrightarrow \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \longrightarrow 0 \tag{185}$$

which is associated to the class $\xi' \in \mathrm{Ext}^1(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}, \mathcal{K})$. This sequence fits into the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \longrightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow f & & \uparrow g & & \\
0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{L}^\vee & \longrightarrow & 0,
\end{array}$$

where f is the map given by the contraction by the sections $(-1)^{n+1-i} s_i$, for $i = 1, \dots, n+1$, and g is given by the global section $\sigma \in H^0(X, \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W})$ corresponding to the adjoint Ω (cf. 4.1.4). Tensoring by $\mathcal{L}$ we obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{K}} & \longrightarrow & \bigwedge^n W \otimes \mathcal{L} & \longrightarrow & \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}^\vee \otimes \mathcal{L} & \longrightarrow & \mathcal{E}^\vee \otimes \mathcal{L} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0.
\end{array}$$

We have the standard factorization

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{K}} & \longrightarrow & \bigwedge^n W \otimes \mathcal{L} & \longrightarrow & \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \tilde{\mathcal{K}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}^\vee \otimes \mathcal{L} & \longrightarrow & \mathcal{E}^\vee \otimes \mathcal{L} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

where the sequence in the middle is associated to the class $\xi'' \in H^1(X, \tilde{\mathcal{K}})$ which is the image of $\xi \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$ through the map $H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X, \tilde{\mathcal{K}})$. In particular we obtain the commutative square

$$\begin{array}{ccc}
H^0(X, \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W}) & \longrightarrow & H^1(X, \tilde{\mathcal{K}}) \\
\uparrow & & \| \\
H^0(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \tilde{\mathcal{K}}).
\end{array}$$

By commutativity we immediately have that the image of σ through the coboundary map

$$H^0(X, \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W}) \to H^1(X, \tilde{\mathcal{K}})$$

is ξ''. Tensoring by $\mathcal{L} \otimes \det \mathcal{E}$, the map $\mathcal{F}^\vee \to \mathcal{K}$ gives

$$\begin{array}{ccc} \mathcal{F}^\vee \otimes \mathcal{L} \otimes \det \mathcal{E} & \longrightarrow & \tilde{\mathcal{K}} \otimes \det \mathcal{E} \\ \| & \nearrow_{\Gamma} & \\ \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2} & & \end{array} \tag{186}$$

and, since $\xi \cdot \Omega \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L} \otimes \det \mathcal{E})$ is sent to $\xi'' \cdot \Omega \in H^1(X, \tilde{\mathcal{K}} \otimes \det \mathcal{E})$, we have that

$$H^1(\Gamma)(\xi \cup \Omega) = \xi'' \cdot \Omega, \tag{187}$$

where $\xi \cup \Omega$ is the cup product. By hypothesis $\xi \cup \Omega = 0 \in H^1(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2})$, so also $\xi'' \cdot \Omega = 0 \in H^1(X, \tilde{\mathcal{K}} \otimes \det \mathcal{E})$, hence the global section $\sigma \cdot \Omega \in H^0(X, \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{E})$ is in the kernel of the coboundary map

$$H^0(X, \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{F}) \to H^1(X, \tilde{\mathcal{K}} \otimes \det \mathcal{E})$$

associated to the sequence

$$\begin{aligned} 0 \to \tilde{\mathcal{K}} \otimes \det \mathcal{E} \to \bigwedge^n W \otimes \mathcal{L} \otimes \det \mathcal{E} \to \\ \to \det \mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \det \mathcal{E} \to 0. \end{aligned}$$

This occurs iff there exist $L_i^\sigma \in H^0(X, \mathcal{L} \otimes \det \mathcal{E})$, $i = 1, \dots, n+1$ such that

$$\sigma \cdot \Omega = \sum_{i=1}^{n+1} L_i^\sigma \cdot \tilde{\omega}_i. \tag{188}$$

Since by hypothesis $H^0(X, \det \mathcal{E}) \otimes H^0(X, \mathcal{L}) \to H^0(X, \det \mathcal{E} \otimes \mathcal{L})$ is surjective, we can write $L_i^\sigma = E_i^\sigma \cdot l_i^\sigma$ where $E_i^\sigma \in H^0(X, \det \mathcal{E})$ and $l_i^\sigma \in H^0(X, \mathcal{L})$. The relation (188) gives the following relation in $H^0(X, \det \mathcal{E} \otimes \det \mathcal{E})$:

$$\Omega \cdot \Omega = \sum_{i=1}^{n+1} (E_i^\sigma \cdot l_i^\sigma) \cdot \omega_i = \sum_{i=1}^{n+1} E_i^\sigma \cdot (l_i^\sigma \cdot \omega_i). \tag{189}$$

Assume now that Ω is not in the image of $H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E})$. Then (189) gives a nontrivial adjoint quadric in $H^0(Y, \mathcal{I}_Y(2))$. A contradiction. □

Corollary 4.2.3. *Under the same hypotheses assume also that* $n = \operatorname{rk}\mathcal{F} = \dim X$, *if* $\xi \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$ *is such that* $\delta_\xi = 0$ *and* $\xi \cup \Omega = 0$, *where* Ω *is the adjoint form associated to a generic* $n+1$*-dimensional subspace* $W \subset H^0(X, \mathcal{F})$, *then* ξ *is a supported deformation, that is,* $\xi_{D_\mathcal{F}}$ *is trivial.*

Proof. Take W a generic $n+1$-dimensional subspace of $H^0(X, \mathcal{F})$. By Theorem 4.2.2 we have that $\Omega \in \operatorname{Im}\Psi_\mathcal{B}$, hence we apply Theorem 4.1.12 and we deduce that $\xi_{D_W} = 0$. By the fact that $D_\mathcal{F} = D_W$ (see Remark 3.5.4), we conclude. □

Recall the following definition ([30]):

Definition 4.2.4. *We say that a property holds for a sufficiently ample line bundle* $\mathcal{L}$ *on a projective variety* X *if there exists an ample line bundle* $\mathcal{L}_0$ *such that the property holds for all* $\mathcal{L}$ *with* $\mathcal{L} \otimes \mathcal{L}_0^{-1}$ *ample.*

A key observation is given in the following general lemma

Lemma 4.2.5. *Let* X *be a projective variety and* $\mathcal{F}$ *a locally free sheaf of rank* n *on* X. *If* $\mathcal{L}$ *is a sufficiently ample line bundle, then the natural map*

$$\bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}) \to H^0(X, \det(\mathcal{F} \otimes \mathcal{L})) \tag{190}$$

is surjective.

Proof. First take an ample line bundle $\mathcal{L}_1$ such that $\mathcal{F} \otimes \mathcal{L}_1$ is generated by its global sections; see [40, Definition on page 153]. That is we have an exact sequence

$$0 \to \mathcal{K}_0 \to H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}_1 \to 0.$$

The wedge of this sequence is

$$0 \to \mathcal{K}_1 \to \bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{O}_X \to \det(\mathcal{F} \otimes \mathcal{L}_1) \to 0.$$

Since $H^1(X, \mathcal{K}_1)$ may not be zero, we take $\mathcal{L}_2$ such that $H^1(X, \mathcal{K}_1 \otimes \mathcal{L}_2^n) = 0$; cf. [40, Chapter III, Theorem 5.2 (b)]. We have the exact sequence

$$0 \to \mathcal{K}_1 \otimes \mathcal{L}_2^n \to \bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{L}_2^n \to \det(\mathcal{F} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \to 0.$$

By hypothesis the map

$$\bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2^n) \to H^0(X, \det(\mathcal{F} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2))$$

is surjective. Now consider the following commutative diagram

$$\begin{array}{ccc} \bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2^n) & \longrightarrow & H^0(X, \det(\mathcal{F} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)) \\ \uparrow & & \uparrow \\ \bigwedge^n (H^0(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2)) & \longrightarrow & \bigwedge^n H^0(X, \mathcal{F} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \end{array}$$

where all the maps are natural. The top horizontal map is surjective by what we have just seen. The first vertical map is induced by

$$v_1 \otimes u_1 \wedge \ldots \wedge v_n \otimes u_n \to (v_1 \wedge \ldots \wedge v_n) \otimes (u_1 \otimes \ldots \otimes u_n)$$

and we can assume that it is surjective since $\mathcal{L}_2$ is ample enough to have that

$$\mathrm{Sym}^n(H^0(X, \mathcal{L}_2)) \to H^0(X, \mathcal{L}_2^n)$$

is surjective. Hence the second vertical arrow is surjective and taking $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ the lemma is proved. □

Remark 4.2.6. *This lemma, applied to the case we are interested in, shows that if we tensor the sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0 \tag{191}$$

by an invertible sheaf $\mathcal{L}$ sufficiently ample, then every section of $H^0(X, \det(\mathcal{E} \otimes \mathcal{L}))$ can be obtained by an element of $\bigwedge^{n+1} H^0(X, \mathcal{E} \otimes \mathcal{L})$. This means that every element of $H^0(X, \det(\mathcal{E} \otimes \mathcal{L}))$ is a linear combination of suitable adjoint forms.

We have then

Theorem 4.2.7. *Let $\xi \in H^1(X, \mathcal{F})$, $\xi \neq 0$ be a nontrivial extension class associated to the exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0. \tag{192}$$

If $\mathcal{L}$ is sufficiently ample, then there exist $N \in \mathbb{N}$ such that for every $n \geqslant N$

1. $\bigwedge^{n+1} H^0(X, \mathcal{E} \otimes \mathcal{L}^{\otimes n}) \to H^0(X, \det(\mathcal{E} \otimes \mathcal{L}^{\otimes n}))$ *is surjective*
2. *We can construct an adjoint Ω for the twisted sequence*

$$0 \to \mathcal{L}^{\otimes n} \to \mathcal{E} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \to 0$$

and there exist $E_i, \sigma_i \in H^0(X, \det(\mathcal{E} \otimes \mathcal{L}^{\otimes n}))$ for $i = 1, \ldots, n+1$, such that $\Omega^2 = \sum E_i \cdot \sigma_i$, i.e. there are adjoint quadrics.

Proof. The first part is a direct consequence of the previous lemma.

For the second part, note that for $\mathcal{L}$ sufficiently ample the sequence

$$0 \to \mathcal{L}^{\otimes n} \to \mathcal{E} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \to 0$$

satisfies

- $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \geqslant n+1$,
- $H^0(X, \det(\mathcal{E} \otimes \mathcal{L}^{\otimes n})) \otimes H^0(X, \mathcal{L}^{\otimes n}) \to H^0(X, \det(\mathcal{E} \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes n})$ is surjective,
- $H^1(X, \mathcal{L}^{\otimes n}) = 0$.

We can assume that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections, hence $D_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}} = 0$ and $\xi_{D_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}} = \xi$. By Corollary 4.2.3, we conclude.

□

4.2.1 *Adjoint quadrics and Syzygies*

There is another way to look at adjoint quadrics. Following [28], we recall the notion of syzygy.

Consider a field k and a finite dimensional vector space V over k. $S(V)$ will denote the symmetric algebra on V. Then if we take a graded $S(V)$-module $B = \bigoplus_{q \in \mathbb{Z}} B_q$ we have the standard Koszul complex

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \overset{d_{p-1,q+1}}{\to} \bigwedge^{p} V \otimes B_q \overset{d_{p,q}}{\to} \bigwedge^{p-1} V \otimes B_{q+1} \to \dots \tag{193}$$

The maps $d_{p,q}$ are constructed as follows.

The identity $i \in V^{\vee} \otimes V$ gives the contraction

$$\bigwedge^{p} V \to \bigwedge^{p-1} V \otimes V$$

which, composed with the multiplication

$$V \otimes B_q \to B_{q+1}$$

is the map $d_{p,q}$:

$$\begin{array}{ccc} \bigwedge^p V \otimes B_q & \longrightarrow & \bigwedge^{p-1} V \otimes V \otimes B_q \, . \\ & \searrow^{d_{p,q}} & \downarrow \\ & & \bigwedge^{p-1} V \otimes B_{q+1} \end{array}$$

The cohomology groups $K_{p,q}$ associated to this complex are called Koszul cohomology groups and they are strictly related to the so called syzygies of B.

The syzygies can be introduced as follows. Consider a minimal free resolution of B of the form

$$\cdots \bigoplus_{q\geqslant q_1} S(V)(-q)\otimes M_{1,q} \to \bigoplus_{q\geqslant q_1} S(V)(-q)\otimes M_{0,q} \to B \to 0. \tag{194}$$

This resolution exists provided that $\dim_k B_q < \infty$ for all q and $\{q \in \mathbb{Z} \mid B_q \neq 0\}$ is bounded from below.

Definition 4.2.8. *The* syzygies *of order* p *and weight* q *for the* $S(V)$-*module* B *are the vector spaces* $M_{p,q}$.

In particular $M_{0,q}$ is the vector space generated by the degree q generators of B as $S(V)$-module, and $M_{1,q}$ is generated by the primitive relations of weight q among the generators of B. That is, an element of $M_{1,q}$ is a relation $\sum_i u_i x_i$ such that x_i are generators of degree e_i and u_i are elements of $S(V)$ of degree $q-e_i$. The term primitive means that this relation is not a linear combination of relations of lower weight.

The relation between syzygies and Koszul cohomology is clear thanks to the following theorem: c.f. [28, Theorem 1.b.4].

Theorem 4.2.9. *As vector spaces over* k, $K_{p,q} \cong M_{p,p+q}$.

We are interested in the following application. Consider X a smooth projective variety and $\xi \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$ associated to the extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0. \tag{195}$$

As usual we also assume that $\mathcal{F}$ is locally free and generically globally generated of rank n such that $h^0(X,\mathcal{F}) \geqslant n+1$.

With the notation above, take

$$V := \operatorname{Im}\left(H^0(X,\mathcal{L})\otimes H^0(X,\det\mathcal{F}) \to H^0(X,\det\mathcal{E})\right)$$

and

$$B = \bigoplus_{q\in\mathbb{Z}} H^0(X, q\det\mathcal{E}).$$

Then B is an $S(V)$-module and we have the following

Proposition 4.2.10. *Assume that* V *is strictly contained in* $H^0(X,\det\mathcal{E})$ *and that* $H^0(X,\mathcal{L})\otimes H^0(X,\det\mathcal{E}) \to H^0(X,\det\mathcal{E}\otimes\mathcal{L})$ *is surjective. Assume that the deformation* ξ *satisfies* $\delta_\xi = 0$ *and* $\xi \cup \Omega = 0$, *where* Ω *is the adjoint form associated to a generic* $n+1$-*dimensional subspace* $W \subset H^0(X,\mathcal{F})$. *If* $K_{1,1} \cong M_{1,2} = 0$, *then* $\xi_{D_{\mathcal{F}}} = 0$.

Proof. Consider a generic subspace $W \subset H^0(X, \mathcal{F})$ and an associated generalized adjoint Ω. If

$$\Omega \in \operatorname{Im}\left(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E})\right),$$

we are done by Theorem 4.1.12 and Remark 3.5.4. Otherwise Ω and σ_i can be taken as generators of B, hence every adjoint quadric

$$\Omega^2 - \sum L_i \cdot \sigma_i$$

can be seen as an element of $M_{1,2}$. Since this is zero by hypothesis it follows that $\Omega^2 - \sum L_i \cdot \sigma_i$ is not a primitive relation among the generators of B. Hence there are no adjoint quadrics and, by Corollary 4.2.3, the deformation is supported on the base locus $D_{\mathcal{F}}$. □

5 APPLICATIONS OF THE GENERALIZED ADJOINT FORM: SMOOTH HYPERSURFACES

In this chapter the theory of generalized adjoint forms is applied to the case of smooth projective hypersurfaces and of smooth sufficiently ample divisors of a projective variety. See [61].

5.1 INFINITESIMAL TORELLI THEOREM FOR PROJECTIVE HYPERSURFACES

We start with the study of smooth hypersurfaces of the projective space $\mathbb{P}^n$.

5.1.1 *Meromorphic 1-forms on a smooth projective hypersurface*

Let $V \subset \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ of degree $\deg F = d$. An infinitesimal deformation $\xi \in \operatorname{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ of V gives an exact sequence for the sheaf of differential forms Ω^1_V:

$$0 \to \mathcal{O}_V \to \Omega^1_{\mathcal{V}|V} \to \Omega^1_V \to 0. \tag{196}$$

We assume that $n \geqslant 3$, hence $H^0(V, \Omega^1_V) = 0$ and we are facing the problem described at the beginning of Chapter 4. As we have seen the idea is to twist (196) by a suitable integer a such that $\Omega^1_V(a)$ has at least $n = \operatorname{rank}(\Omega^1_V) + 1$ global sections. Using the exact sequence

$$0 \to \mathcal{O}_V(-d) \to \Omega^1_{\mathbb{P}^n|V} \to \Omega^1_V \to 0 \tag{197}$$

one can compute that $a = 2$ is enough for this purpose, that is $h^0(V, \Omega^1_V(2)) \geqslant n$. So from now on we will consider the sequence

$$0 \to \mathcal{O}_V(2) \to \Omega^1_{\mathcal{V}|V}(2) \to \Omega^1_V(2) \to 0 \tag{198}$$

which again corresponds to

$$\xi \in \operatorname{Ext}^1(\Omega^1_V(2), \mathcal{O}_V(2)) \cong \operatorname{Ext}^1(\Omega^1_V, \mathcal{O}_V) \cong H^1(V, T_V).$$

Denote by $\mathcal{J}$ the Jacobian ideal of F, that is the ideal of $\mathbb{C}[x_0, \dots, x_n]$ generated by the partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 0, \dots, n$. As we have seen in Section 1.5, the deformation ξ is given by a class $[R]$ of degree d in the quotient $\mathbb{C}[x_0, \ldots, x_n]/\mathcal{J}$. If we choose a representative R for this class, then $F + tR = 0$, for small t, is the equation of the hypersurface that is the associated deformation of V.

From now on fix an infinitesimal deformation ξ and a representative R of its corresponding class.

Putting together sequence (198) and the conormal exact sequence (197) we obtain the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & \Omega^1_{\mathbb{P}^n|V}(2) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \mathcal{O}_V(2-d) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

which can be completed as follows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{\mathbb{P}^n|V}(2) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_V(2-d) & = & \mathcal{O}_V(2-d) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array} \qquad (199)$$

Since the deformation ξ of (198) comes from $R \in H^0(\mathbb{P}^n, \mathcal{O}^n_{\mathbb{P}}(d))$, then it gives the zero element in $H^1(V, T_{\mathbb{P}^n|V})$, hence we have that the sheaf $\mathcal{G}$ in (199) is a direct sum $\mathcal{G} = \mathcal{O}_V(2) \oplus \Omega^1_{\mathbb{P}^n|V}(2)$. This can be seen also from the fact that $\mathcal{G}$ is the restriction to V of the tensor product with $\mathcal{O}_{\mathbb{P}^n}(2)$ of the sheaf of Kähler differentials on the product of $\mathbb{P}^n$ and a disk. Hence we have a natural morphism $\phi\colon \Omega^1_{\mathbb{P}^n|V}(2) \to \Omega^1_{\mathcal{V}|V}(2)$ which fits in the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & \| & & \uparrow & \nwarrow^{\phi} & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{\mathbb{P}^n|V}(2) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_V(2-d) & = & \mathcal{O}_V(2-d) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{200}$$

The morphism ϕ gives in a natural way a morphism

$$\phi^n\colon H^0(V, \det(\Omega^1_{\mathbb{P}^n|V}(2))) \to H^0(V, \det(\Omega^1_{\mathcal{V}|V}(2)))$$

hence, since

$$H^0(V, \det(\Omega^1_{\mathbb{P}^n|V}(2))) \cong H^0(V, \mathcal{O}_V(n-1)) \tag{201}$$

and

$$H^0(V, \det(\Omega^1_{\mathcal{V}|V}(2))) \cong H^0(V, \mathcal{O}_V(n+d-1)), \tag{202}$$

ϕ^n can be seen as a map

$$\phi^n\colon H^0(V, \mathcal{O}_V(n-1)) \to H^0(V, \mathcal{O}_V(n+d-1)).$$

We can write explicitly the isomorphism (201). Note that

$$H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(2n)) \to H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$$

is surjective, so we will focus on the rational n-forms on $\mathbb{P}^n$. These forms may be written as $\omega = \frac{P\Psi}{Q}$ where

$$\Psi = \sum_{i=0}^{n} (-1)^i x_i (dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n)$$

gives a generator of $H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(n+1))$ and $\deg Q = \deg P + (n+1)$. This comes for example by [34, Corollary 2.11] and it is easy to see. In fact take a rational n-form with poles along

$Q = 0$. On the chart $x_0 \neq 0$ with coordinates $z_i = x_i/x_0$ it can be written as $\frac{P(z)}{Q(z)} dz_1 \wedge \dots \wedge dz_n$ for some polynomial P. Since

$$dz_1 \wedge \dots \wedge dz_n = d(x_1/x_0) \wedge \dots \wedge d(x_n/x_0) = \frac{1}{x_0^{n+1}} \Psi$$

we immediately obtain by homogenization that the rational form is

$$\frac{x_0^{\deg Q} P(x)}{x_0^{\deg P + n+1} Q(x)} \Psi.$$

In particular $\deg Q = \deg P + n + 1$.

In our case Q is a polynomial of degree $2n$, hence P has degree $n-1$. This identification depends on the (noncanonical) choice of the polynomial Q and gives an isomorphism

$$H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n)) \to H^0(V, \mathcal{O}_V(n-1))$$

defined by $\omega_{|V} \mapsto P$.

Proposition 5.1.1. *ϕ^n is given via the multiplication by the polynomial R (modulo F).*

Proof. Locally we can see $\mathcal{V}$ in the product $\Delta \times \mathbb{P}^n$ of the projective space with a disk; here $\mathcal{V}$ is defined by the equation $F + tR = 0$. Hence $d(F + tR) = 0$ in $\Omega^1_{\mathcal{V}}$, that is $dF + dt \cdot R + dR \cdot t = 0$.

Call $F_i := \frac{\partial F}{\partial x_i}$. Since V is smooth, there exist i such that $U_i = (F_i \neq 0)$ is a nontrivial open subset; let for example U_1 be nontrivial. Take local coordinates $z_i = \frac{x_i}{x_0}$ in the open set $(x_0 \neq 0) \cap U_1$. Then we have

$$dz_1 = -\frac{Rdt}{F_1} - \frac{tdR}{F_1} - \sum_{i>1} \frac{F_i}{F_1} dz_i \tag{203}$$

which gives in V (that is for $t = 0$)

$$dz_1 = -\frac{Rdt}{F_1} - \sum_{i>1} \frac{F_i}{F_1} dz_i \tag{204}$$

The image $\phi^n(\omega_{|V})$ is then obtained by the substitution of (204) in $\frac{P(z)}{Q(z)} dz_1 \wedge \ldots \wedge dz_n$, which is the local form of $\frac{P(\xi)\Psi}{Q(\xi)}$. Hence

$$\frac{P(z)}{Q(z)} dz_1 \wedge \ldots \wedge dz_n = -\frac{P(z)R(z)}{Q(z)F_1(z)} dt \wedge dz_2 \wedge \ldots \wedge dz_n.$$

If we homogenize we obtain on U_1

$$\frac{P\Psi}{Q} = -\frac{PR}{QF_1} \sum_{i \neq 1} (-1)^{i-1} \operatorname{sgn}(i-1) x_i dt \wedge dx_0 \wedge \widehat{dx_1} \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

Hence

$$\phi^n(\omega_{|V}) = -\frac{PR}{QF_1}\sum_{i\neq 1}(-1)^{i-1}\mathrm{sgn}(i-1)x_i dt\wedge dx_0\wedge\widehat{dx_1}\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_n$$

and it is clear that ϕ^n is given by multiplication with R. □

5.1.2 *A canonical choice of adjoints on a hypersurface of degree* $d > 2$

We want now to construct adjoint forms associated to the sequence (198).

Remember that $n \geqslant 3$, so that $H^1(V, \mathcal{O}_V(2)) = H^1(V, \mathcal{O}_V(2-d)) = 0$, and we can lift all the global sections of $H^0(V, \Omega^1_V(2))$ both in the horizontal and in the vertical sequence of (200).

We take $\eta_1, \ldots, \eta_n \in H^0(V, \Omega^1_V(2))$ global forms and we want to find liftings $s_1, \ldots, s_n \in H^0(V, \Omega^1_{\mathcal{V}|V}(2))$. This can be done since $H^1(V, \mathcal{O}_V(2))$ is zero. A generalized adjoint is then the global section of the sheaf $\det(\Omega^1_{\mathcal{V}|V}(2)) = \mathcal{O}_V(n+d-1)$ given by $\Omega := \Lambda^n(s_1\wedge\ldots\wedge s_n) \in H^0(V, \det(\Omega^1_{\mathcal{V}|V}(2)))$.

We point out another interesting way to compute this generalized adjoint form using Proposition 5.1.1.

Consider the sequence (197), that is the vertical sequence in (200). Since $H^1(V, \mathcal{O}_V(2-d)) = 0$, we can find liftings $\tilde{s}_1, \ldots, \tilde{s}_n \in H^0(V, \Omega^1_{\mathbb{P}^n|V}(2))$ of the sections $\eta_1, \ldots, \eta_n$. Furthermore they are unique if $d > 2$. We can thus consider the adjoint form associated to (197) given by $\widetilde{\Omega} := \Lambda^n(\tilde{s}_1\wedge\ldots\wedge\tilde{s}_n)$. This adjoint is independent from the deformation ξ; it depends only on V and its embedding in $\mathbb{P}^n$. If $d > 2$, then $\widetilde{\Omega}$ is unique.

To describe $\widetilde{\Omega}$ explicitly we first consider the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n}(2-d) \to \Omega^1_{\mathbb{P}^n}(2) \to \Omega^1_{\mathbb{P}^n|V}(2) \to 0.$$

If $d > 2$, the vanishing of $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2-d))$ and $H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2-d))$ (cf. Bott Formulas), gives the isomorphism $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)) = H^0(V, \Omega^1_{\mathbb{P}^n|V}(2))$. Hence, the forms $\tilde{s}_i$ are the restriction on V of global rational 1-forms. By [34, Theorem 2.9] we can write

$$\tilde{s}_i = \frac{1}{Q}\sum_{j=0}^{n} L^i_j dx_j \tag{205}$$

where $\deg Q = 2$ and L^i_j is a homogeneous polynomial of degree 1 which does not contain x_j in its expression. Hence

$$\widetilde{\Omega} = \Lambda^n(\tilde{s}_1\wedge\ldots\wedge\tilde{s}_n) = \frac{1}{Q^n}\sum_{i=0}^{n} M_i dx_0\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$$

where M_i is the determinant of the matrix obtained by

$$\begin{pmatrix} L_0^1 & \dots & L_0^n \\ \vdots & & \vdots \\ L_n^1 & \dots & L_n^n \end{pmatrix} \tag{206}$$

removing the i-th row. Since $\widetilde{\Omega}$ is a rational n-form on $\mathbb{P}^n$, following [34, Corollary 2.11] it can be written as $\frac{P\Psi}{Q^n}$, and we deduce that

$$\frac{M_i}{(-1)^i x_i} = P \tag{207}$$

for all $i = 0, \dots, n$. P is a polynomial of degree $n-1$ and it corresponds to $\widetilde{\Omega}$ via the isomorphism $H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n)) \cong H^0(V, \mathcal{O}_V(n-1))$. Hence by Proposition 5.1.1 we have that the form $\Omega \in H^0(V, \mathcal{O}_V(n+d-1))$ given by PR is a canonical choice of adjoint form for $W = \langle \eta_1, \dots, \eta_n \rangle$ and ξ.

Remark 5.1.2. *Alternatively this can be seen using the Euler sequence on V:*

$$0 \to \mathcal{O}_V \to \bigoplus^{n+1} \mathcal{O}_V(1) \to T_{\mathbb{P}^n|V} \to 0. \tag{208}$$

This sequence, dualized and conveniently tensorized gives

$$0 \to \Omega^1_{\mathbb{P}^n|V}(2) \to \bigoplus_{i=1}^{n+1} \mathcal{O}_V(1) \to \mathcal{O}_V(2) \to 0. \tag{209}$$

The sections $\tilde{s}_i$ are associated via the first morphism to an $n+1$-uple of linear polynomials $(L_i^0, \dots, L_i^n)$. Then, taking the wedge product of (209) we obtain an exact sequence

$$0 \to \Omega^n_{\mathbb{P}^n|V}(2n) \cong \mathcal{O}_V(n-1) \to \bigwedge^n \left(\bigoplus^{n+1} \mathcal{O}_V(1) \right) = \bigoplus^{n+1} \mathcal{O}_V(n) \to$$
$$\to \Omega^{n-1}_{\mathbb{P}^n|V}(2n) \to 0$$

where the morphism $\mathcal{O}_V(n-1) \to \bigoplus^{n+1} \mathcal{O}_V(n)$ is given by

$$G \mapsto (Gx_0, \dots, (-1)^n G x_n).$$

Since $\widetilde{\Omega} = \Lambda^n(\tilde{s}_1 \wedge \dots \wedge \tilde{s}_n) \in H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$ is sent exactly to $(L_0^0, \dots, L_0^n) \wedge \dots \wedge (L_n^0, \dots, L_n^n) = (M_0, \dots, M_n)$ (using the same notation as above), then we conclude that $\widetilde{\Omega}$ corresponds in $H^0(V, \mathcal{O}_V(n-1))$ to a polynomial P which satisfies

$$\frac{M_i}{(-1)^i x_i} = P.$$

5.1.3 *The adjoint sublinear systems obtained by meromorphic 1-forms*

To study the conditions given in (176) and (177), we need to describe the sections

$$\widetilde{\Omega_i} := \Lambda^{n-1}(\tilde{s}_1 \wedge \ldots \wedge \widehat{\tilde{s}_i} \wedge \ldots \wedge \tilde{s}_n) \in H^0(V, \Omega^{n-1}_{\mathbb{P}^n|V}(2n-2))$$

(cf. (173)) and their images in

$$H^0(V, \Omega^{n-1}_V(2(n-1))) = H^0(V, \mathcal{O}_V(n+d-3))$$

that we have denoted by ω_i.

A computation similar to the above shows that

$$\widetilde{\Omega_i} = \Lambda^{n-1}(\tilde{s}_1 \wedge \ldots \wedge \widehat{\tilde{s}_i} \wedge \ldots \wedge \tilde{s}_n) = = \frac{1}{Q^{n-1}} \sum_{j<k} M^i_{jk} dx_0 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \tag{210}$$

where M^i_{jk} is the determinant of the matrix obtained by (206) removing the i-th column and the j-th and k-th rows. On the other hand, rearranging the expression of [34, Theorem 2.9] we can write

$$\widetilde{\Omega_i} = \frac{1}{Q^{n-1}} \sum_j A^i_j (\sum_{k \neq j} (-1)^{k+j} \mathrm{sgn}(k-j) x_k dx_0 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n) \tag{211}$$

with $\deg A^i_j = n-2$.

Comparing (210) and (211) gives

$$M^i_{jk} = (-1)^{j+k}(A^i_j x_k - x_j A^i_k). \tag{212}$$

As before this can be computed also via the Euler sequence.

We call

$$\Xi_j := \sum_{k \neq j} (-1)^{k+j} \mathrm{sgn}(k-j) x_k dx_0 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n.$$

Note that the sections Ξ_j, for $j = 0, \ldots, n$ give a basis of $H^0(V, \Omega^{n-1}_{\mathbb{P}^n|V}(n))$.

Proposition 5.1.3. $\omega_i = \sum_j A^i_j \cdot F_j$ *in* $H^0(V, \mathcal{O}_V(n+d-3))$

Proof. It is enough to show that the image of Ξ_j through the morphism $\Omega^{n-1}_{\mathbb{P}^n|V}(n) \to \mathcal{O}_V(d-1)$ is F_j. Consider the exact sequence of the tangent sheaf of V:

$$0 \to T_V \to T_{\mathbb{P}^n|V} \to \mathcal{O}_V(d) \to 0. \tag{213}$$

The beginning of the Koszul complex is

$$\bigwedge^{n} T_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-d) \to \bigwedge^{n-1} T_{\mathbb{P}^n|V}$$

which, tensored by $\mathcal{O}_V(-n)$, gives

$$\bigwedge^{n} T_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n-d) \to \bigwedge^{n-1} T_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n). \tag{214}$$

This is exactly the dual of $\Omega^{n-1}_{\mathbb{P}^n|V}(n) \to \mathcal{O}_V(d-1)$. Hence we only need to show that the morphism (214) composed with the contraction by Ξ_i

$$\bigwedge^{n-1} T_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n) \xrightarrow{\Xi_i} \mathcal{O}_V$$

is the multiplication by F_i. This is easy to see by a standard local computation. □

Remark 5.1.4. *We immediately have that the polynomials associated to the sections ω_i are in the Jacobian ideal of V.*

Condition (176), that is

$$\Omega \in \operatorname{Im}\left(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1))\right),$$

can be written, modulo F, as

$$RP = \sum \omega_i \cdot S_i = \sum_{i,j} A^i_j \cdot F_j \cdot S_i, \tag{215}$$

where $\deg S_i = 2$. In particular this implies that RP is in the Jacobian ideal of V.

Proposition 5.1.5. *The base locus D_W of the linear system $|\lambda^n W|$ is zero for the generic W.*

Proof. By [55, Proposition 3.1.6] it is enough to prove that $H^0(V, \Omega^1_V(2))$ generically generates the sheaf $\Omega^1_V(2)$ and that $D_{H^0(V,\Omega^1_V(2))} = 0$. We have an explicit basis for $H^0(V, \Omega^1_V(2))$ given by

$$\frac{x_i dx_j - x_j dx_i}{Q} \tag{216}$$

where $i < j$ and $\deg Q = 2$. The vector space $\lambda^n H^0(V, \Omega^1_V(2)) \subset H^0(V, \mathcal{O}_V(n+d-3))$ is obviously nonzero, hence $H^0(V, \Omega^1_V(2))$ generically generates the sheaf $\Omega^1_V(2)$.

It remains to prove that $D_{H^0(V,\Omega^1_V(2))} = 0$. An easy computation (for example by induction) shows that $\lambda^n H^0(V, \Omega^1_V(2))$ contains all the polynomials of the form

$$x_{i_1} x_{i_2} \dots x_{i_{n-2}} \frac{\partial F}{\partial x_j} \tag{217}$$

where $\{i_1, \dots, i_{n-2}\} \subset \{1, \dots, n+1\}$ and $j \notin \{i_1, \dots, i_{n-2}\}$. Since V is smooth, these polynomials do not vanish simultaneously on a divisor, hence $D_{H^0(V,\Omega^1_V(2))} = 0$, and we are done. □

5.1.4 *On Griffiths' proof of infinitesimal Torelli Theorem*

As we have seen in Section 1.5, the deformation ξ is trivial if and only if R lies in the Jacobian ideal $\mathcal{J}$ of the variety V. Furthermore the infinitesimal Torelli for hypersurfaces states that the differential of the period map is injective if and only if this possibility does not occur. In this section we want to find equivalent conditions to these facts involving generalized adjoint forms. As we have pointed out at the beginning of this chapter, we remark that even if we consider the exact sequence (198)

$$0 \to \mathcal{O}_V(2) \to \Omega^1_{\mathcal{V}|V}(2) \to \Omega^1_V(2) \to 0,$$

and its corresponding adjoint forms, via the isomorphism

$$\mathrm{Ext}^1(\Omega^1_V(2), \mathcal{O}_V(2)) \cong \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \cong H^1(V, T_V)$$

we will obtain information on the original (untwisted) sequence, and hence on R.

The following lemma gives the first important translation into the setting of adjoint forms

Lemma 5.1.6. R *is in the Jacobian ideal* $\mathcal{J}$ *if and only if for the generic adjoint* Ω *it holds that*

$$\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1))).$$

Proof. If $\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1)))$, then by the Generalized Adjoint Theorem, $\xi_{D_W} = 0$. Since $D_W = 0$ by the previous proposition, the deformation is trivial, hence R lies in the Jacobian ideal.

Viceversa if $R \in \mathcal{J}$, the deformation is trivial and by Theorem 4.1.13, we have that $\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1)))$. □

Our theory gives another characterization for the class $[R]$ to be trivial.

Proposition 5.1.7. *Assume that* $\deg R = d > 3$. *Then* R *is in the Jacobian ideal* $\mathcal{J}$ *if and only if* $RP \in \mathcal{J}$ *for every polynomial* $P \in H^0(V, \mathcal{O}_V(n-1))$ *corresponding to a generalized adjoint* $\widetilde{\Omega} \in H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$.

Proof. One implication is trivial.

To prove the other one the idea is to show that every monomial of $H^0(V, \mathcal{O}_V(n-1))$ corresponds to a suitable generalized adjoint. Hence, if $RP \in \mathcal{J}$ for every polynomial $P \in H^0(V, \mathcal{O}_V(n-1))$ corresponding to a generalized adjoint, we have that

$$R \cdot H^0(V, \mathcal{O}_V(n-1)) \subset \mathcal{J}$$

and we are done by Macaulay Theorem (cf. Theorem 1.5.3 and [67, Theorem 6.19 and Corollary 6.20]).

We work by induction, since $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n-1)) \to H^0(V, \mathcal{O}_V(n-1))$ is surjective. The base of the induction is for $n = 2$. A simple computation shows that the map

$$\bigwedge^2 H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$$

is surjective because its image contains the canonical basis of degree one monomials.

For the general case we show that every monomial of degree $n-1$ is given by a generalized adjoint. Consider the natural homomorphism:

$$\bigwedge^n H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n-1))$$

and take a monomial M with $\deg M = n-1$. There is a variable x_i which does not appear in M. We restrict to the hyperplane $x_i = 0$ and we use induction on $\frac{M}{x_j}$, where x_j appears in M. There exist $s_1, \dots, s_{n-1} \in H^0(\mathbb{P}^{n-1}, \Omega^1_{\mathbb{P}^{n-1}}(2))$ with $s_1 \wedge \dots \wedge s_{n-1}$ which corresponds to $\frac{M}{x_j}$, that is

$$s_1 \wedge \dots \wedge s_{n-1} = \frac{M\Psi'}{x_j \cdot Q^{n-1}}$$

where

$$\Psi' = \sum_{k=0, k\neq i}^{n} (-1)^k x_k (dx_0 \wedge \dots \wedge \widehat{dx_i} \dots \wedge \widehat{dx_k} \dots \wedge dx_n)$$

gives a basis of $H^0(\mathbb{P}^{n-1}, \Omega^{n-1}_{\mathbb{P}^{n-1}}(n))$. It is easy to see that

$$s_1 \wedge \dots \wedge s_{n-1} \wedge \frac{(x_j dx_i - x_i dx_j)}{Q} = \frac{M\Psi}{Q^n},$$

i.e. M corresponds to a generalized adjoint, which is exactly our thesis. □

Putting together these results and the classical facts recalled in Section 1.5, we obtain the following theorem

Theorem 5.1.8. *For a smooth hypersurface* X *of degree* d *in* $\mathbb{P}^n$ *with* $n \geqslant 3$ *and* $d > 3$ *the following are equivalent:*

i) the differential of the period map is zero on the infinitesimal deformation

$$[R] \in (\mathbb{C}[\xi_0, \dots, \xi_n]/\mathcal{J})_d \simeq H^1(X, \Theta_X)$$

ii) R *is an element of the Jacobian ideal* $\mathcal{J}$

iii) $\Omega \in \operatorname{Im}(H^0(X, \mathcal{O}_X(2)) \otimes \lambda^n W \to H^0(X, \mathcal{O}_X(n+d-1)))$ *for the generic generalized adjoint* Ω

iv) The generic generalized adjoint Ω *lies in* $\mathcal{J}$*.*

This theorem is a new formulation of the Griffiths' infinitesimal Torelli for smooth projective hypersurfaces ([34, Theorem 9.8]) in the setting of adjoint forms.

5.2 INFINITESIMAL TORELLI THEOREM FOR SMOOTH SUFFICIENTLY AMPLE HYPERSURFACES

The same ideas of the previous section work in general for a sufficiently ample divisor of a smooth algebraic variety.

Take an n-dimensional smooth variety Y and a sufficiently ample line bundle L on Y. Let $s \in H^0(Y, L)$ be a global section and X the corresponding divisor. Assume that X is smooth. As in the case of smooth projective hypersurfaces, we can describe the infinitesimal variation of Hodge structure via a ring multiplication.

5.2.1 *Pseudo-Jacobi Ideal*

In the case that we are studying, the usual Jacobian ideal can be replaced by the so called pseudo-Jacobi ideal introduced in [30] and [27]. We briefly recall how it is constructed.

Given a line bundle L on Y, consider the extension

$$0 \to \mathcal{O}_Y \to \Sigma_L \xrightarrow{\tau} T_Y \to 0 \tag{218}$$

with extension class $-c_1(L) \in H^1(Y, \Omega^1_Y)$. Σ_L is a sheaf of differential operators of order less or equal to 1 on the sections of L. In an open subset of Y with coordinates $x_1, \dots, x_n$ this sheaf is free and is generated by the constant section 1 and the sections D_i, for $i = 1, \dots, n$, which operates on the sections of L by

$$D_i(f \cdot l) = \frac{\partial f}{\partial x_i} \cdot l$$

where l is a trivialization of L. The operators D_i are sent to $\frac{\partial}{\partial x_i}$ in T_Y.

In particular to a global section s of L, we can associate a global section $\widetilde{ds}$ of $L \otimes \Sigma_L^\vee$. If locally $s = f \cdot l$, then $\widetilde{ds}$ is given by

$$\widetilde{ds} = f \cdot l \cdot 1^\vee + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot l \cdot D_i^\vee \tag{219}$$

where $\{1^\vee, D_1^\vee, \dots, D_n^\vee\}$ is a local basis of $\Sigma_L^\vee$ dual to $\{1, D_1, \dots, D_n\}$.

Given a line bundle E, the contraction by $\widetilde{ds}$ gives a map

$$E \otimes \Sigma_L \otimes L^\vee \to E.$$

To give an idea in the case $E = \mathcal{O}_Y$, the contraction $\Sigma_L \otimes L^\vee \to \mathcal{O}_Y$ is given explicitly in local coordinates by

$$a_0 \cdot l^\vee \otimes 1 + \sum_{i=1}^{n} a_i \cdot l^\vee \otimes D_i \mapsto a_0 \cdot f + \sum_{i=1}^{n} a_i \cdot \frac{\partial f}{\partial x_i}.$$

Definition 5.2.1. *The* pseudo-Jacobi ideal $\mathcal{J}_{E,s}$ *is the image of the map*

$$H^0(Y, E \otimes \Sigma_L \otimes L^\vee) \to H^0(Y, E). \tag{220}$$

Definition 5.2.2. *The quotient* $H^0(Y, E)/\mathcal{J}_{E,s}$ *is denoted by* $R_{E,s}$ *and is called pseudo-Jacobi ring.*

The usual Jacobian ideal of a homogeneous polynomial F of degree d is recovered taking $L = \mathcal{O}_{\mathbb{P}^n}(d)$ and $E = \mathcal{O}_{\mathbb{P}^n}(k)$. In this case it easy to see that $\Sigma_L = \oplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1)$ and sequence (218) is the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0. \tag{221}$$

The pseudo-Jacobi ideal $\mathcal{J}_{\mathcal{O}_{\mathbb{P}^n}(k),F} \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ is generated by $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}$, that is the degree k part of the usual Jacobian ideal.

In the case of a smooth algebraic variety Y of dimension n with a smooth hypersurface X, we take L to be the sheaf $\mathcal{O}_Y(X)$, the section $s \in H^0(Y, L)$ is such that $X = \mathrm{div}(s)$, and $E = L = \mathcal{O}_Y(X)$. We consider the deformations of X inside of the ambient space Y. Exactly as in the case of projective hypersurfaces, such an infinitesimal deformation of X is given by $X + tR = 0$, $t^2 = 0$, where $R \in H^0(Y, L)$. Define $\mathrm{Aut}(Y, L) = \{f \colon Y \to Y \text{ such that } f^*(L) = L\}$. The Kuranishi family for X is constructed as in the case of smooth projective hypersurfaces using the incidence variety (see Section 1.5) and its base is $S/\mathrm{Aut}(Y, L)$ where

$$S = \{(s) \in |L| \mid \text{the corresponding divisor is smooth}\}.$$

We have

Proposition 5.2.3. *The tangent space to $S/Aut(Y, L)$ at X is $R_{L,s}$.*

See [27, Corollary page 48]. This means that an infinitesimal deformation is trivial if and only if R is an element of the pseudo-Jacobi ideal $\mathcal{J}_{L,s}$. We will denote by $\mathcal{P}$ the period map associated to the Kuranishi family. The fact that L is sufficiently ample allows to prove the following theorem, [27, Proposition page 45].

Theorem 5.2.4. *The derivative of the period map is given by the multiplication*

$$R_{L,s} \to \bigoplus_q \mathrm{Hom}(R_{K_Y \otimes L^{q+1},s}, R_{K_Y \otimes L^{q+2},s}). \tag{222}$$

Hence, as in the previous section, the infinitesimal Torelli problem is reduced to the study of the injectivity of a map given by multiplication. A positive solution is given by a suitable generalization of Macaulay Theorem 1.5.3 (see [30] and [27]).

Theorem 5.2.5 (Generalized Macaulay's Theorem)**.** *Let Y and X as above. Then*

1. $R_{K_Y^2 \otimes L^{n+1},s} \cong \mathbb{C}$
2. *For any fixed bundle A, for L sufficiently ample, the multiplication map*

$$R_{L^k \otimes A,s} \otimes R_{K_Y^2 \otimes L^{n+1-k} \otimes A^\vee,s} \to R_{K_Y^2 \otimes L^{n+1},s} \cong \mathbb{C} \tag{223}$$

is is a perfect pairing provided that

$$H^{k-1}(Y, A \otimes \bigwedge^{k} \Sigma_L) = H^k(Y, A \otimes \bigwedge^{k} \Sigma_L) = 0 \quad \textit{if} \quad k \neq 0, 1, n \tag{224}$$

or

$$H^1(Y, A \otimes \Sigma_L) = 0 \quad \textit{if} \quad k = 1 \tag{225}$$

or

$$H^{n-1}(Y, A \otimes \bigwedge^{n} \Sigma_L) = 0 \quad \textit{if} \quad k = n \tag{226}$$

The infinitesimal Torelli theorem for sufficiently ample smooth divisors can be also proved directly without using Macaulay's theorem, see [30].

5.2.2 *Infinitesimal Torelli and generalized adjoint*

We study the deformations of a smooth sufficiently ample hypersurface X in a smooth algebraic variety Y of dimension $n > 2$ using the theory of generalized adjoint forms. The goals are the same of Section 5.1.4.

An infinitesimal deformation $\xi \in \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \cong H^1(X, T_X)$ of X gives an exact sequence

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0. \tag{227}$$

As in the case of the hypersurfaces of $\mathbb{P}^n$, we consider deformations inside the ambient Y, that is deformations given by $R \in \mathbb{P}(H^0(Y, \mathcal{O}_Y(X)))$. Hence we have that ξ is in the image of the map $H^0(Y, \mathcal{O}_Y(X)) \to H^1(X, T_X)$ coming from the normal exact sequence

$$0 \to T_X \to T_{Y|X} \to \mathcal{O}_X(X) \to 0$$

and the restriction sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(X) \to \mathcal{O}_X(X) \to 0.$$

Take the tensor of sequence (227) by an ample divisor H, such that

1. $X - H$ is ample
2. the cohomology groups $H^i(Y, \mathcal{O}_Y(H))$ vanish for $i \geqslant 1$
3. the sheaf $\mathcal{O}_Y(K_Y + nH)$ is generated by global sections, that is we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \otimes H^0(\mathcal{O}_Y(K_Y + nH)) \to \mathcal{O}_Y(K_Y + nH) \to 0 \tag{228}$$

4. $H^1(Y, \mathcal{I} \otimes \mathcal{O}_Y(K_Y + n(X - H))) = 0$

5. $h^0(Y, \Omega^1_Y(H)) \geqslant n$

6. $D_{\Omega^1_X(H)} = 0$ i.e. $\Omega^1_X(H)$ is generated by global sections.

This can be done since X is sufficiently ample.

Consider the diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(H) & \longrightarrow & \Omega^1_{\mathcal{X}|X}(H) & \longrightarrow & \Omega^1_X(H) & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & \Omega^1_{Y|X}(H) & & \\ & & & & & & \uparrow & & \\ & & & & & & \mathcal{O}_X(H-X) & & \\ & & & & & & \uparrow & & \\ & & & & & & 0. & & \end{array} \tag{229}$$

It easy to see using the hypotheses on H and the Kodaira Vanishing theorem that $H^1(X, \mathcal{O}_X(H)) = H^1(X, \mathcal{O}_X(H-X)) = 0$. Hence all the global meromorphic 1-forms of $\Omega^1_X(H)$ can be lifted both to $H^0(X, \Omega^1_{\mathcal{X}|X}(H))$ and $H^0(X, \Omega^1_{Y|X}(H))$.

Diagram (229) can be completed as follows

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(H) & \longrightarrow & \Omega^1_{\mathcal{X}|X}(H) & \longrightarrow & \Omega^1_X(H) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(H) & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{Y|X}(H) & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \mathcal{O}_X(H-X) & = & \mathcal{O}_X(H-X) & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0. & & \end{array} \tag{230}$$

Since our deformation comes from $H^0(Y, \mathcal{O}_Y(X))$ by hypothesis, then the horizontal sequence completing diagram (230) is associated to the zero element of $H^1(X, T_{Y|X})$. Therefore we have the splitting of the second row and a map ϕ as follows

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X(H) & \longrightarrow & \Omega^1_{\mathcal{X}|X}(H) & \longrightarrow & \Omega^1_X(H) & \longrightarrow & 0 \\
 & & \| & & \uparrow & \nwarrow^{\phi} & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X(H) & \longrightarrow & \mathcal{O}_X(H) \oplus \Omega^1_{Y|X}(H) & \longrightarrow & \Omega^1_{Y|X}(H) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_X(H-X) & = & \mathcal{O}_X(H-X) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}
\tag{231}
$$

Note that $\det(\Omega^1_{Y|X}(H)) \cong \Omega^{n-1}_X(-X+nH)$ and $\det(\Omega^1_{\mathcal{X}|X}(H)) \cong \Omega^{n-1}_X(nH)$.

Proposition 5.2.6. *The map*

$$\phi^n \colon H^0(X, \Omega^{n-1}_X(-X+nH)) \to H^0(X, \Omega^{n-1}_X(nH))$$

is given by the section $R_{|X} \in H^0(X, \mathcal{O}_X(X))$.

Proof. This is a local computation. Take on Y local coordinates $x_1, \dots, x_{n-1}, y$ such that X is given by $y = 0$. Then locally the deformation of X is given by $y + tr = 0$, where r is a local equation of R. From $d(y + tr) = 0$ we obtain on X that $dy = -rdt$. Hence if a section of $H^0(X, \det(\Omega^1_{Y|X}))$ is locally given by $a \cdot dx_1 \wedge \dots \wedge dx_{n-1} \wedge dy$, then its image in $H^0(X, \det(\Omega^1_{\mathcal{X}|X}))$ is $-radx_1 \wedge \dots \wedge dx_{n-1} \wedge dt$. Tensoring by nH gives our thesis. $\square$

Consider n global sections $\eta_1, \dots, \eta_n \in H^0(X, \Omega^1_X(H))$. By our hypotheses on H, we can choose unique liftings $\tilde{s}_1, \dots, \tilde{s}_n \in H^0(X, \Omega^1_{Y|X}(H))$. Call $\widetilde{\Omega} \in H^0(X, \Omega^{n-1}_X(-X+nH))$ the associated generalized adjoint form. If we take $s_1 := \phi(\tilde{s}_1), \dots, s_n := \phi(\tilde{s}_n) \in H^0(X, \Omega^1_{\mathcal{X}|X}(H))$, we have that the generalized adjoint $\Omega \in H^0(X, \det(\Omega^1_{\mathcal{X}|X}(H))) = H^0(X, \Omega^{n-1}_X(nH))$ corresponding to $s_1 \wedge \dots \wedge s_n$ is $\Omega = \widetilde{\Omega} \cdot R$. We point out that $\widetilde{\Omega}$ does not depend on the deformation ξ, while Ω obviously does.

We can prove a theorem analogous to 5.1.6.

Theorem 5.2.7. *Assume that* $W = \langle \eta_1 \dots, \eta_n \rangle$ *is a generic subspace in* $H^0(X, \Omega^1_X(H))$ *with* $\lambda^n W \neq 0$. *Then* R *is in the pseudo-Jacobi ideal* $\mathcal{J}_{\mathcal{O}_Y(X),s}$ *if and only if the adjoint form* Ω *is in the image of* $H^0(X, \mathcal{O}_X(H)) \otimes \lambda^n W \to H^0(X, \Omega^{n-1}_X(nH))$.

Proof. Since $D_{\Omega^1_X(H)} = 0$ by our hypotheses on H, by Remark 3.5.4 it follows that $D_W = 0$. If R is in the pseudo-Jacobi ideal, then the deformation ξ is zero, hence $\Omega \in \operatorname{Im} H^0(X, \mathcal{O}_X(H)) \otimes \lambda^n W \to H^0(X, \Omega^{n-1}_X(nH))$ by Theorem 4.1.13. Viceversa if $\Omega \in \operatorname{Im} H^0(X, \mathcal{O}_X(H)) \otimes \lambda^n W \to H^0(X, \Omega^{n-1}_X(nH))$, then the deformation is supported on D_W by Theorem 4.1.12. We have already seen that $D_W = 0$, hence we are done. □

The generalized version of Macaulay's Theorem 5.2.5, gives the following result.

Proposition 5.2.8. *Let* X *be a sufficiently ample divisor on* Y. *Assume that the hypotheses (225) holds for* $A = \mathcal{O}_Y$. *Then the infinitesimal deformation* R *is in the pseudo-Jacobi ideal* $\mathcal{J}_{\mathcal{O}_Y(X),s}$ *if and only if* $R\widetilde{\Omega} \in \mathcal{J}_{\Omega^n_Y(nH+X),s}$ *for every section* $\widetilde{\Omega} \in H^0(Y, \Omega^n_Y(nH))$ *which restricts to a generalized adjoint relative to the vertical exact sequence of diagram (229).*

Proof. Note that a generalized adjoint relative to the vertical sequence of diagram (229) is in fact an element of

$$H^0(X, (\Omega^n_Y(nH))_{|X}) = H^0(X, \Omega^{n-1}_X(-X + nH)).$$

We want to apply the generalized version of Macaulay theorem mentioned above with $A = \mathcal{O}_Y$ and $k = 1$. We only know that $R\widetilde{\Omega} \in \mathcal{J}_{\Omega^n_Y(nH+X),s}$ for $\widetilde{\Omega} \in H^0(Y, \Omega^n_Y(nH))$ which restricts to a generalized adjoint. So now we prove that this is enough to have that

$$R \cdot H^0(Y, \Omega^n_Y(nH)) \subset \mathcal{J}_{\Omega^n_Y(nH+X),s}.$$

Consider the restriction sequence

$$0 \to \Omega^n_Y(nH - X) \to \Omega^n_Y(nH) \to \Omega^n_Y(nH)_{|X} \to 0.$$

By the hypotheses on H it is easy to see that

$$0 \to H^0(Y, \Omega^n_Y(nH - X)) \to H^0(Y, \Omega^n_Y(nH)) \to H^0(X, \Omega^n_Y(nH)_{|X}) \to 0$$

is exact. By Lemma 4.2.5 we can also assume that all the global sections of $H^0(X, \Omega^n_Y(nH)_{|X})$ are in fact linear combinations of generalized adjoints. Hence our hypothesis that $R\widetilde{\Omega} \in \mathcal{J}_{\Omega^n_Y(nH-X),s}$

for every section $\widetilde{\Omega} \in H^0(Y, \Omega^n_Y(nH))$ which restricts to a generalized adjoint in $H^0(X, \Omega^n_Y(nH)_{|X})$, together with the fact that the map

$$H^0(Y, \Omega^n_Y(nH - X)) \to H^0(Y, \Omega^n_Y(nH)))$$

is given by the multiplication by s, which is an element of the pseudo-Jacobi ideal, implies that

$$R \cdot H^0(Y, \Omega^n_Y(nH)) \subset \mathcal{J}_{\Omega^n_Y(nH+X),s}.$$

Now we apply Macaulay theorem to deduce that R is in the pseudo-Jacobi ideal. It is enough to show that

$$R_{\Omega^n_Y(nX-nH),s} \otimes R_{\Omega^n_Y(nH),s} \to R_{(\Omega^{2n}_Y)(nX),s}$$

is surjective. This follows from the surjectivity at the level of the H^0:

$$H^0(Y, \Omega^n_Y(nX - nH)) \otimes H^0(Y, \Omega^n_Y(nH)) \to H^0(Y, (\Omega^{2n}_Y)(nX)).$$

In fact take the sequence (228)

$$0 \to \mathcal{I} \to \mathcal{O}_Y \otimes H^0(\Omega^n_Y(nH)) \to \Omega^n_Y(nH) \to 0$$

and twist it by the sheaf $\Omega^n_Y(n(X - H))$ to obtain

$$0 \to \mathcal{I} \otimes \Omega^n_Y(n(X - H)) \to \Omega^n_Y(n(X - H)) \otimes H^0(\Omega^n_Y(nH)) \to$$
$$\to (\Omega^{2n}_Y)(nX) \to 0.$$

Since, by our choice of H, $h^1(\mathcal{I} \otimes \Omega^n_Y(n(X - H))) = 0$, we conclude. □

Putting the results together we have the following theorem that is a version of the infinitesimal Torelli theorem for a sufficiently ample smooth divisor in the setting of generalized adjoint forms.

Theorem 5.2.9. *The following are equivalent for a smooth hypersurface X if X is sufficiently ample.*

i) The differential of the period map $d\mathcal{P}$ (see Theorem 5.2.4) is zero on the infinitesimal deformation $R \in H^0(Y, \mathcal{O}_Y(X))$

ii) R is an element of the pseudo-Jacobi ideal $\mathcal{J}_{\mathcal{O}_Y(X),s}$

iii) $\Omega \in \operatorname{Im} H^0(X, \mathcal{O}_X(H)) \otimes \lambda^n W \to H^0(X, \Omega^{n-1}_X(nH))$ for the generic generalized adjoint Ω

if furthermore condition (225) holds for $A = \mathcal{O}_Y$*, then i),ii),iii) are also equivalent to*

iv) $R\widetilde{\Omega} \in \mathcal{I}_{\Omega^n_Y(nH+X),s}$ *for* $\widetilde{\Omega} \in H^0(Y, \Omega^n_Y(nH))$ *which restricts to a generalized adjoint form in* $H^0(X, (\Omega^n_Y(nH))_{|X})$.

Remark 5.2.10. *If* $A = \mathcal{O}_Y$*, condition (225) holds trivially for* $Y = \mathbb{P}^n$ *since in that case* $\Sigma_L = \bigoplus \mathcal{O}_{\mathbb{P}^n}(1)$*. This condition is also satisfied when* $Y = \mathrm{Grass}(k, n)$ *is a Grassmannian variety. In fact from the exact sequence*

$$0 \to \mathcal{O}_Y \to \Sigma_L \to T_Y \to 0$$

and the vanishing $H^1(\mathcal{O}_Y) = H^1(T_Y) = 0$ *we have that* $H^1(Y, \Sigma_L) = 0$.

6 GENERALIZED ADJOINT FORM FOR A FIBRATION OVER A CURVE

The theory of adjoint forms presented in the previous chapters is mostly used to deal with infinitesimal deformations $\mathcal{X} \to \operatorname{Spec}(\mathbb{C}[\epsilon])$ and to study the restriction sequence arising from them:

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0. \tag{232}$$

A first extension to the case of a fibration over an algebraic curve $\mathcal{X} \to B$ was made by González-Alonso in [26]. In the last chapter of this thesis we extend the theory of generalized adjoint forms seen in Chapter 4 to the case of a fibration over a smooth curve of genus g.

6.1 A "GLOBAL" DEFORMATION

From now on take $f\colon \mathcal{X} \to B$ a proper morphism of smooth complex varieties, where the base B is a smooth curve and $\mathcal{X}$ has dimension $m+1$. Denote by $B^0 \subset B$ the open set of regular values, that is the set such that a fiber X_b is smooth if and only if $b \in B^0$. We will call $\mathcal{X}^0 = f^{-1}(B^0)$ the union of the smooth fibers. As customary, we assume that the f is not isotrivial, that is its smooth fibers are not isomorphic.

Consider an exact sequence on $\mathcal{X}$

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{233}$$

where $\mathcal{L}$ is locally free of rank one, $\mathcal{E}$ is locally free of rank $n+1$ and $\mathcal{F}$ is locally free of rank n on $\mathcal{X}^0$. The example to have in mind is the exact sequence defining the sheaf of relative differentials $\Omega^1_{\mathcal{X}/B}$

$$0 \to f^*\omega_B \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/B} \to 0. \tag{234}$$

The restriction of sequence (233) on a smooth fiber X_b is

$$0 \to \mathcal{L}_{|X_b} \to \mathcal{E}_{|X_b} \to \mathcal{F}_{|X_b} \to 0. \tag{235}$$

In the explicit case of (234) this is exactly the well-known sequence

$$0 \to \mathcal{O}_{X_b} \to \Omega^1_{\mathcal{X}|X_b} \to \Omega^1_{X_b} \to 0 \tag{236}$$

that we have deeply studied in the previous chapters. Sequence (235) is associated to the element

$$\xi_b \in H^1(X_b, (\mathcal{L} \otimes \mathcal{F}^\vee)_{|X_b}) = \mathrm{Ext}^1(\mathcal{F}_{|X_b}, \mathcal{L}_{|X_b}). \qquad (237)$$

The key to encode all these extensions in a unique object is the notion of relative $\mathcal{E}xt$ sheaf.

Definition 6.1.1. *Given a morphism of schemes* $f\colon X \to Y$*, the relative* $\mathcal{E}xt$ *sheaves, denoted by* $\mathcal{E}xt^p_f(\mathcal{F}, -)$*, is the* p*-th right derived functor of* $f_*\mathcal{H}om(\mathcal{F}, -)$*.*

For a complete dissertation on relative $\mathcal{E}xt$ sheaves we refer to [8, Chapter 1]; see also [26, Appendix]. Here we only recall a few properties that we will need in the following:

Theorem 6.1.2. *The sheaves* $\mathcal{E}xt^p_f$ *satisfy*

1. *If* f *is projective and* $\mathcal{F}, \mathcal{G}$ *are coherent* $\mathcal{O}_X$*-modules, then* $\mathcal{E}xt^p_f(\mathcal{F}, \mathcal{G})$ *is a coherent* $\mathcal{O}_X$*-module*
2. *For any* $U \subset Y$ *open subset, it holds that*

$$\mathcal{E}xt^p_f(\mathcal{F}, \mathcal{G})_{|U} \cong \mathcal{E}xt^p_f(\mathcal{F}_{|f^{-1}(U)}, \mathcal{G}_{|f^{-1}(U)})$$

3. $\mathcal{E}xt^p_f(\mathcal{O}_X, \mathcal{G}) = R^p f_* \mathcal{G}$
4. *If* $\mathcal{L}$ *and* $\mathcal{N}$ *are locally free sheaves of finite rank on* X *and* Y*, respectively, then*

$$\mathcal{E}xt^p_f(\mathcal{F} \otimes \mathcal{L}, - \otimes f^*\mathcal{N}) \cong \mathcal{E}xt^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee \otimes f^*\mathcal{N}) \cong$$
$$\cong \mathcal{E}xt^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee) \otimes \mathcal{N}$$

5. *For any* $\mathcal{O}_X$*-modules* $\mathcal{F}, \mathcal{G}$ *there is a spectral sequence*

$$E_2^{p,q} = R^p f_* \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}) \Longrightarrow \mathcal{E}xt^{p+q}_f(\mathcal{F}, \mathcal{G})$$

where $\mathcal{E}xt^q$ *is the usual* $\mathcal{E}xt$ *sheaf (see Notation 2.3.2).*

This spectral sequence is referred as *local to global spectral sequence*.

In the case we are studying, the right sheaf to consider is

$$\mathcal{G} := \mathcal{E}\mathrm{xt}^1_f(\mathcal{F}, \mathcal{L})$$

where $\mathcal{F}, \mathcal{L}$ are the same appearing in sequence (233). Note that $\mathcal{G}^\vee$ is torsion free over B and hence it is locally free.

Lemma 6.1.3. *There is an injection*

$$R^1f_*(\mathcal{L}\otimes\mathcal{F}^\vee)\to\mathcal{G}$$

which is an isomorphism over an open dense subset of B^0*. In particular, for a general* $b\in B^0$ *we have the isomorphism*

$$\mathcal{G}\otimes\mathbb{C}(b)\cong H^1(X_b,(\mathcal{L}\otimes\mathcal{F}^\vee)_{|X_b})\cong \mathrm{Ext}^1(\mathcal{F}_{|X_b},\mathcal{L}_{|X_b}).$$

Proof. From the local to global spectral sequence

$$E_2^{p,q}=R^pf_*\mathcal{E}xt^q(\mathcal{F},\mathcal{L})\Longrightarrow\mathcal{E}xt_f^{p+q}(\mathcal{F},\mathcal{L}) \tag{238}$$

we have the associated five term exact sequence

$$0\to E_2^{1,0}\to H^1\to E_2^{0,1}\to E_2^{2,0}\to H^2. \tag{239}$$

The firs two terms give the desired injection

$$0\to R^1f_*\mathcal{H}om(\mathcal{F},\mathcal{L})\to\mathcal{E}xt_f^1(\mathcal{F},\mathcal{L}). \tag{240}$$

Furthermore note that on $\mathcal{X}^0$, $\mathcal{F}$ is locally free, hence by the previous theorem we have

$$\begin{aligned}\mathcal{G}_{|B^0}\cong\mathcal{E}xt_f^1(\mathcal{F}_{|\mathcal{X}^0},\mathcal{L}_{|\mathcal{X}^0})\cong\mathcal{E}xt_f^1(\mathcal{O}_{\mathcal{X}^0},(\mathcal{F}^\vee\otimes\mathcal{L})_{|\mathcal{X}^0})\cong\\ \cong R^1f_*((\mathcal{L}\otimes\mathcal{F}^\vee)_{|\mathcal{X}^0})=R^1f_*(\mathcal{L}\otimes\mathcal{F}^\vee)_{|\mathcal{X}^0}.\end{aligned} \tag{241}$$

The last statement comes from the proper base change theorem [40, Theorem 12.11]. □

From the Grothendieck spectral sequence, see [48, Theorem 12.10], applied to the functor of global sections Γ and the functor $f_*\mathcal{H}om$ we obtain the following spectral sequence:

$$E_2^{p,q}=H^p(B,\mathcal{E}xt_f^q(\mathcal{F},\mathcal{L}))\Longrightarrow\mathrm{Ext}^{p+q}(\mathcal{F},\mathcal{L}). \tag{242}$$

The associated five term exact sequence is

$$\begin{aligned}0\to H^1(B,f_*\mathcal{H}om(\mathcal{F},\mathcal{L}))\to\mathrm{Ext}^1(\mathcal{F},\mathcal{L})\xrightarrow{\rho}H^0(B,\mathcal{E}xt_f^1(\mathcal{F},\mathcal{L}))\to\\ \to H^2(B,f_*\mathcal{H}om(\mathcal{F},\mathcal{L}))\to\mathrm{Ext}^2(\mathcal{F},\mathcal{L}).\end{aligned}$$

The fourth term is zero because B is a curve, hence ρ is surjective. Calling $\xi\in\mathrm{Ext}^1(\mathcal{F},\mathcal{L})$ the element corresponding to sequence (233), we have that the map

$$\rho\colon\mathrm{Ext}^1(\mathcal{F},\mathcal{L})\to H^0(B,\mathcal{G}) \tag{243}$$

maps ξ to a global section $\rho(\xi)$ of $\mathcal{G}$ which associates to the general $b\in B^0$ the element $\xi_b\in H^1(X_b,(\mathcal{L}\otimes\mathcal{F}^\vee)_{|X_b})$ as defined in (237); see [44, Lemma 2.1].

Definition 6.1.4. *Take $\mathcal{D} \subset \mathcal{X}$ a divisor and define*

$$\mathcal{F}(-\mathcal{D}) := \operatorname{Ker}(\mathcal{F} \to \mathcal{F}_{|\mathcal{D}}) \tag{244}$$

and

$$\mathcal{G}_{\mathcal{D}} := \mathcal{E}xt^1_f(\mathcal{F}(-\mathcal{D}), \mathcal{L}). \tag{245}$$

Denote by D_b the restriction $\mathcal{D}_{|X_b}$ of $\mathcal{D}$ to the fiber X_b.

Since we have the inclusion $\mathcal{F}(-\mathcal{D}) \to \mathcal{F}$, by the fact that the relative $\mathcal{E}xt$ functors are contravariant in the second component, we obtain a map $\mathcal{G} \to \mathcal{G}_{\mathcal{D}}$.

Remark 6.1.5. *By the same arguments of Lemma 6.1.3, we have that $\mathcal{G}_{\mathcal{D}} \otimes \mathbb{C}(b) \cong \operatorname{Ext}^1(\mathcal{F}_{|X_b}(-D_b), \mathcal{L}_{|X_b})$ for general $b \in B$.*

Comparing with Chapter 4, the sheaf $\mathcal{G}$ takes the place of $\operatorname{Ext}^1(\mathcal{F}, \mathcal{L})$ and $\mathcal{G}_{\mathcal{D}}$ that of $\operatorname{Ext}^1(\mathcal{F}(-D), \mathcal{L})$. Therefore the condition of being supported given in Definition 4.1.2, that is

$$\xi \in \operatorname{Ker} \operatorname{Ext}^1(\mathcal{F}, \mathcal{L}) \to \operatorname{Ext}^1(\mathcal{F}(-D), \mathcal{L}),$$

is replaced by

$$\rho(\xi) \in \operatorname{Ker} H^0(X, \mathcal{G}) \to H^0(X, \mathcal{G}_{\mathcal{D}}). \tag{246}$$

By what we have seen so far, if $\mathcal{G}_{\mathcal{D}}$ is torsion-free, (246) is equivalent to ξ_b being supported on D_b (in the sense of Definition 4.1.2) for the general $b \in B$.

If $\rho(\xi) \in \operatorname{Ker} H^0(X, \mathcal{G}) \to H^0(X, \mathcal{G}_{\mathcal{D}})$, we have that in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \hat{\mathcal{E}} & \longrightarrow & \mathcal{F}(-\mathcal{D}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array} \tag{247}$$

the top row splits when restricted to the general fiber. Of course this does not mean that the top row itself splits.

6.2 THE ADJOINT MAP

Following [26], we reformulate condition (176) in a more intrinsic way. Take sequence (235)

$$0 \to \mathcal{L}_{|X_b} \to \mathcal{E}_{|X_b} \to \mathcal{F}_{|X_b} \to 0 \tag{248}$$

where X_b is a smooth fiber and take $W \subset H^0(X_b, \mathcal{F}_{|X_b})$ a subspace of dimension $n+1$ contained in the kernel of the coboundary map

$$\delta_{\xi_b} \colon H^0(X_b, \mathcal{F}_{|X_b}) \to H^1(X_b, \mathcal{L}_{|X_b}).$$

Calling $\widetilde{W} \subset H^0(X_b, \mathcal{E}_{|X_b})$ the preimage of W, we have the short exact sequence

$$0 \to H^0(X_b, \mathcal{L}_{|X_b}) \to \widetilde{W} \to W \to 0 \tag{249}$$

and hence the resolution

$$H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^{n} \widetilde{W} \xrightarrow{\triangle} \bigwedge^{n+1} \widetilde{W} \to \bigwedge^{n+1} W \to 0. \tag{250}$$

As in the previous chapters call $\lambda^n W$ the image of $\bigwedge^n W$ in $H^0(X_b, \det \mathcal{F}_{|X_b})$, then we have the following commutative diagram

$$\begin{array}{ccccccc} H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^n \widetilde{W} & \longrightarrow & \bigwedge^{n+1} \widetilde{W} & \longrightarrow & \bigwedge^{n+1} W & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{\nu}_W & & \downarrow \nu_W & & \\ H^0(X_b, \mathcal{L}_{|X_b}) \otimes \lambda^n W & \xrightarrow{\mu_W} & H^0(X_b, \det \mathcal{E}_{|X_b}) & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{251}$$

where $A := H^0(X_b, \det \mathcal{E}_{|X_b})/(\mathrm{Im}\,\mu_W)$. The map $\tilde{\nu}_W$ is given by the wedge product of sections of $\mathcal{E}_{|X_b}$ and it defines ν_W since the first square is commutative.

Definition 6.2.1. *The map ν_W is called* adjoint map *associated to W.*

Condition (176), that is

$$\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}_{|X_b}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}_{|X_b})),$$

can be written now as $\mathrm{Im}\,\mu_W = \mathrm{Im}\,\tilde{\nu}_W$, or, equivalently, as $\nu_W = 0$. Note that $\mathrm{Im}\,\mu_W \subset \mathrm{Im}\,\tilde{\nu}_W$ is always true.

The Generalized Adjoint Theorem 4.1.12 together with its inverse 4.1.13 in this language is

Theorem 6.2.2. *If $\nu_W = 0$, then ξ_b is supported on D_W. Viceversa if ξ_b is supported on D_W and $H^0(\mathcal{L}_{|X_b}(D_W)) = H^0(\mathcal{L}_{|X_b})$, then $\nu_W = 0$.*

Instead of considering only one subspace $W \subset \mathrm{Ker}\,\delta_{\xi_b}$, we can consider the previous construction at the level of the Grassmannian variety $G := \mathrm{Grass}(n+1, \mathrm{Ker}\,\delta_{\xi_b})$. Given a vector space V, denote by V_G the trivial vector bundle over G of fiber V. Call $\mathcal{S} \subset H^0(\mathcal{F}_{X_b})_G$ the tautological bundle over G and $\widetilde{\mathcal{S}} \subset H^0(\mathcal{E}_{X_b})_G$ its preimage. The exact sequence of vector bundles

$$H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^{n} \widetilde{\mathcal{S}} \to \bigwedge^{n+1} \widetilde{\mathcal{S}} \to \bigwedge^{n+1} \mathcal{S} \to 0. \tag{252}$$

gives (250) on every point $W \in G$.

We have an analogous of Diagram (251)

$$\begin{array}{ccccccc} H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^n \widetilde{\mathcal{S}} & \longrightarrow & \bigwedge^{n+1} \widetilde{\mathcal{S}} & \longrightarrow & \bigwedge^{n+1} \mathcal{S} & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{\nu} & & \downarrow \nu & & \\ H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^n \mathcal{S} & \xrightarrow{\mu} & H^0(X_b, \det \mathcal{E}_{|X_b})_G & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \end{array} \tag{253}$$

where μ is given by the composition of

$$H^0(X_b, \mathcal{L}_{|X_b}) \otimes \bigwedge^n \mathcal{S} \to H^0(X_b, \mathcal{L}_{|X_b}) \otimes H^0(X_b, \det \mathcal{F}_{|X_b})_G$$

and

$$H^0(X_b, \mathcal{L}_{|X_b}) \otimes H^0(X_b, \det \mathcal{F}_{|X_b})_G \to H^0(X_b, \det \mathcal{E}_{|X_b})_G$$

and $\mathcal{A}$ denotes the quotient $H^0(X_b, \det \mathcal{E}_{|X_b})_G/(\operatorname{Im} \mu)$. Note that μ is not necessarily an injection and $\mathcal{A}$ is not necessarily a vector bundle.

Definition 6.2.3. *The map* ν *is called* adjoint map.

By construction, the map ν gives ν_W at any point $W \in G$.

6.3 GLOBAL ADJOINT MAP

We go back to the entire fibration $f \colon \mathcal{X} \to B$ and to the exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0. \tag{254}$$

Recall that $\mathcal{F}$ is locally free only on $\mathcal{X}^0$ and define

$$\tilde{\mathcal{F}} := \det \mathcal{E} \otimes \mathcal{L}^\vee. \tag{255}$$

We have that $\tilde{\mathcal{F}}$ is a line bundle and $\tilde{\mathcal{F}}_{|\mathcal{X}^0} \cong \det \mathcal{F}_{|\mathcal{X}^0}$. In the explicit case where (254) is

$$0 \to f^*\omega_B \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/B} \to 0, \tag{256}$$

the sheaf $\tilde{\mathcal{F}}$ is the relative dualizing sheaf $\omega_{\mathcal{X}/B} = \omega_{\mathcal{X}} \otimes f^*\omega_B^\vee$.

We restrict sequence (254) to the general fiber and we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}_{|X_b} & \longrightarrow & \mathcal{E}_{|X_b} & \longrightarrow & \mathcal{F}_{|X_b} & \longrightarrow & 0. \end{array}$$

Assumption 6.3.1. *Assume that the map*

$$H^0(\mathcal{X},\mathcal{E})/H^0(\mathcal{X},\mathcal{L}) \to H^0(X_b,\mathcal{F}_{|X_b})$$

is injective for the general fiber. Call $V := H^0(\mathcal{X},\mathcal{E})/H^0(\mathcal{X},\mathcal{L})$.

Of course we have that V is a subset of $\operatorname{Ker}\delta_{\xi_b}$. The key idea is that this space does not depend on the fiber and gives a common subspace in order to apply the adjoint theory.

Note that on the case of (256) this assumption is true since $V = H^0(\mathcal{X},\Omega^1_{\mathcal{X}})/f^*H^0(B,\omega_B)$ naturally injects in $H^0(X_b,\Omega^1_{X_b})$ by standard properties of abelian varieties; see [26].

Now denote by $G := \operatorname{Grass}(n+1,V)$ the Grassmannian variety of $n+1$-dimensional subspaces of V and by Y the product $Y := B\times G$. Call $p_B\colon Y\to B$ and $p_G\colon Y\to G$ the natural projections. The tautological bundle $\mathcal{S}_V\subset\mathcal{O}_G\otimes V$ on G gives via pullback the bundle $\mathcal{S} := p_G^*\mathcal{S}_V$ on Y. Of course $\mathcal{S}$ is a vector subbundle of $\mathcal{O}_Y\otimes V$.

Denote by $\widetilde{\mathcal{S}}$ the preimage of $\mathcal{S}$ inside $\mathcal{O}_Y\otimes H^0(\mathcal{X},\mathcal{E})$. We have the exact sequence of vector bundles

$$0\to H^0(\mathcal{X},\mathcal{L})\otimes\mathcal{O}_Y\to\widetilde{\mathcal{S}}\to\mathcal{S}\to 0 \tag{257}$$

which, for a point $(b,W)\in Y$ with $b\in B^0$, is basically (249). From (257) we obtain

$$H^0(\mathcal{X},\mathcal{L})\otimes\bigwedge^{n}\widetilde{\mathcal{S}}\to\bigwedge^{n+1}\widetilde{\mathcal{S}}\to\bigwedge^{n+1}\mathcal{S}.$$

Define

$$\widetilde{\nu}\colon\ \bigwedge^{n+1}\widetilde{\mathcal{S}}\to p_B^*f_*\det\mathcal{E}$$

via the composition of the wedge product

$$\bigwedge^{n+1}H^0(\mathcal{X},\mathcal{E})\otimes\mathcal{O}_Y\to H^0(\mathcal{X},\det\mathcal{E})\otimes\mathcal{O}_Y$$

and the evaluation map

$$H^0(\mathcal{X},\det\mathcal{E})\otimes\mathcal{O}_Y\to p_B^*f_*\det\mathcal{E}.$$

This map sends the image of $H^0(\mathcal{X},\mathcal{L})\otimes\bigwedge^n\widetilde{\mathcal{S}}$ into the image of $p_B^*f_*\mathcal{L}\otimes\bigwedge^n\mathcal{S}$, that is the following diagram is commutative

$$\begin{array}{ccccccc}
H^0(\mathcal{X},\mathcal{L})\otimes\bigwedge^n\widetilde{\mathcal{S}} & \longrightarrow & \bigwedge^{n+1}\widetilde{\mathcal{S}} & \longrightarrow & \bigwedge^{n+1}\mathcal{S} & \longrightarrow & 0\\
\downarrow & & \downarrow{\scriptstyle\widetilde{\nu}} & & \downarrow{\scriptstyle\nu} & & \\
p_B^*f_*\mathcal{L}\otimes\bigwedge^n\mathcal{S} & \longrightarrow & p_B^*f_*\det\mathcal{E} & \longrightarrow & \mathcal{A} & \longrightarrow & 0
\end{array}$$

where $\mathcal{A} = p_B^* f_* \det \mathcal{E} / \operatorname{Im}(p_B^* f_* \mathcal{L} \otimes \bigwedge^n \mathcal{S})$. It is clear that over a point $(b, W) \in Y$ with $b \in B^0$, the map ν is the adjoint map of Diagram (251), whereas its restriction to $\{b\} \times G$ gives the adjoint map of Diagram (253). Note that the map

$$p_B^* f_* \mathcal{L} \otimes \bigwedge^n \mathcal{S} \to p_B^* f_* \det \mathcal{E}$$

is not necessarily an inclusion, hence $\mathcal{A}$ is not necessarily a vector bundle on Y. If we want $\mathcal{A}$ to be a vector bundle we have to make the further assumption that the image of

$$\bigwedge^n \mathcal{S} \to p_B^* f_* \tilde{\mathcal{F}}$$

has all the fibers of the same dimension.

Definition 6.3.2. *The map* ν *is the* global adjoint map *of the fibration* $f \colon \mathcal{X} \to B$.

Since the global adjoint map is constructed by gluing together the adjoint maps of each fiber, it allows to control, to a certain extent, $\rho(\xi)$.

Theorem 6.3.3. *If for the general* $b \in B^0$ *there exists* $W \subset V$ *such that* ν *is zero over the point* $(b, W) \in Y$*, then* ξ_b *is supported on* D_b*. Furthermore if the divisors* $D_b \subset X_b$ *glue together to a divisor* $\mathcal{D} \subset \mathcal{X}$ *and if* $\mathcal{G}_{\mathcal{D}}$ *is torsion-free, then* $\rho(\xi) \in \operatorname{Ker} H^0(B, \mathcal{G}) \to H^0(B, \mathcal{G}_{\mathcal{D}})$.

CONCLUSIONS AND FUTURE WORK

There are still many open questions that may lead to interesting results. In particular we point out two of them.

The first one, concerning Chapter 3, is the study of the adjoint quadrics vanishing on the canonical image of a variety X. In the last section of Chapter 3 we have considered some examples of algebraic varieties without adjoint quadrics through their canonical image. Still it should be possible to find more interesting and exotic cases, for example studying surfaces with specific invariants. This could provide new examples of varieties for which the infinitesimal Torelli theorem holds; see Corollary 3.5.7.

The second question is suggested by Remark 5.2.10. In Chapter 5, comparing the section on smooth projective hypersurfaces and the section on smooth sufficiently ample divisors of a variety Y, one can see that in the second case a lot of the explicitness is lost. For example the sheaf $\mathcal{O}_{\mathbb{P}^n}(2)$ which allows to apply the generalized adjoint theory is replaced by a certain ample $\mathcal{O}_Y(H)$. Since in the case $Y = \text{Grass}(n, k)$ the cohomologies of the twists of the cotangent sheaf are classically known, see [64], it should be possible to achieve almost the same level of explicitness of $\mathbb{P}^n$.

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