Abstract

The *land of modal temporal logics* is an interesting and surprising one. Travelling through it, one may bump into the *point-based temporal logics wood*, which has been largely explored and studied, and whose peculiarities have been mostly discovered. The exploration of this part of the land is not finished yet, but most people expect that it cannot reserve too bad surprises.

On the other hand, moved by the curiosity, one may be also tempted to venture into the *interval-based (propositional) temporal logics swamp*. This is a dangerous place. It is full of strange creatures (the Philosophical Questions about Point vs. Intervals), of scaring beings (the Highly Undecidable Propositional Interval Logics), and of still mysterious entities (such as the Fragments of the Logic of Allen's Relations, or the Strict Propositional Interval Logics), between others. Survivors of past explorations, however, tell us that sometimes and somewhere it is possible to find more pacific animals, such as Local Propositional Interval Logics, and Logics interpreted over Non-Standard Structures (such as the Split Logics). Moreover, they tell that there are zones where one may expect to find a bit more hospitable environment, such as the so-called Tableaux Cave. Much work, in any case, has still to be done to make it possible to safely travel in this part of the land. This work describes some of the adventures we have been living in exploring the area of Propositional Temporal Interval Logics.

La *terra delle logiche modali temporali* è interessante e sorprendente. Viaggiando attraverso essa, uno può imbattersi nel *bosco delle logiche temporali basate sui punti*, che è stato largamente esplorato e studiato, e le cui peculiarità sono state scoperte per la maggior parte. L' esplorazione di questa parte della terra non è ancora terminata, ma si ritiene che non possa più riservare brutte sorprese.

D'altra parte, spinto dalla curiosità, uno può essere tentato di avventurarsi nella *palude delle logiche (proposizionali) ad intervalli.* Questo é un posto pericoloso. È pieno di strane creature (le Questioni Filosofiche riguardo a Punti contro Intervalli), di esseri spaventosi (le Logiche Proposizionali ad Intervalli Altamente Indecidibili), e di entità ancora misteriose (come i Frammenti della Logica delle Relazioni di Allen, o le Logiche Proposizionali ad Intervalli Strette), tra altri. I sopravvissuti di passate spedizioni, comunque, raccontano che talvolta, da qualche parte, è possibile trovare animali più pacifici, come le Logiche Proposizionali ad Intervalli Locali, e Logiche interpretate su Strutture Non-Standard (come le Logiche Split). Inoltre, raccontano

the esistono zone dove uno può aspettarsi di trovare un ambiente più ospitale, come quella che chiamano la Caverna dei Tableaux. In ogni caso, deve essere fatto ancora molto per rendere sicuro il viaggiare in questa parte della terra. Questo lavoro descrive le nostre avventure durante l'esplorazione dell'area delle Logiche Proposizionali Temporali ad Intervalli.

Acknowledgments

When I started my PhD program, in October 2000, I got no idea of what *research* was. My principal supervisor, Angelo Montanari, has been decidedly brave (or unconscious?) when he proposed me to start a line of research in Interval Logics without knowing me at all. But his tenancy led him to finish what he had began, and finally, after a non-trivial work, he taught me to be precise, careful, and to *read it again*. Even if I did not learn all the lessons yet, I must strongly thank him for his patience. In his hard task, Angelo asked the help of the one who became my second supervisor, Valentin Goranko. The period that I spent in Johannesburg under his responsibility turned out to be much more useful than I thought before starting it. Over all, I always remember Val's favorite comment to my work, that is, *the Devil stays into details*. So, my special acknowledgement is for him too.

Moreover, a number of people, shared with me something during these three years; I will just cite a few, but I remember them all. My very friend Enrico Marzano followed me on the "research way", and I know that he will succeed anyway. My (past and current) roommates and/or colleagues Massimo Francheschet, Raffaella Gentilini, Alberto Casagrande, Nicola Vitacolonna, Subramanian Venkataraman, Yoko Motohama, and Lorenzo Turicchia helped me, anyone in his (her) own way, and deserve a special mention just for having stood me.

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Introduction

"Computer science is no more about computers
than astronomy is about telescopes".
Edsger W. Dijkstra

Time and Computer Science

As it has been largely pointed out and discussed (see, for example, [87, 34, 92]), time is one of the most paradoxical concept our minds have to deal with. To quote from the *Confessions* of St. Augustine:

> *Quid es ergo tempus? Si nemo me quaerat, scio; si quaerenti explicare velim, nescio*[1].

One of time's most puzzling aspects concerns its ontological status: on the one hand, it is a subjective and relative notion, based on our conscious experience of successive events; on the other hand, our civilization and technology are based on the understanding that something like objective, absolute Time exists. Some philosophers have taken this paradox so far as they conclude that time is unreal; others, accepting the existence of absolute time, have engaged in heated debates regarding its structure, be it linear or circular, bounded or unbounded, dense or discrete, and so on.

But even if we leave these metaphysical issues aside for the time being, it is obvious that time plays a crucial role in our thinking and, thus, that there is a clear need for *precise reasoning* about it. Not surprisingly, the problem of representing temporal knowledge and reasoning about it has been addressed in many academic disciplines beside philosophy as well. In *physics* time and change have always been central notions, although in mathematics modeling time has usually not received higher ontological status than being one of many dimensions. It is quite different in formal *linguistics*, where a lot of work has gone into modelling temporal phenomena (and necessarily so, if only by the observation that every single sentence utter is put in some temporal perspective through *tense*).

Computer scientists are thinking of the behavior of programs in terms of the step by step evolution of an artificial system; thus, specifying the behavior of of programs or reasoning about their correctness naturally involve a temporal component. In particular, *artificial intelligence* is permeated with temporal concerns. In one way or another indeed every area of artificial intelligence has to do with time. To cite a few: medical diagnosis systems reason about the time at which the virus infected the

[1] Translated: "What, then, is time? If no one asks me, I know; if I wish to explain it to someone, I don't know.", Book XI, Chapter XIV.

blood system; device troubleshooting systems look at how long it takes a capacitor to saturate; in *automatic programming*, the time at which a variable becomes bound is important; in *robot planning* one wants to achieve one goal before another, to meet deadlines, and so on. At a more general level, one can identify several classes of tasks in artificial intelligence that require reasoning about time. We simply recall some of them: *prediction*: given a description of the world over some period of time, and the set of rules governing change, predict the world at some future time; *explanation*: given a description of the world over some period of time, and the set of rules governing change, produce a description of the world at some earlier time that accounts for the world being the way it is at the later time; *planning*: given a description of some desiderated state of the world over some period of time, and the rules governing change, produce a sequence of actions that would result in a world fitting that description; *learning new rules*: given a description of the world at different times, produce the rules governing change which account for the observed regularities in the world. Finally, *temporal databases* (which are one natural temporal evolution of classical databases) and *computational linguistics* deal with time as a central component.

While these enterprises are not necessarily concerned with the same concept of time, they all could go under the heading of Temporal Logics. However, often a more restricted, technical definition is used in which temporal logic is a branch of *modal* logic. The modal approach towards logics of time was initiated about forty years ago by Prior. His aim was to develop temporal logics that are close to natural language and could be used for investigating philosophical problems related to time, such as determinism. Prior's work has been a source of inspiration for modal logicians; the core of our knowledge of (meta)logical properties of modal formalism for reasoning about time stems from the sixties and seventies, through the work of van Benthem, Burgess, Gabbay, Kamp, Segerberg, and others. Later, theoretical computer science has become the main area for applying modal logic of time; pioneering work in this area has been done by Pnueli. This change of setting also influenced the agenda of temporal logicians: next to expressiveness, completeness and decidability issues, also computational aspects are investigated.

An Interval-Based Representation of Time

In the present work we committed to an *interval-based* representation of time, instead of a *point-based* one. It is worth to remind that this choice is not free from philosophical consequences, and many authors argued about such a choice, giving reasons for and against it. Interval-based temporal logics stem from four major scientific areas:

Philosophy. The philosophical roots of interval temporal logics can be traced back to Zeno and Aristotle. The nature of time has always been a favourite subject in philosophy, and in particular, the discussion whether time instants or time periods should be regarded as the primary objects of temporal ontology has a distinct philosophical flavour. Consider, for instance the following sentences [92]: *The fire is burning; now it is out.* What happened at the boundary instant dividing these two successive states? Is the fire burning or not at the dividing instant? There seem not to be reasons for preferring either, which means that we must choose both, incurring in a contradiction, or neither, incurring in a truth value gap. To dramatize the matter, think of *the instant of death*. I am alive now, and I will be dead sometime later. But is the boundary point between these two different states the last moment of my life or the first moment of my state of death? Or are things more mysterious than that, and are they contradictory, or incomplete states of being alive or dead[2]? Some of the modern formal logical treatments of interval-based structure of time include: Hamblin's work [43], providing a philosophical analysis of interval ontology and interval based tense logics; Humberstone's paper [49], which elaborates on Hamblin's work, introducing a sequent calculus for an interval tense logic over precedence and sub-interval relations; the study by Roeper [83], which is a follow-up on Humberstone's work, discussing and analyzing persistency (preservation of truth in sub-intervals) and homogeneity; Burgess' investigation [11], proposing axiomatic systems for interval-based tense logics of the rationals and reals, already studied earlier in [83]. A comprehensive study and a logical analysis of point-based and interval-based ontologies, languages, and logical systems can be found in [91].

Linguistics. Interval-based logical formalisms have been extensively investigated in the study of natural languages since the seminal work of Reichenbach [81]. They arise as suitable frameworks for modeling progressive tenses and expressing various language constructions involving time periods and event duration which cannot be adequately grasped by point-based temporal languages. Period-based temporal languages and logics have been proposed and studied e.g. in [25, 50, 82]. The linguistic aspects of interval logics will not be treated here, apart from some discussion on the expressiveness of various interval-based temporal languages.

Artificial intelligence. Interval temporal languages and logics have sprung up from *expert systems*, *planning systems*, *temporal databases*, *theory of events*, *natural language analysis and processing*, and many other fields as formal tools for temporal representation and reasoning in artificial intelligence. Some of the notable contributions in that area include: the seminal work by Allen [2], proposing a temporal logic for reasoning about time intervals and identifying the thirteen possible relations between intervals in a linear ordering; Allen and Hayes' contribution [4], providing an axiomatization and a representation result for interval structures based on the *meets* relation between intervals, further advanced and studied by Ladkin in [54], which also presents a completeness theorem and algorithms for satisfiability checking for Allen's Interval Algebra viewed as a first-order theory; Galton's critical analysis of Allen's framework [32], arguing

[2]Indeterminacy may be a reality in some countries. When Einstein was considering to move to Leiden University in Holland, the famous physicist Ehrenfest tried to dissuade him from moving, by telling him that the Dutch university professor is the only species on this planet where the distinction between life and death is completely undetectable [92].

the necessity of considering points and intervals on a par, and Allens and Ferguson‘s work [3], developing interval-based theory of *actions* and *events*. The theories based on actions and events study the evolution of a portion of the world as the result of the occurrence of a set of actions and/or events. The basic reasoning mechanisms are concerned with *predicting* the effects of the occurrence of events as well as *explaining* a given situation in terms of possible causes. This kind of reasoning is called *temporal projection*, which can be distinguished in *forward* and *backward* (explication of the future or of the past). These approaches are based on some kind of (first-order) temporal logic with a given notion of underlying time (which can be either point-based and interval-based). These framework are endowed with a semantics and a number of tools in order to efficiently solve the problems of prediction, planning and explanation, among others. Good references here are [60, 53, 3]. A recent survey on temporal representation and reasoning in artificial intelligence, including also Vilain and Kautz's Point Algebra [96], van Beek's Continuous Endpoint Algebra [90], and Nebel and Bürckert's ORD-Horn Algebra [72], can be found in [16].

Computer science. One of the first applications of interval temporal logics to computer science, viz. for specification and design of hardware components, was proposed in [41] and [65], and further developed in [71, 67, 68, 69]. Later, other systems and applications of interval logics were proposed in [19, 20, 23, 24, 8, 100, 78]. In particular, model checking tools and techniques for interval logics were developed in [12]. Interesting generalizations of interval logics for specification and verification of real-time processes in computer science are the *duration calculi* (see [102, 88, 46, 103, 45, 99]). They extend interval logics to allow one to represent and reason about time durations for which a system is in a given state. For an up-to-date survey on duration calculi see Hansen paper [44].

Interval Logics: What is (and What is Not) This Work about?

Interval temporal logics are modal logics whose semantics reflect the structure of time. At the propositional level, an interval logic simply consists of a set of propositional variables, the classical propositional connectives, and a set of *modalities* allowing one to move through the structures. As an example, one can express something like "the property p is true over the current interval and the property q is true over all intervals starting at the endpoint of the current one". Roughly speaking, a propositional interval temporal logics can be viewed as a variant of a point-based modal temporal logics, where the notion of *satisfiability* at a state is replaced by the notion of satisfiability with respect to an ordered pair of states, that is, an interval. At the first-order level, the language is enriched with variables and functions, which are usually divided into two classes depending on their interpretation, being *flexible* (i.e., it changes over time) or *rigid*. Duration calculi are extensions of first-order interval temporal logics, endowed with the additional notion of *state*. Each state is denoted by means of a state expression, and it is characterized by a *duration*. The duration of a state is (the length of) the time period during which the system remains in the state.

In our work, rather then concentrating on solving particular problems for specific applications, we consider interval logics as a general framework to reason about time. Typical problems of interest from this point of view are (un)decidability of the satisfiability/validity problem, complexity issues, finding sound and complete axiomatizations, developing well behaved proof methods. In particular we address three main goals:

(i) giving a reasonably complete survey of known propositional interval logics, focusing on their expressive power and their logical properties (decidability, axiomatic systems, etc.);

(ii) looking for 'well-behaved' propositional interval logics, either by concentrating on particular subsets of the modalities or by restricting the underlying structures in some way;

(iii) devising classical semantic tableaux methods for propositional interval logics.

Each one of the three objectives can be considered to some extent new. As for the first one, the literature of interval logics is quite extended and, so far, there was no attempt to give a general picture. As for the second point, once researchers realized that, in general, interval logics are difficult to deal with, they have usually chosen a very restrictive way to look for positive results, namely the *locality* principle. It consists in reducing the evaluation of formulas over an interval to the evaluation of the same formula over the first point of that interval. On the contrary, we have been exploring different alternatives, without resorting to any form of locality. As for the last point, so far, in the interval logic literature, tableau methods for interval logics had been focused on very specific systems, and many of them have been confined to propositional interval logics with the locality principle. Instead, our tableau method is an easy implementable and very general one, which can be adapted to a number of logics present in the literature.

This thesis obviously does not cover every topic of interest in the field. One relevant omission is that of many 'non-pure' interval logics. The class of interval logics can be divided into two main classes: 'pure' interval logics, where the semantics is essentially interval-based, that is, propositional variables are evaluated over intervals, and 'non-pure' interval logics, where the semantics is essentially point-based and intervals are only auxiliary entities. In this thesis, we have focused our attention on 'pure' interval logics, even though interval logics with locality can be viewed, to a certain extent, as a particular case of non-pure interval logics. Important contributions in the area of 'non-pure' interval logics have been made by Dillon, Kutty, Moser, Melliar-Smith, and Ramakrishna (see [19, 23, 24, 22, 21], where Future Interval Logic and Graphical Interval Logic are described). Complexity results on 'non-pure' propositional interval logics have been obtained by Aaby and Narayana [1], while applications of these logics have been explored in Ramakrishna's PhD thesis [76]. There are other contributions that we have not discussed here, such as, for instance, probabilistic interval logics, such as those reported in [40]. Moreover, we have not discussed programming languages and related systems, based on interval logics [68, 26]. Finally, we have not discussed other types of deductive systems for interval logics such as sequent calculi and natural deduction, that have been proposed in the literature, e.g. in [95, 79, 77].

Structure of The Dissertation

The dissertation is organized as follows.

- In Chapter 1 we introduce some preliminaries concerning temporal structures for intervals. We define various settings, and we distinguish between those including point-intervals and those excluding them. Then, we raise the problem of representing temporal structures by taking *intervals* as primitive objects, and we briefly recall some of the important representation theorems in the literature. Finally, we state representation theorems for the structures that play a major role in this thesis, namely, the *neighborhood frames.* This chapter is (partially) based on [37];

- In Chapter 2 we outline main developments and results on interval temporal logics and duration calculi. We present various formal systems studied in the literature and discuss their distinctive features, emphasizing on expressiveness, axiomatic systems, and (un)decidability results. Since an up-to-date survey on duration calculi con be found in [44], we will survey this topic rather briefly, while going in more detail on interval logics, mainly on propositional level. This chapter is essentially based on [38] and on [97];

- In Chapter 3 we focus our attention on propositional interval temporal logics with temporal modalities for neighboring intervals over linear orders. We study the class of Propositional Neighborhood Logics ($\mathcal{PNL}$) over the two natural semantics (strict and non-strict). We develop complete axiomatic systems for logics in $\mathcal{PNL}$. Source for this chapter are [36] and [37];

- In Chapter 4 we introduce a generalization of the logic CDT for (non-strict) partial orderings, called BCDT^+, which actually extends most of the propositional interval temporal logics proposed in the literature. Then, we provide a tableau method for BCDT^+ which combines features of explicit tableau methods for modal logics with constraint label management and the classical tableau method for first-order logic, and we prove its soundness and completeness. Due to the generality of this logic, the above method can be easily adapted to many of the propositional systems recalled in Chapter 2, providing a general tableau method for propositional interval logics. This chapter is based on [38];

- Chapter 5 is devoted to the investigation of different approach to the decision problem for propositional interval logics. The class of Split Logics ($\mathcal{SL}$) is introduced and their properties are studied. Split logics are interpreted on *non-standard* structure that do not include *all* the intervals of a linear ordering. We give decidability results for some split logics, and we discuss small examples of applications. The contents of this chapter have been presented in [64];

- Finally, in Chapter 6 we list the major open problems in interval logics which, in our view, deserve to be studied by further research.

1

Temporal Structures for Time Intervals

"Minds are just like parachutes.
Either they open, or they crash. "
Anonymous

Interval-based settings for interval logics are subject to the same ontological dilemmas as the instant-based settings, viz.:

- Should the time structure be considered *linear* or *branching*?
- *Discrete* or *dense*?
- *With* or *without beginning*? etc.

In addition, new dilemmas arise regarding the nature of the intervals:

- *Should intervals include their end-points or not*?
- *Can they be unbounded*?
- *Are point-intervals (i.e. with coincident endpoints) admissible or not*?
- *How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it*?

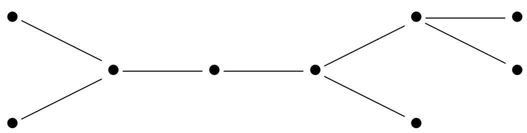
The last question is of particular importance when semantics of interval logics are defined. In this chapter, we first introduce concrete point-based interval structures and we illustrate their main properties. Then, we take into consideration more abstract interval frames, and we give a number of representation theorems for relevant classes of them. In particular, we establish new representation theorems relating interval neighborhood frames and structures, for both strict and non-strict semantics.

1.1 Preliminaries

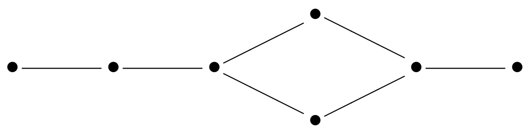
Given a strict partial ordering $\mathbb{D} = \langle D, < \rangle$, an **interval** in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. $[d_0, d_1]$ is a **non-point interval** if $d_0 < d_1$. Intervals of the type $[d, d]$ will be called **point-intervals**. A point d **belongs to an interval** $[d_0, d_1]$ if $d_0 \leq d \leq d_1$ (i.e. the endpoints of an intervals are included in it). The set of all intervals on $\mathbb{D}$ will be denoted by $\mathbb{I}(\mathbb{D})^+$, while the set of all non-point-intervals will be denoted by $\mathbb{I}(\mathbb{D})^-$. By $\mathbb{I}(\mathbb{D})$ we will denote either of these. For the purpose of this thesis, we will call a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ an **interval structure**, although later in this work we will be using a more general definition, considering only subsets of $\mathbb{I}(\mathbb{D})$. In all systems considered here the intervals will be assumed linear, although this restriction can often be relaxed without essential complications. Thus, we will concentrate on partial orderings with the **linear intervals property:**

$$\forall x \forall y (x < y \rightarrow \forall z_1 \forall z_2 (x < z_1 < y \wedge x < z_2 < y \rightarrow z_1 < z_2 \vee z_1 = z_2 \vee z_2 < z_1)),$$

that is, orderings in which every interval is linear. Clearly every linear ordering falls here. An example of a non-linear ordering with this property is:



while a non-example is:



An interval structure is: **linear** if every two points are comparable; **discrete** if every point with a successor/predecessor has an immediate successor/predecessor along every path starting from/ending in it, that is,

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z \leq y \wedge \forall w (x < w \wedge w \leq y \rightarrow z \leq w))),$$

and

$$\forall x \forall y (x < y \rightarrow \exists z (x \leq z \wedge z < y \wedge \forall w (x \leq w \wedge w < y \rightarrow w \leq z)));$$

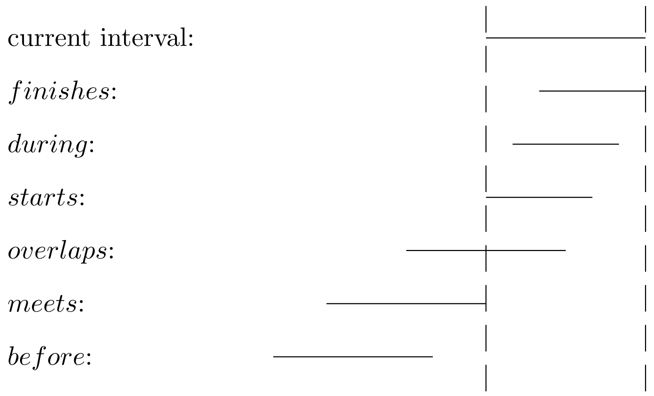


Figure 1.1: Allen's relations between two intervals.

dense if for every pair of different comparable points there exists another point in between, that is:

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y));$$

unbounded above (resp. **below**) if every point has a successor (resp. predecessor); **Dedekind complete** if every non-empty and bounded above set of points has a least upper bound. Besides interval logics over the classes of linear, (un)bounded, discrete, dense, and Dedekind complete interval structures, we will be discussing those interpreted on the single structures $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ with their usual orderings.

1.2 Relations between Intervals

It is well known that there are thirteen different binary relations between intervals on a linear ordering (and quite a few more on a partial ordering) [2]. They are the so-called Allen's relations, which are depicted in Figure 1.1 (the *inverse* relation can be obtained in the obvious way). In the context of Allen's Interval Algebra (IA), the relation between any given intervals can be represented as a suitable subset of the above relations; as an example, the fact that two intervals i, j are such that i is *before* or *after* j is denoted by $i\{b, bi\}j$ (the inverse relations are denoted by $\{si, bi, fi, di, mi, oi\}$). In this way, 2^{13} different relations, including the empty one, can be expressed: in particular, in case of complete lack of knowledge about the actual relation between two intervals, the entire set of relations is assumed. Interval relations can be expressed in terms of relations between their endpoints. For example, given an interval structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ and two intervals $[d_0, d_1]$ and $[d_2, d_3]$ belonging to it, we can state that $[d_0, d_1]$ is *before* $[d_2, d_3]$ if and only if $d_1 < d_2$. In the literature, different semantics (and different notations) for the above relations have been given. Consider, for instance, the case of the relation *during*. Such a relation can assume at

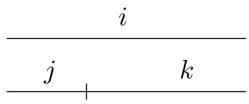


Figure 1.2: Graphical representation of the ternary relation A.

least three different forms: $[d_2, d_3]$ is a **sub-interval** of $[d_0, d_1]$ if $d_0 \leq d_2$ and $d_3 \leq d_1$; $[d_2, d_3]$ is a **proper sub-interval** of $[d_0, d_1]$, if $[d_2, d_3]$ is a sub-interval of $[d_0, d_1]$ and $[d_2, d_3] \neq [d_0, d_1]$; and, $[d_2, d_3]$ is a **strict sub-interval** of $[d_0, d_1]$ if $d_0 < d_2$ and $d_3 < d_1$. As one can easily realize, different interpretations of a relation may lead to different properties of a logic based on that relation. This is, for example, the case of logics based on the *meets* and *met-by* relations (see Chapter 3).

Among all the possible ternary relations between intervals (in a linear ordering), we will be particularly interested in those relations corresponding to the three possible situations occurring when an extra point is added in one of the three possible distinct positions with respect to the two endpoints of an interval (*before, between*, and *after*). In this last case, one can define a ternary relation [95] modelling the fact that a given interval is *chopped* into two consecutive intervals. This relation (denoted by A) can be (first-order) defined by means of Allen's *meets* relation, as follows:

- $A(j, k, i) \triangleq j\ meets\ k \wedge j\ starts\ i \wedge k\ ends\ i$,

and graphically depicted as in Figure 1.2. As an aside, notice that in [95] there is a small imprecision: the relation A is defined in terms of Allen's relations (through first-order logic), but Allen's point-based interpretation usually excludes point-intervals, while, as it is clear from the given semantics of *chop*, Venema includes them.

Modalities of interval logics for time intervals mainly correspond to the possible relations between intervals. In the literature, both *unary* and *binary* modalities have been proposed. Unary modalities correspond to (different interpretations of) Allen's relations, while binary modalities are based on the relation A.

1.3 Representation Theorems

As we already pointed out, one of the major issues in interval temporal logic is the choice of points or intervals as primitive objects. In the rest of the chapter, we will focus our attention on abstract characterizations (taking intervals as primitive objects) and representation theorems that have been given for a number of meaningful interval structures. More precisely, abstract characterizations and representation theorems have been established for the following classes of interval structures:

- Allen and Hayes [4]: a representation theorem for dense unbounded strict linear *meets*-structures;

- Ladkin [54]: a refinement of the above result (for both convex and non-convex intervals), for arbitrary strict linear *meets/met-by*-structures (also called *interval neighborhood* structures);
- Venema [93]: a representation theorem for arbitrary non-strict linear *starts /finishes*-structures;
- Goranko, Montanari, and Sciavicco [37]: representation theorems for both strict and non-strict interval neighbourhood structures.

To complete the picture, cfr. Van Benthem [90].

1.3.1 M-frames and Structures

Definition 1 *An interval* **M-frame** *is a pair* $\langle \mathbb{I}, M \rangle$, *where* $\mathbb{I}$ *is a non-empty set, and* M *is a binary relation on* $\mathbb{I}$ *respecting the following Allen and Hayes' famous (first-order) conditions:*

(MF1) $\forall x, y, k, w(xMy \wedge xMw \wedge kMy \rightarrow kMw)$;

(MF2) $\forall x, y, k, w(xMy \wedge kMw \rightarrow xMw \uplus \exists t(xMtMw) \uplus \exists t(kMtMy))$;

(MF3) $\forall x \exists y, k(yMxMk)$;

(MF4) $\forall x, y, k, w(xMyMk \wedge xMwMk \rightarrow y = w)$;

(MF5) $\forall x, y(xMy \rightarrow \exists k, w, z(kMx \wedge yMw \wedge kMzMw)))$,

where $\uplus$ *is the exclusive logical disjunction.*

As an aside, we recall that Galton [33] showed that any one of the above conditions is independent from the others, unlike previously claimed by Ladkin in [55].

The concrete structures are defined as follows:

Definition 2 *An interval* **M-structure** *is a strict linear interval structure* $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle$. *The relation* M *is defined as follows: for any pair of intervals* $[d_0, d_1]$ *and* $[d_2, d_3]$, $[d_0, d_1]M[d_2, d_3]$ *if* $d_1 = d_2$.

Theorem 3 *Any interval M-frame is isomorphic to an interval M-structure.*

The proof of the above theorem can be found in [4].

1.3.2 BE-frames and Structures

Definition 4 *An interval* **BE-frame** *is a triple* $\langle \mathbb{I}, B, E \rangle$, *where* $\mathbb{I}$ *is a non-empty set, and* B, E *are binary relations defined on it, in such a way that they respect the following conditions:*

(BEF1) *Transitivity of* B *and* E;

(BEF2) *Left linearity of B and E:* $\forall x \forall y \forall z(xBz \wedge yBz \rightarrow xBy \vee x = y \vee yBx)$*, and likewise for E;*

(BEF3) *Atomicity for B and E:* $\forall x(\neg\exists z(zBx) \vee \exists y(yBx \wedge \neg\exists z(zBy))$*, and likewise for E;*

(BEF4) *An interval has no proper ends if and only if it has no proper beginnings:*

$$\forall x(\neg\exists z(zBx) \leftrightarrow \neg\exists z(zEx))$$

where $\leftrightarrow$ is the usual shorthand;

(BEF5) *Unique directedness of intervals:*

$$\forall x \forall y \forall z(xBy \wedge xEz \rightarrow \exists! u(zBu \wedge yEu),$$

$$\forall x \forall y \forall z(xBy \wedge zEx \rightarrow \exists! u(zBu \wedge uEy),$$

$$\forall x \forall y \forall z(xEy \wedge zBx \rightarrow \exists! u(uBy \wedge zEu);$$

(BEF6) *No overlap of B and E :* $\neg\exists x \exists y(xBy \wedge xEy)$.

As for concrete structures, we give the following definition:

Definition 5 *An interval* ***BE-structure*** *is an interval structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+ \rangle$ such that two binary relations B, E over $\mathbb{I}(\mathbb{D})^+$ are defined in such a way to correspond to the interval relations* begin *and* end*, i.e.:*

- *iBj holds if i is a* ***proper beginning*** *of j, i.e. $j = [d_0, d_1]$ and $i = [d_0, d_2]$ for some $d_0, d_1, d_2 \in D$ such that $d_0 \leq d_1 < d_2$;*
- *iEj holds if i is a* ***proper end*** *of j, i.e. $i = [d_1, d_2]$ and $j = [d_0, d_2]$ for some $d_0, d_1, d_2 \in D$ such that $d_0 < d_1 \leq d_2$.*

The detailed proof of the following theorem can be found in [93]; here we give a sketch, since the proofs of the representation theorems in the next sections are quite similar to this one.

Theorem 6 *Any interval BE-frame is isomorphic to an interval BE-structure.*

To give an intuition, the proof goes as follows. One direction is trivial. So, let $\mathbf{F} = \langle \mathbb{I}, B, E \rangle$, where $\mathbb{I}$ is a non-empty set of objects, and B, E are two binary relations over $\mathbb{I}$ that satisfy the given conditions. The (sub-)set of $\mathbb{I}$ containing only point-intervals is defined by $D = \{x \in \mathbb{I} \mid \neg\exists y(yBx)\}$. A relation $<$ can be defined over D in the following way: $\forall x, y \in D\ x < y \leftrightarrow \exists z(xBz \wedge yEz)$. It is possible to show that this relation is irreflexive, transitive and asymmetric, that is, that $\mathbb{D} = \langle D, < \rangle$ is a strict partial ordering. Since for all non-point intervals $y \in D$, it can be shown (by left-linearity) that there is a unique point-interval $x \in \mathbb{D}$ such that xBy (and similarly for E), one can define a mapping $f : \mathbb{I} \mapsto \mathbb{I}(\mathbb{D})^+$ in such a way that $\forall x \in \mathbb{I}\ (f(x) = [bp(x), ep(x)])$, where $bp(x)$ and $ep(x)$ are the (unique) beginning and ending point-intervals of x. Once proved that f is an isomorphism, the proof is completed.

1.3.3 Interval Neighborhood Frames and Structures

In this section, we give the basic notions of interval neighborhood frames and structures. In the next one, we provide representation theorems for both strict and non-strict semantics.

Definition 7 *A* ***neighborhood frame*** *is a triple* $\mathbf{F} = \langle \mathbb{I}, R, L\rangle$ *where* $\mathbb{I}$ *is a non-empty set and* R, L *are binary relations on* $\mathbb{I}$*.*

For every sequence $S_1, ..., S_k \in \{R, L\}$, we denote the composition of the relations $S_1, ..., S_k$ by $S_1 ... S_k$. Also, we put:

$\mathbf{B_F} = \{w \in \mathbb{I} \mid$ there is no $v \in \mathbb{I}$ such that $wLv\}$,

$\mathbf{B^2_F} = \{w \in \mathbb{I} \mid$ there are no $u, v \in \mathbb{I}$, with $u \neq v$, such that wLv and $wLu\}$,

$\mathbf{E_F} = \{w \in \mathbb{I} \mid$ there is no $v \in \mathbb{I}$ such that $wRv\}$, and

$\mathbf{E^2_F} = \{w \in \mathbb{I} \mid$ there are no $u, v \in \mathbb{I}$, with $u \neq v$, such that wRv and $wRu\}$.

Consider the following conditions:

(NF1) $\forall x, y(xRy \leftrightarrow yLx)$;

(NF2) $\forall x\forall y(\exists z(xLz \wedge zRy) \rightarrow \forall z(xLz \rightarrow zRy))$ and $\forall x\forall y(\exists z(xRz \wedge zLy) \rightarrow \forall z(xRz \rightarrow zLy))$;

(NF3) $RL \subseteq LRR \cup LLR \cup E$ on $\mathbb{I} - \mathbf{B^2_F}$ and $LR \subseteq RLL \cup RRL \cup E$ on $\mathbb{I} - \mathbf{E^2_F}$, where E is the equality, i.e. $\forall x\forall y(\exists z\exists u(xRz \wedge zLu) \wedge \exists z(xRz \wedge zLy) \rightarrow x = y \vee \exists w\exists z((xLw \wedge wRz \wedge zRy) \vee (xLw \wedge wLz \wedge zRy)))$ and $\forall x\forall y(\exists z\exists u(xRz \wedge zRu) \wedge \exists z(xLz \wedge zRy) \rightarrow x = y \vee \exists w \exists z((xRw \wedge wLz \wedge zLy) \vee (xRw \wedge wRz \wedge zLy)))$;

(NF4) $RRR \subseteq RR$, i.e. $\forall w\forall x\forall y\forall z(wRx \wedge xRy \wedge yRz \rightarrow \exists u(wRu \wedge uRz))$.

Definition 8 *An* ***interval neighborhood frame*** *is a neighborhood frame* $\mathbf{F} = \langle \mathbb{I}, R, L\rangle$ *satisfying the conditions NF1,...,NF4.*

Notice that, assuming **NF1, NF4** is equivalent to $\forall w\forall x\forall y\forall z(wLx \wedge xLy \wedge yLz \rightarrow \exists u(wLu \wedge uLz))$.

Definition 9 *An interval neighborhood frame* $\mathbf{F} = \langle \mathbb{I}, R, L\rangle$ *is said to be:*

- ***Strict,*** *if the relation* LRR *is irreflexive, and* ***non-strict*** *if the relation* LRR *is reflexive (note that 'not strict' does not imply 'non-strict');*
- ***Open,*** *if it satisfies the condition* $\forall x(\exists y(xLy) \wedge \exists y(xRy))$*;*
- ***Rich,*** *if it satisfies the condition* $\forall x(\exists y(xRy \wedge yRy) \wedge \exists y(xLy \wedge yLy))$*.*
- ***Normal,*** *if it satisfies the condition* $\forall x\forall y(\forall z(zRx \leftrightarrow zRy) \wedge \forall z(zLx \leftrightarrow zLy) \rightarrow x = y)$*;*

- ***Tight,*** *if it satisfies the condition* $\forall x\forall y((xRRy \wedge yRRx) \rightarrow x = y)$*;*
- ***Weakly left-connected*** *(resp.,* ***weakly right-connected****) if the relation* $LR \cup LRR \cup LLR$ *(resp.,* $RL \cup RRL \cup RLL$*) is an equivalence relation on* $\mathbb{I} - \mathbf{B_F}$ *(resp.,* $\mathbb{I} - \mathbf{E_F}$*);* ***left-connected*** *(resp.,* ***right-connected****) if that relation is the universal relation on* $\mathbb{I} - \mathbf{B_F}$ *(resp.,* $\mathbb{I} - \mathbf{E_F}$*);*
- ***Weakly connected*** *if each of the relations* $LR \cup LRR \cup LLR$ *and* $RL \cup RRL \cup RLL$ *is an equivalence relation on* $\mathbb{I}$*;* ***connected,*** *if each of these relations is the universal relation on* $\mathbb{I}$*.*

Now, consider the following definitions:

(NF5) NF2 implies $LRL \subseteq L$ and $RLR \subseteq R$, i.e., $\forall x\forall y((xLRLy \rightarrow xLy) \wedge (xRLRy \rightarrow xRy))$;

(NF6) Assuming NF2, normality implies $\forall x\forall y(\exists z(zRx \wedge zRy) \wedge \exists z(zLx \wedge zLy) \rightarrow x = y)$, i.e., $\forall x\forall y(xLRy \wedge xRLy \rightarrow x = y)$. Assuming also openness, normality becomes equivalent to that condition;

(NF7) In every non-strict interval neighborhood frame, $RR = RRR$ and $LL = LLL$;

(NF8) Every rich interval neighborhood frame is non-strict and open;

(NF9) Every non-strict interval neighborhood frame is weakly connected. Every strict interval neighborhood frame is weakly left- and right-connected;

(SNF) In every strict interval neighborhood frame each of L, R, LLR, RRL, and RLL is irreflexive, too;

(NNF) An interval neighborhood frame is non-strict iff either of $LRR \cup LLR$ and $RLL \cup RRL$ is an equivalence relation on $\mathbb{I}$.

Proposition 10 *NF5, NF6, NF7, NF8, SNF, and NNF are consequences of the definitions.*

In what follows, linear interval structures will be referred to as **interval neighborhood structures**.

Theorem 11 $\mathbf{F}$ *is a tight, rich, connected, and normal interval neighborhood frame if and only if* $\mathbf{F}$ *is isomorphic to a non-strict interval neighborhood structure.*

Proof.

Let $\mathbf{F} = \langle \mathbb{I}, R, L \rangle$ be a tight, rich, connected, and normal interval neighborhood frame. We construct an underlying linear ordering for $\mathbf{F}$ and then we show that $\mathbf{F}$ is isomorphic to the non-strict interval neighborhood structure over that ordering.

Let $\mathbf{P}(\mathbb{I}) = \{u \in \mathbb{I} | uRu\}$. Note that $\mathbf{P}(\mathbb{I})$ is non-empty and uLu for every $u \in \mathbf{P}(\mathbb{I})$. We will show that for every $u, v \in \mathbf{P}(\mathbb{I})$,

$$uLRv \text{ iff } u = v.$$

Indeed, $uLuRu$, i.e. $uLRu$. Conversely, let $uLRv$. Note that, by NF5, LR is an equivalence relation on $\mathbf{P}(\mathbb{I})$. Furthermore, if $uLRv$ then $uRuLRv$, i.e. $uRLRv$, so uRv, hence $uRLv$ and so, likewise, uLv. Now, for every w, vRw implies $uRLRw$, hence uRw. Likewise, uRw implies vRw. Analogously, uLw implies vLw and vice versa. Then, by normality, $u = v$. From this, it follows that for every $w \in \mathbb{I}$ there is a unique $v \in \mathbf{P}(\mathbb{I})$, hereafter denoted by $\mathbf{b}(w)$, such that wLv. Likewise, there is a unique $v \in \mathbf{P}(\mathbb{I})$, hereafter denoted by $\mathbf{e}(w)$, such that wRv. We now define a relation $<$ on $\mathbf{P}(\mathbb{I})$ as follows:

$$u < v \text{ iff } uRRv.$$

The relation $<$ is a linear ordering on $\mathbf{P}(\mathbb{I})$: reflexivity is obvious, transitivity follows from NF7 and NF8, and anti-symmetry follows from tightness. As for the linearity: for any $u, v \in \mathbf{P}(\mathbb{I})$, $uLRRv$ or $uLLRv$ since $LRR \cup LLR$ is the universal relation on $\mathbb{I}$. Suppose $uLRRv$. Then $uRuLRRv$, i.e., $uRLRRv$, hence $uRRv$, i.e., $u < v$. Likewise, if $uLLRv$ then $uLLv$, hence $vRRu$, i.e., $v < u$. Note that for every $w \in \mathbb{I}$, $\mathbf{b}(w)RwR\mathbf{e}(w)$, hence $\mathbf{b}(w) < \mathbf{e}(w)$. Now, we define a mapping μ from $\mathbb{I}$ to the non-strict interval neighborhood structure $\langle \mathbb{I}^+(\mathbf{P}(\mathbb{I})), \mathbf{L}, \mathbf{R}\rangle$ over $\langle \mathbf{P}(\mathbb{I}), <\rangle$ as follows:

$$\mu(w) = (\mathbf{b}(w), \mathbf{e}(w)).$$

1. μ is an injection. If $\mu(w_1) = \mu(w_2)$, then let $\mathbf{b}(w_1) = \mathbf{b}(w_2) = b$ and $\mathbf{e}(w_1) = \mathbf{e}(w_2) = e$. Then, for every $x \in \mathbb{I}$, w_1Rx implies $w_2R\mathbf{e}(w_2)(= \mathbf{e}(w_1))Lw_1Rx$, i.e., w_2RLRx, hence w_2Rx. Likewise, w_2Rx implies w_1Rx. Analogously, w_1Lx iff w_2Lx. Then, by normality, $w_1 = w_2$.

2. μ is onto. If $u, v \in \mathbf{P}(\mathbb{I})$ and $u < v$, then $uRRv$, i.e., $uRwRv$ for some $w \in \mathbb{I}$ and hence $\mu(w) = (u, v)$.

3. μ is an isomorphism. If w_1Rw_2, then $\mathbf{e}(w_1)R\mathbf{e}(w_1)Lw_1Rw_2$, i.e., $\mathbf{e}(w_1)\ RLRw_2$. Hence $\mathbf{e}(w_1)Rw_2$, and thus $\mathbf{e}(w_1) = \mathbf{b}(w_2)$ by uniqueness of $\mathbf{b}(w_2)$. It follows that $\mu(w_1)\mathbf{R}\mu(w_2)$. Conversely, if $\mu(w_1)\mathbf{R}\mu(w_2)$, then $w_1R\mathbf{e}(w_1)L\mathbf{e}(w_1)$ $(= \mathbf{b}(w_2))Rw_2$, i.e., $w_1\ RLRw_2$, and hence w_1Rw_2. Likewise, w_1Lw_2 iff $\mu(w_1)$ $\mathbf{L}\ \mu(w_2)$.

This completes the proof. ■

Theorem 12 [1]

1. *$\mathbf{F}^-$ is a weakly connected, strict and normal interval neighborhood frame if and only if $\mathbf{F}^-$ is isomorphic to a strict interval neighborhood structure.*

2. *$\mathbf{F}^-$ is a connected, open, strict and normal interval neighborhood frame if and only if $\mathbf{F}^-$ is isomorphic to a strict unbounded interval neighborhood structure.*

[1] Similar representation results can be found in [54].

Proof.

We prove 2 (the proof can be easily modified for 1). Let $\mathbf{F}^- = \langle \mathbb{I}, R, L\rangle$ be a connected, open, strict, and normal interval neighborhood frame. We construct an underlying point-based linear ordering and we show that $\mathbf{F}^-$ is isomorphic to the strict unbounded interval neighborhood structure over that ordering.

First, for every $w \in \mathbb{I}$, we define $[w]_b = \{v \in \mathbb{I} \mid wLRv\}$ and $[w]_e = \{v \in \mathbb{I} \mid wRLv\}$. By NF5, we have that $LRL \subseteq L$ and $RLR \subseteq R$. Hence, the relations LR and RL are equivalence relations in $\mathbb{I}$, and thus the sets $P_b = \{[w]_b \mid w \in \mathbb{I}\}$ and $P_e = \{[w]_e \mid w \in \mathbb{I}\}$ are partitions of $\mathbb{I}$. Now, we define the mapping $\theta : P_e \mapsto P_b$ as follows:

$$\theta([w]_e) = [v]_b \text{ where } wRv.$$

First, notice that the definition is correct: if $[w_1]_e = [w_2]_e$, $[v_1]_b = [v_2]_b$, and w_1Rv_1 then $w_2RLRLRv_2$. By NF5, we obtain w_2RLRv_2 and thus w_2Rv_2 by NF5 again. Then, θ is a function: if wRv_1 and wRv_2 then v_1LRv_2, i.e., $[v_1]_b = [v_2]_b$; also, if wRv and $w_1 \in [w]_e$, then w_1RLw. Hence w_1RLRv, and thus w_1Rv. Furthermore, θ is a bijection between P_e and P_b. Indeed, if $\theta([w_1]_e) = \theta([w_2]_e) = [v]_b$, then w_1Rv and w_2Rv, and hence w_1RLw_2, i.e., $[w_1]_e = [w_2]_e$. The surjectivity immediately follows from the definition of P_b. From now on, we will identify P_e with P_b via θ and we will only deal with P_b. We define a relation $<$ on P_b as follows:

$$[w]_b < [v]_b \text{ iff } wLRRv.$$

Correctness of the definition: if $[w_1]_b = [w_2]_b$, $[v_1]_b = [v_2]_b$, and $w_1LRR\ v_1$, then $w_2LRw_1LRRv_1LRv_2$, i.e., $w_2(LRL)R(RLR)v_2$, and thus w_2LRRv_2 by NF5. Now we show that the relation $<$ is a strict linear ordering on P_b:

1. Irreflexivity holds because $\mathbf{F}^-$ is strict;
2. Transitivity: let $w_1LRRw_2LRRw_3$, i.e., $w_1LR(RLR)Rw_3$. Hence, we have that $w_1\ LRRRw_3$ by NF5, and thus $w_1\ LRRw_3$ by NF4;
3. Linearity: we have to show that for every $[w]_b, [v]_b \in P_b$, $[w]_b < [v]_b$ or $[w]_b = [v]_b$ or $[v]_b < [w]_b$, i.e., $wLRRv$ or $wLRv$ or $vLRRw$, that is, $wLLRv$, which is precisely the connectedness condition on $\mathbf{F}^-$.

Notice that $\langle P_b, <\rangle$ is open: for every $[w]_b \in P_b$ there exists $v \in \mathbb{I}$ such that vRw and there exists $u \in \mathbb{I}$ such that vLu. Hence, $vLRRw$, i.e., $[v]_b < [w]_b$. Likewise, there exists $[v]_b$ such that $[w]_b < [v]_b$. It remains to show that the strict interval structure on $\langle P_b, <\rangle$ is isomorphic to $\mathbf{F}^-$. The isomorphism is given by the mapping $\mu : \mathbf{F}^- \mapsto \mathbb{I}(P_b)^-$ determined by

$$\mu(w) = ([w]_b, \theta([w]_e)).$$

Let $\theta([w]_e) = [v]_b$ where wRv. We have that $wLRRv$, and thus $[w]_b < [v]_b$. Hence, μ associates intervals from $\mathbb{I}(P_b)^-$ with every $w \in \mathbf{F}^-$. Now, if $[w_1]_b = [w_2]_b$ and $\theta([w_1]_e) = \theta([w_2]_e)$, then w_1LRw_2, and w_1Rv_1 and w_2Rv_2, for v_1, v_2 such that $[v_1]_b = [v_2]_b$ and thus v_1LRv_2. Hence w_1RLRLw_2, and thus w_1RLw_2 by NF5. From w_1LRw_2

and $w_1 RL w_2$, it follows that $w_1 = w_2$ by NF6, that is, NF2 plus normality. Finally, for every interval $([w]_b, [v]_b)$ in $\mathbb{I}(P_b)^-$, we have $[w]_b < [v]_b$, i.e., $wLRRv$, and thus $wLRu$ and uRv for some $u \in \mathbf{F}^-$. Then $[u]_b = [w]_b$ and $\theta([u]_e) = [v]_b$, i.e., $([w]_b, [v]_b) = \mu(u)$. Thus, μ is an isomorphism and the proof is completed. ■

2

A Road Map on Interval Temporal Logics

"The question of whether a computer can
think is no more interesting than
the question of whether a submarine can swim."
Edsger W. Dijkstra

Even if the interval logic literature is not very large, compared to that of other fields in computer science, it is not an easy job to provide a general picture of research in this area, in order to better understand what has already been done, and what still remains to be investigated.

2.1 Parameters for Interval Logics

We highlight various classes of parameters that completely characterize an interval logic, which we are going to discuss now.

1. Do we want to consider *all* intervals (resp., *all strict* intervals) of a given structure, or not? Given a partial ordering $\mathbb{D} = \langle D, < \rangle$, most interval logics take into consideration all intervals $\mathbb{I}(\mathbb{D})^{+}$ or, at least, all strict intervals $\mathbb{I}(\mathbb{D})^{-}$. This is not the only possible choice. Indeed, one may consider a definition of interval structure which is more general than the one given in Chapter 1, such as $\langle \mathbb{D}, \mathbb{H}(\mathbb{D}) \rangle$, where $\mathbb{H}(\mathbb{D})$ is a subset of $\mathbb{I}(\mathbb{D})$, determined by suitable rules. In Chapter 5, we will see an example of interval logics interpreted over restricted interval structures.

   Moreover, the semantics of interval temporal logics is sometimes subjected to restrictions related to the specific applications for which it has been developed, such as:

   - **Locality**, meaning that all atomic propositions are point-wise and truth at an interval is defined as truth at its initial point;

- **Homogeneity**, requiring that truth of a formula at an interval implies truth of it at every sub-interval;

2. Are we considering a *propositional* interval temporal logic, a *first-order* interval temporal logic, or a *duration calculus*? The generic language of propositional interval logics includes the set of propositional letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\wedge$ (all others, including the propositional constants $\top$ and $\bot$, are definable as usual), and a specific set of interval temporal operators (modalities). The first-order languages for interval logics extend the propositional ones essentially the same way as in classical logic, but accounting for the fact that the first-order domain may change over time. Formally, these languages involve **terms** built as usual from variables, constants and functional symbols. Constants and functional symbols are classified as **global** or **rigid** (whose interpretation does not depend on the time) and **temporal** or **flexible** (whose interpretation can vary over time). Predicate symbols (also classified as global or temporal) are denoted by $p_n, q_m, \ldots$, where $n, m \ldots$ represent the arities. Among the constants, there is a specific and important one, present in most first-order interval logics and duration calculi, namely, the flexible constant l denoting the **length of the current interval**. Often it is combined with a structure of an additive group (typically, the additive group of reals) as part of the temporal domain, which allow one to compute lengths of concatenated intervals, etc. A specific additional feature of the syntax of duration calculi is the special category of terms called **state expressions** which are meant to represent the duration for which a system stays in a particular state;

3. Is our interval logic interpreted over the class of *all* partial orderings (as already pointed out, we confine ourselves to partial orderings with the linear interval property), or over a class of all structures that satisfy a given set of properties? Interesting classes that we will take into consideration are:

   - The linear orderings, denoted by **lin**;
   - The unbounded orderings, denoted by **u**;
   - The dense orderings, denoted by **de**;
   - The discrete orderings, denoted by **di**;
   - The Dedekind complete orderings, denoted by **c**;
   - The unbounded and dense orderings, denoted by **ude**;
   - The unbounded and discrete orderings, denoted by **udi**;
   - The unbounded and Dedekind complete orderings, denoted by **uc**;

4. Is our semantic strict or non-strict? For a given interval logic at least two variants are possible, namely the **strict** one (where point-interval are excluded) and the **non-strict** one (where point-intervals are admitted);

5. Are the modalities of our language unary or binary? We will see that in most cases few modalities are enough to define all the others, although this is not always the case. Moreover, in the non-strict semantics, including or excluding the modal constant π (defined later in this chapter) for point-intervals is, sometimes, a relevant parameter.

In general, for a given interval logic L interpreted over the class of all interval structures, we will denote by $\mathcal{L}$ the class of the logics with the same language and abstract grammar as L and interpreted over all possible classes of interval structures. Some interval logics are usually confined to linear structures; in such a case we will be omitting the superscript ^{lin}, and we will denote by BL the variant of L interpreted over partial orderings. Notice that sometimes we will be using specific superscripts to indicate the presence of particular modalities, such as π.

From now on in this work, with the exception of Chapter 5, we will take into consideration interval logics whose semantics include either all intervals or all strict intervals.

2.2 Syntax and Semantics of Interval Logics

A **model** for a propositional interval logic is built up from an interval structure, provided by a valuation function for propositional variables. A **non-strict interval model** is a pair $\mathbf{M}^{+}=\langle\mathbb{D},\mathbb{I}(\mathbb{D})^{+},V\rangle$, where $V:\mathbb{I}(\mathbb{D})^{+}\rightarrow\mathbf{P}(\mathcal{AP})$. A **strict interval model** is a pair $\mathbf{M}^{-}=\langle\mathbb{D},\mathbb{I}(\mathbb{D})^{+},V\rangle$, where V takes $\mathbb{I}(\mathbb{D})^{-}$ as domain. The semantics of a propositional interval logic in the case that not all intervals are included, can be simply obtained by substituting the set $\mathbb{I}(\mathbb{D})$ with the opportune set of intervals. The semantics of the modal part must be given case-by-case, depending on the model being strict or non strict, and on the interpretation of the considered relation.

A propositional interval logic's well formed **formula** (usually denoted by ϕ_0,ϕ_1, $\ldots$, ψ_0, $\psi_1,\ldots$) is generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi\wedge\psi \mid \nabla_n^1(\phi_1,\ldots,\phi_n) \mid \ldots \mid \nabla_m^k(\phi_1,\ldots,\phi_m)$$

where ∇_l^i is the $i-$th $l-$ary modality. The formal semantics (i.e., the **truth** relation) of a propositional interval formula, in a given model $\mathbf{M}=\langle\mathbb{D},\mathbb{I}(\mathbb{D}),V\rangle$ and for a given reference interval $[d_0,d_1]$, depends on the assumption we make over it. The propositional part is as in the classical case:

(P0) $\mathbf{M},[d_0,d_1]\Vdash p$ if and only if $p\in V([d_0,d_1])$;

(P1) $\mathbf{M},[d_0,d_1]\Vdash\neg\phi$ if and only if it is not the case that $\mathbf{M},[d_0,d_1]\Vdash\phi$;

(P2) $\mathbf{M},[d_0,d_1]\Vdash\phi\wedge\psi$ if and only if $\mathbf{M},[d_0,d_1]\Vdash\phi$ and $\mathbf{M},[d_0,d_1]\Vdash\psi$.

The abstract syntax of formulas of a generic first-order interval language includes the clauses:

$$\phi ::= p_n(\theta_1,\ldots,\theta_n) \mid \exists x\phi \mid\ \neg\phi \mid \phi\wedge\psi \mid \nabla_n^1(\phi_1,\ldots,\phi_n) \mid \ldots \mid \nabla_m^k(\phi_1,\ldots,\phi_m)$$

where $\theta_1, \ldots, \theta_n$ are terms.

The semantics of first-order interval formulas is a combination of the standard semantics of a first-order (temporal) logic with the semantics of the specific underlying propositional interval logic.

2.2.1 Modalities for Time Intervals

Modalities for time intervals are obviously associated to the various relations between intervals. In the following, we generically discuss the semantics of an interval temporal logic. Since the non-strict semantics is the most widely adopted, we will give the *non-strict version* of the formal rules; where not otherwise specified, it is not difficult to adapt them to the strict case. In this section we only give the informal semantics of the modalities; the formal rules will be given during the presentation of the various logics.

The most widely accepted point-based interpretation of the interval relations are, $[d_0, d_1]$ being the current interval:

- *Met by* relates the current interval to an interval $[d_1, d_2]$ with $d_1 < d_2$, and the corresponding modality is usually denoted by $\langle A \rangle$ (and its inverse, corresponding to the relation *meets*, by $\langle \overline{A} \rangle$); notice that, only for historical reasons, we use the symbols $\Diamond_r$ and $\Diamond_l$ for the above modalities when they capture also the point-intervals;

- *Starts*, or *begins*, relates the current interval to an interval $[d_0, d_2]$ with $d_0 \leq d_2 < d_1$, and the corresponding modality is usually denoted by $\langle B \rangle$ (while its inverse, corresponding to the relation *started by*, or *begun by*, by $\langle \overline{B} \rangle$);

- *Finishes*, or *ends*, relates the current interval to an interval $[d_2, d_1]$ with $d_0 < d_2 \leq d_1$, and the corresponding modality is usually denoted by $\langle E \rangle$ (its inverse, corresponding to the relation *finished by*, or *ended by*, by $\langle \overline{E} \rangle$);

- *During* relates the current interval to an interval $[d_2, d_3]$ with $d_0 < d_2 \leq d_3 < d_1$, and the corresponding modality is usually denoted by $\langle D \rangle$ (its inverse, corresponding to the relation *includes*, by $\langle \overline{D} \rangle$). Anyway, as we have seen in the previous chapter, such a relation can be interpreted in at least three versions; in this version, this relation is also called *strict sub-interval* (and denoted by $\sqsubset$), while the *proper sub-interval*, denoted by $\overline{\sqsubset}$, corresponds to an interval $[d_2, d_3]$ with $d_0 \leq d_2 \leq d_3 < d_1$ or $d_0 < d_2 \leq d_3 \leq d_1$, and the *sub-interval* relation by $\sqsubseteq$, corresponds to an interval $[d_2, d_3]$ with $d_0 \leq d_2 \leq d_3 \leq d_1$;

- *Overlaps* relates the current interval to an interval $[d_2, d_3]$ with $d_2 < d_0 \leq d_3 < d_1$, and the corresponding modality is usually denoted by $\langle O \rangle$ (while its inverse, corresponding to the relation *overlapped by*, by $\langle \overline{O} \rangle$);

- *After* relates the current interval to an interval $[d_2, d_3]$ with $d_1 < d_2$, and the corresponding modality is usually denoted by $\langle L \rangle$ (while its inverse, corresponding to the relation *before*, by $\langle \overline{L} \rangle$);

- *Chop* captures two intervals $[d_0, d_2]$ and $[d_2, d_1]$ such that $d_0 \leq d_2 \leq d_1$, and the corresponding modality is usually denoted by C (there are two more modalities, corresponding to the two other possible positions for d_2, and they are usually denoted by D and T).

Historically speaking, names of modalities came from Halpern and Shoham's paper ([42]), where authors use alternative names for Allen's relations; indeed, $\langle A \rangle$ stands for *after*, which is used instead of *meets*, as well $\langle L \rangle$ stands for *later*, which is used instead of *after*. Finally, in Chapter 5, we will use the modality $\langle F \rangle$ (*future*), interpreted as the relation $\{m, b\}$.

In some interval logics in the literature a modal constant it is also present for point-intervals, usually denoted by π. Formally, its semantics (obviously making sense only in the case of non-strict semantics) is:

(π) $\mathbf{M}^+, [d_0, d_1] \Vdash \pi$ if $d_0 = d_1$.

The presence of π in the language allows for interpretation of the strict semantics into the non-strict one (at least for languages with only unary modalities), by means of the translation:

- $\tau(p) = p$ for $p \in \mathcal{AP}$;
- $\tau(\neg\phi) = \neg\tau(\phi)$;
- $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$;
- $\tau(\nabla_1^i \phi) = \nabla_1^i(\neg\pi \wedge \tau(\phi))$ for any (unary) interval modality ∇_1^i.

The interpretation is effected by the following claim, proved by a straightforward induction on ϕ:

Proposition 13 *For every interval model* $\mathbf{M}$, *strict interval* $[d_0, d_1]$ *in* $\mathbf{M}$, *and a formula* ϕ, $\mathbf{M}^-, [d_0, d_1] \Vdash \varphi$ *iff* $\mathbf{M}^+, [d_0, d_1] \Vdash \tau(\phi)$.

2.3 Interval Temporal Logics with Unary Modalities

In this section, we concentrate on those propositional interval logics that have been studied in the literature and that are characterized by having only unary modalities (and possibly the modal constant π). For interval logics with unary modalities we restrict ourselves to the propositional level, since first-order interval logics have mostly been developed for binary modalities. Since the non-strict semantics is generally the standard, we will be omitting the superscript $^+$ for the names of the logics.

2.3.1 The Class $\mathcal{D}$

Perhaps the most natural relations between intervals are those of *sub-interval* and *meets*. The latter corresponds to the neighborhood logics which will be discussed later. We denote the interval logic based on the former and interpreted over the class of all linear orderings by D. Notice that, because of the linear interval hypothesis and that this logics only look inward the current interval, we can restrict ourselves to the class of linear structures. The abstract syntax of the simplest version of D is:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle D \rangle \phi,$$

but one could also include in the language the modal constant π.

The sub-interval relation and the temporal logics associated with it were studied, from the perspective of philosophical temporal logics, in [43, 83], [49] (together with precedence), and [91]. In the computer science literature, it was apparently first mentioned in [42] and its expressiveness (interpreted over linear non-strict models) discussed in [56].

Besides the strict and non-strict versions, the logic D allows essential semantic variations, depending on which sub-interval relation is assumed. Accordingly, the truth definition for D is based on the clause:

($\langle D \rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle D \rangle \phi$ if there exists a sub-interval $[d_2, d_3]$ of $[d_0, d_1]$ such that $\mathbf{M}^+, [d_2, d_3] \Vdash \phi$.

At present, we are not aware of any specific published results about expressive power, axiomatic systems, and decidability for (variants of) the logic D, but we note that, at least in the cases of proper and strict versions, non-trivial valid formulas expressible in D arise, associated with length vs depth (maximal length of chains of nested sub-intervals). To give an intuition, we may list some valid formulas in the logic D, where the sub-interval relation is the strict one:

(A-D1) $\langle D \rangle \langle D \rangle p \rightarrow \langle D \rangle p$;

(A-D2^n) $\bigwedge_{i=1}^{d(n)} \langle D \rangle \left(p_i \wedge \bigwedge_{j \neq i} \langle D \rangle \neg p_j \right) \rightarrow \langle D \rangle^n \top$ (for a large enough $d(n)$).

One can easily find out that an axiomatic system including all propositional tautologies, the K-axiom for $[D]$, the formula (A-D1), and the standard derivation rules (that is, the interval version of the axiomatic system for $K4$) cannot derive the (infinite) axiom schema (A-D2^n). The problem of finding sound and complete axiomatic systems for (variants of) D is still essentially not studied.

2.3.2 The Class $\mathcal{BE}$

The logics in $\mathcal{BE}$ feature the two modalities $\langle B \rangle$ and $\langle E \rangle$, and their formulas are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B\rangle\phi \mid \langle E\rangle\phi.$$

Here we give the semantic clauses for the non-strict case, that are readily adaptable to the strict one:

($\langle B\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle B\rangle\phi$ if there exists d_2 such that $d_0 \leq d_2 < d_1$ and $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$;

($\langle E\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle E\rangle\phi$ if there exists d_2 such that $d_0 < d_2 \leq d_1$ and $\mathbf{M}^+, [d_2, d_1] \Vdash \phi$.

The modal constant π is definable (in the non-strict case) as follows:

- $\pi \triangleq [B]\bot$ (or, alternatively, by a similar formula using the modality $\langle E\rangle$);

Accordingly, the point-intervals that respectively begins and ends the current interval can be captured as follows:

- $[[BP]]\phi \triangleq (\phi \wedge \pi) \vee \langle B\rangle(\phi \wedge \pi)$, and
- $[[EP]]\phi \triangleq (\phi \wedge \pi) \vee \langle E\rangle(\phi \wedge \pi)$.

When interpreted on the same class of structures, the logic BE is strictly more expressive than D. Indeed, if we assume the strict sub-interval relation for the modality $\langle D\rangle$ (the other cases can be dealt with in the same way) can be defined as

- $\langle D\rangle\phi \triangleq \langle B\rangle\langle E\rangle\phi$.

The undefinability of $\langle B\rangle$ and $\langle E\rangle$ in D has been conjectured by Lodaya in [56], and it can be easily proved. For example, we assume the non-strict semantics, we interpret the modality $\langle D\rangle$ with the strict sub-interval relation, and we consider the class of all linear structures.

Theorem 14 *The (non-strict) modalities $\langle B\rangle$ and $\langle E\rangle$ cannot be defined in the logic D interpreted in the class of all non-strict linear structures.*

Proof.

Let $\langle \mathbb{I}(\mathbb{D})^+, \sqsubset, V\rangle$ be a D-model, where $\mathbb{I}(\mathbb{D})^+$ is the set of all non-strict intervals over $\mathbb{D}$, $\sqsubset$ is the strict sub-interval relation, and V is the valuation function (notice that, given a non-strict model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V\rangle$, we can always build up a D-model in a straightforward way). Clearly, the notions of p-morphism and bisimulation between D-models are defined as standard, and the usual truth-preservation properties well-known for modal/temporal logics are respected. So, consider two D-models $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, \sqsubset, V\rangle$ and $\mathbf{M'}^+ = \langle \mathbb{I}(\mathbb{D}')^+, \sqsubset', V'\rangle$, such that:

1. $\mathbb{D} = \{d_0, d_1\}$ and $\mathbb{D}' = \{d'_0\}$ are linearly ordered sets, where $d_0 < d_1$;
2. $\mathbb{I}(\mathbb{D})^+ = \{[d_0, d_0], [d_1, d_1], [d_0, d_1]\}$, and $\mathbb{I}(\mathbb{D}')^+ = \{[d'_0, d'_0]\}$;

3. the valuations of all interval in both model are equal to $\{p\}$.

Consider the relation $R \subseteq \mathbb{D} \times \mathbb{D}' = \{(d_0, d_0'), (d_1, d_0')\}$. It induces a bisimulation $f \subseteq \mathbb{I}(\mathbb{D})^+ \times \mathbb{I}(\mathbb{D}')^+$ between $\mathbf{M}^+$ and $\mathbf{M'}^+$; indeed we have that:

1. all valuations are equal, so two f-related intervals satisfy the same atomic propositions;

2. the strict sub-interval relation is empty in both models, and thus the back and the forth conditions are trivially satisfied.

Since $\mathbf{M}^+, [d_0, d_1]$ satisfies $\langle B \rangle p$, $\mathbf{M}^+, [d_0, d_1]$ satisfies $\langle E \rangle p$, while $\mathbf{M'}^+, [d_0, d_1]$ does not, it immediately follows then the $\langle B \rangle$ and the $\langle E \rangle$ modalities cannot be defined in D. ∎

BE interpreted over the class of all structures is expressive enough to capture some relevant conditions on the underlying interval structure (as originally pointed out by Halpern and Shoham in [42]). First, one can constrain an interval structure to be discrete by means of the formula

- **discrete**$\triangleq \pi \lor l1 \lor (\langle B \rangle l1 \land \langle E \rangle l1)$,

where $l1$ is true over an interval $[d_0, d_1]$ if and only if $d_0 < d_1$ and there are no points between d_0 and d_1. Such a condition can be expressed in BE by means of:

- **l1**$\triangleq \langle B \rangle \top \land [B][B]\bot$.

It is not difficult to show that an interval structure is discrete if and only if the formula *discrete* is valid in it. Furthermore, one can easily force an interval structure to be dense by constraining the formula:

- **dense** $\triangleq \neg l1$

to be valid. Finally, one can constrain an interval structure to be Dedekind complete by means of the formula

- **compl** $\triangleq (\langle B \rangle cell \land [[EP]]\neg q \land [E]([[BP]]q \rightarrow \langle B \rangle cell))$
$\rightarrow \langle B \rangle([E](\neg\pi \rightarrow \langle D \rangle cell))$,

where *cell* is true over an interval $[d_0, d_1]$ if and only if its begin and end points satisfy a given proposition letter q (the cell delimiters), all sub-intervals satisfy a proposition letter p (the cell content), and there exists at least one sub-interval satisfying p, that is:

- **cell** $\triangleq [[BP]]q \land [[EP]]q \land [D]p \land \langle D \rangle p$.

BE also allows one to define the *universal modality* $[All]$ (the application of $[All]$ to a formula ϕ constrains ϕ to hold over every interval of the model), which is captured by the following formula:

- $[All]\phi \triangleq \phi \wedge [B]\phi \wedge [E]\phi \wedge [B][E]\phi$.

As for (un)decidability results, Lodaya [56] proves the following theorem, which tailors the undecidability proof provided by Halpern and Shoham for the logic HS (cfr. Theorem 20).

Theorem 15 *The satisfiability problem for* BE*-formulas interpreted over non-strict dense linear structures (i.e.: the logic* BE^{de+}*) is not decidable.*

Undecidability id proved by reducing the non-halting problem of a Turing Machine on a blank tape to the satisfiability problem for BE^{de+}). According to Halpern and Shoham's approach, any computation of a TM is modeled by an infinite sequence of configuration, called instantaneous descriptions (ID for short). Each ID is a finite sequence of tape cells containing a unique tape symbol, and one of the cells has additional information representing the head position and the state of the machine. A suitable proposition is used to talk about consecutive IDs, e.g. to relate a cell of a given ID to the same cell of the consecutive ID. By exploiting such a proposition, the transition function δ of the Turing Machine can be respected by examining a group of three cells in an ID and determining the value of the same three cells in the successive ID. A suitable formula, parameterized by a Turing Machine can be built in such a way that it is satisfiable if and only if the Turing Machine does not halt on a blank tape. As a matter of fact, most of Halpern and Shoham's proof is carried out by the modalities $\langle B\rangle$ and $\langle E\rangle$, and the other modalities are used only to specify the sequences of IDs and to express the relationships between consecutive IDs. Lodaya shows how to treat the entire infinite computation as being inside a dense interval, which makes it possible to use the $\langle D\rangle$ modality to talk about sequences of IDs.

Corollary 16 *The satisfiability problem for the logic* BE *interpreted in the class of all non-strict linear structures is not decidable.*

The above corollary follows from the definability of the universal modality and of the class of dense structures in the logic BE.

It is worth noticing that, in our knowledge, the (un)decidability problem for specific classes of structures and for the strict case remains open for BE, as well as the problem of finding sound and complete axiomatic systems for any of the logics in the class $\mathcal{BE}$.

2.3.3 The Classes $\mathcal{B}\overline{\mathcal{B}}$ and $\mathcal{E}\overline{\mathcal{E}}$

In general, interval logics are capable to express properties of *pair* of points. In most cases, this prevents one from the possibility of reducing interval-based logics to point-based ones without resorting to any kind of projection principle. However, there are a few exceptions where such a reduction can be defined thanks to an opportune choice of interval modalities, thus allowing one to benefit from the good computational properties of point-based logics. This is the case of the $\mathrm{B}\overline{\mathrm{B}}$ and $\mathrm{E}\overline{\mathrm{E}}$ logics (and of their fragments), both in the strict and in the non-strict semantics.

The logic B$\overline{\text{B}}$ is generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B\rangle\phi \mid \langle\overline{B}\rangle\phi,$$

while the basic logic E$\overline{\text{E}}$ is generated by the following one:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle E\rangle\phi \mid \langle\overline{E}\rangle\phi.$$

In the previous sections we have seen the formal rules for the modalities $\langle B\rangle$ and $\langle E\rangle$. In the non-strict semantics the semantic clauses for the new modalities are:

($\langle\overline{B}\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle\overline{B}\rangle\phi$ if there exists d_2 such that $d_1 < d_2$ and $\mathbf{M}, [d_0, d_2] \Vdash \phi$;

($\langle\overline{E}\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle\overline{E}\rangle\phi$ if there exists d_2 such that $d_2 < d_0$ and $\mathbf{M}, [d_2, d_1] \Vdash \phi$.

For the rest of this section we restrict our attention to the logic B$\overline{\text{B}}$; all results also hold for the other logic too. Notice that we can restrict the interpretation of the logic B$\overline{\text{B}}$ to the class of bounded below structures, without loosing its expressive power.

The decidability of the satisfiability problem as well as other logical properties for B$\overline{\text{B}}$ will be obtained by embedding it in the Propositional Linear Time Logic with Future and Past (PLTL(F,P), for short) [29]. Hereafter, we restrict ourselves to the non-strict semantics, but all results can be adapted to the other case.

Given a set of propositional letters $\mathcal{AP}$, the formulas of PLTL(F,P), denoted by $f, g \ldots$, are given by the following grammar:

$$f ::= p \mid \neg f \mid f \wedge g \mid Pf \mid Ff,$$

where $p \in \mathcal{AP}$, and P and F are the modalities for the past and the future, respectively.

If $\mathbb{D} = \langle D, <\rangle$ is a bounded below and unbounded above linearly ordered set, and $\mathcal{V} : \mathbb{D} \mapsto \mathbf{P}(\mathcal{AP})$ is a valuation function, then a PLTL(F,P)-model is a pair $M = \langle\mathbb{D}, \mathcal{V}\rangle$. The semantics of PLTL(F,P)-formulas is as follows:

- $M, d_0 \Vdash p$ if $p \in \mathcal{V}(d_0)$;
- $M, d_0 \Vdash \neg f$ if it is not the case that $M, d_0 \Vdash f$;
- $M, d_0 \Vdash f \wedge g$ if $M, d_0 \Vdash f$ and $M, d_0 \Vdash g$;
- $M, d_0 \Vdash Pf$ if there exists d_1 such that $d_1 < d_0$ and $M, d_1 \Vdash f$;
- $M, d_0 \Vdash Ff$ if there exists d_1 such that $d_0 < d_1$ and $M, d_1 \Vdash f$.

Satisfiability and validity for PLTL(F,P)-formulas are defined in the standard way.

Consider the following translation τ from formulas of B$\overline{\text{B}}$ to formulas in the logic PLTL(F,P):

- $\tau(p) = p$;

- $\tau(\neg\phi) = \neg\tau(\phi)$;
- $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$;
- $\tau(\langle B\rangle\phi) = P\tau(\phi)$;
- $\tau(\langle \overline{B}\rangle\phi) = F\tau(\phi)$.

Lemma 17 *If $\phi \in$B$\overline{\text{B}}$ is satisfiable, then $\tau(\phi) \in$PLTL(F,P) is satisfiable.*

Proof.
From the satisfiability of ϕ, it follows that there exists a (bounded below) model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V\rangle$ and an interval $[d_0, d_1]$ such that $\mathbf{M}^+, [d_0, d_1] \Vdash \phi$. Without lose of generality, we suppose that d_0 is the minimum point of $\mathbb{D}$. Let $M = \langle \mathbb{D}, \mathcal{V}\rangle$ be a PLTL(F,P)-model whose valuation function $\mathcal{V}$ is defined as follows: $\forall x \in \mathbb{D}, p \in \mathcal{AP}$ $p \in \mathcal{V}(x)$ iff $p \in V([d_0, x])$. We show by structural induction that $M, d_1 \models \tau(\phi)$:

- Suppose $\phi = p$. By definition, $\tau(\phi) = p$, and by hypothesis $\mathbf{M}^+, [d_0, d_1] \Vdash p$. This means that $p \in V([d_0, d_1])$, and by construction $p \in \mathcal{V}(d_1)$, which means that $M, d_1 \Vdash p$;
- The cases of the propositional connectives are trivial;
- Suppose $\phi = \langle B\rangle\psi$. By definition, $\tau(\phi) = P\psi$, and by hypothesis $\mathbf{M}^+, [d_0, d_1] \Vdash \langle B\rangle\psi$, that is, there exists d_2 such that $d_0 \leq d_2 < d_1$ and $\mathbf{M}^+, [d_0, d_2] \Vdash \psi$. By the inductive hypothesis, $M, d_2 \Vdash \tau(\psi)$. So, by construction, $M, d_1 \models P\tau(\psi)$;
- The case of $\phi = \langle \overline{B}\rangle\psi$ is similar.

■

Lemma 18 *For all $\phi \in$ B$\overline{\text{B}}$, if $\tau(\phi) \in$PLTL(F,P) is satisfiable, then ϕ is satisfiable.*

Proof.
From the satisfiability of $\tau(\phi) \in$PLTL(F,P), it follows that there exists a model $M = \langle \mathbb{D}, \mathcal{V}\rangle$ and a point d_1 such that $M, d_1 \Vdash \tau(\phi)$. Let $M = \langle \mathbb{D}, \mathcal{V}\rangle$. We define $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V\rangle$ in such a way that $\forall x \in \mathbb{D}, p \in \mathcal{AP}$ $p \in V([d_0, x])$ iff $p \in \mathcal{V}(x)$, where d_0 is the minimum point of $\mathbb{D}$. Notice that the evaluation of all those intervals with beginning point distinct from d_0 remains unspecified, without affecting our result. We prove by induction that $\mathbf{M}^+, [d_0, d_1] \Vdash \phi$:

- Suppose $\phi = p$. By definition, $\tau(\phi) = p$, and by hypothesis $M, d_1 \Vdash p$. This means that $p \in \mathcal{V}(d_1)$, and by construction $p \in V([d_0, d_1])$, which means that $\mathbf{M}^+, [d_0, d_1] \Vdash p$;
- The cases of the propositional connectives are straightforward;

- Suppose $\phi = \langle B\rangle\psi$. By definition, $\tau(\phi) = P\tau(\psi)$, and by hypothesis $M, d_1 \Vdash P\psi$, that is, there exists d_2 such that $d_2 < d_1$ and $M, d_2 \Vdash \tau(\psi)$. By inductive hypothesis, $\mathbf{M}^+, [d_0, d_2] \Vdash \psi$, so $\mathbf{M}^+, [d_0, d_1] \Vdash \langle B\rangle\psi$;
- The case of $\phi = \langle\overline{B}\rangle\psi$ is similar.

■

Putting together the above lemmas, we obtain the following theorem.

Theorem 19 *The satisfiability problem for the logic* $\mathrm{B\overline{B}}$ *interpreted over a given class of bounded below and unbounded above structures can be reduced to the satisfiability problem for the logic* PLTL(F,P) *interpreted over the same class of structures.*

From the above theorem, the decidability of the former problem follows from the decidability of the latter. As an example, the satisfiability problem for the logic $\mathrm{B\overline{B}}$ interpreted over $\mathbb{N}$ is NP-complete (this is a consequence of the result shown in [58]), hence decidable.

2.3.4 The Class $\mathcal{PNL}$

A propositional interval logic with only neighborhood modalities, either in the strict and the non-strict semantics, is called a Propositional Neighborhood Logic. The class of $\mathcal{PNL}$, the properties of the logics in this class, and sound and complete axiomatic systems for them, are extensively presented in Chapter 3.

2.3.5 The Class $\mathcal{HS}$

The most expressive propositional interval logic with unary modal operators studied in the literature is Halpern and Shoham's logic HS introduced in [42]. The language for HS contains (as primitive or definable) all unary modalities corresponding to Allen's relations. HS features the modalities $\langle B\rangle, \langle E\rangle$ and their inverses $\langle\overline{B}\rangle, \langle\overline{E}\rangle$, which suffice (in the non-strict semantics) to define all other modal operators, so that it can be regarded as the temporal logic of Allen's relations. Unlike most previously studied interval logics, HS was originally interpreted in non-strict models not over linear orderings, but over all partial orderings with the linear intervals property.

Formally, HS-formulas are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B\rangle\phi \mid \langle E\rangle\phi \mid \langle\overline{B}\rangle\phi \mid \langle\overline{E}\rangle\phi.$$

As pointed out by Venema in [94], the neighborhood modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ are definable in the non-strict semantics as follows:

- $\langle A\rangle\phi \triangleq [[EP]]\langle\overline{B}\rangle\phi$, and
- $\langle\overline{A}\rangle\phi \triangleq [[BP]]\langle\overline{E}\rangle\phi$.

HS can express linearity of the interval structure by means of the following formula:

- **linear** $\triangleq (\langle A\rangle p \rightarrow [A](p \vee \langle B\rangle p \vee \langle \overline{B}\rangle p)) \wedge (\langle \overline{A}\rangle p \rightarrow [\langle \overline{A}\rangle](p \vee \langle E\rangle p \vee \langle \overline{E}\rangle p))$,

as well as all conditions that can be expressed in its fragment BE.

As expected, HS is a highly undecidable logic. In [42] the authors have obtained important results about non-axiomatizability, undecidability and complexity of the satisfiability in HS for many natural classes of models. Their idea for proving undecidability is based on using an infinitely ascending sequence in the model to simulate the halting problem for Turing Machines. An **infinitely ascending sequence** is an infinite sequence of points $d_0, d_1, d_2, \ldots$ such that $d_i < d_{i+1}$ for all i. Any unbounded above ordering contains an infinite ascending sequence. A class of ordered structures contains an infinite ascending sequence if at least one of the structures in the class does.

Theorem 20 *The validity problem in* HS *interpreted over any class of ordered structures with an infinitely ascending sequence is r.e.-hard.*

Thus, in particular, HS is undecidable for the class of all (non-strict) models, the class of all linear models (HS^{lin+}), the class of all discrete linear models (HS^{di+}), the class of all dense linear models (HS^{de+}), and the class of all dense and unbounded linear models (HS^{ude+}).

Theorem 21 *The validity problem in* HS *interpreted over any class of Dedekind complete ordered structures having an infinitely ascending sequence is* Π^1_1*-hard.*

For instance, the validity in HS in any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable. Undecidability occurs even without existence of infinitely ascending sequences. A class of ordered structures has **unboundedly ascending sequences** is for every n there is a structure in the class with an ascending sequence of length at least n.

Theorem 22 *The validity problem in* HS *interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.*

Another proof of undecidability of HS, using a tiling problem, is given in [31].

In [94] (see also [59]) Venema has shown that HS interpreted over a linear ordering is at least as expressive as the universal monadic second-order logic, where second-order quantification is only allowed over monadic predicates, and there are cases where it is strictly more expressive. As a corollary, it can be proved that HS is strictly more expressive than every point-based temporal logic on linear orderings.

In the same paper Venema provided an interesting *geometrical* interpretation of HS, using which he obtained sound and complete axiomatic systems for HS with respect to relevant classes of structures. Here is the idea. An interval can be viewed as an ordered pair of coordinates over a $\langle D, <\rangle \times \langle D, <\rangle$ plane, where $\langle D, <\rangle$ is supposed to be linear. Since the ending point of an interval must be greater than or equal to the starting point, only the north-west half-plane is considered. Clearly, this geometrical interpretation has a good meaning only when HS-formulas are interpreted over linear frames. Here is the standard notation:

- $\Diamond_{\downarrow}\phi \triangleq \langle B\rangle\phi$ (ϕ holds at a point right below the current one).
- $\Diamond_{\uparrow}\phi \triangleq \langle \overline{B}\rangle\phi$ (ϕ holds at a point right above the current one).
- $\Diamond_{\rightarrow}\phi \triangleq \langle E\rangle\phi$ (ϕ holds somewhere to the right of the current point).
- $\Diamond_{\leftarrow}\phi \triangleq \langle \overline{E}\rangle\phi$ (ϕ holds somewhere to the left of the current point).
- $\Diamond_{\updownarrow}\phi \triangleq \Diamond_{\downarrow}\phi \vee \phi \vee \Diamond_{\uparrow}\phi$ (ϕ holds at a point with the same latitude and a different longitude).
- $\Diamond_{\leftrightarrow}\phi \triangleq \Diamond_{\rightarrow}\phi \vee \phi \vee \Diamond_{\leftarrow}\phi$ (ϕ holds at a point with the same longitude and a different latitude).

Notice that, in order to obtain the mirror image (inverse) of a formula written in the geometrical notation, one should simultaneously replace all $\Diamond_{\downarrow}$ by $\Diamond_{\leftarrow}$ and all $\Diamond_{\uparrow}$ by $\Diamond_{\rightarrow}$, and vice versa. Using this geometrical interpretation, Venema has axiomatized HS over the class of all structures, the class of all linear structures, the class of all discrete structures, and $\mathbb{Q}$. The basic axiomatic system for HS includes the following axioms and their mirror-images:

(A-HS1) enough propositional tautologies;

(A-HS2a) $\Box_{\downarrow}(p \rightarrow q) \rightarrow (\Box_{\downarrow}p \rightarrow \Box_{\downarrow}q)$;

(A-HS2b) $\Box_{\uparrow}(p \rightarrow q) \rightarrow (\Box_{\uparrow}p \rightarrow \Box_{\uparrow}q)$;

(A-HS3a) $\Diamond_{\downarrow}\Diamond_{\downarrow}p \rightarrow \Diamond_{\downarrow}p$;

(A-HS3b) $\Diamond_{\uparrow}\Diamond_{\uparrow}p \rightarrow \Diamond_{\uparrow}p$;

(A-HS4a) $\Diamond_{\downarrow}\Box_{\uparrow}p \rightarrow p$;

(A-HS4b) $\Diamond_{\uparrow}\Box_{\downarrow}p \rightarrow p$;

(A-HS5) $\Diamond_{\downarrow}\top \rightarrow \Diamond_{\downarrow}\Box_{\downarrow}\bot$;

(A-HS6) $\Box_{\downarrow}\bot \rightarrow \Box_{\rightarrow}\bot$;

(A-HS7a) $\Diamond_{\uparrow}\Diamond_{\leftarrow}p \rightarrow \Diamond_{\leftarrow}\Diamond_{\uparrow}p$;

(A-HS7b) $\Diamond_{\downarrow}\Diamond_{\rightarrow}p \leftrightarrow \Diamond_{\rightarrow}\Diamond_{\downarrow}p$;

(A-HS7c) $\Diamond_{\rightarrow}\Diamond_{\uparrow}p \rightarrow \Diamond_{\uparrow}\Diamond_{\rightarrow}p$;

(A-HS8) $(\Diamond_{\downarrow}p \wedge \Diamond_{\downarrow}q) \rightarrow [\Diamond_{\downarrow}(p \wedge \Diamond_{\downarrow}q) \vee \Diamond_{\downarrow}(p \wedge q) \vee \Diamond_{\downarrow}(\Diamond_{\downarrow}p \wedge q)]$, and the following inference rules: Modus Ponens, Generalization for $\Box_{\uparrow}, \Box_{\downarrow}, \Box_{\leftarrow}$, and $\Box_{\rightarrow}$, and a pair of additional, un-orthodox rules which guarantee that all vertical and horizontal lines in the model are 'syntactically represented':

$$\frac{hor(p) \rightarrow \phi}{\phi} \quad \frac{ver(q) \rightarrow \psi}{\psi},$$

where p, q do not occur in ϕ, ψ respectively, and

- $hor(\phi) \triangleq \phi \wedge \Box_{\rightarrow}\phi \wedge \Box_{\leftarrow}\phi \wedge \Box_{\downarrow}(\neg\phi \wedge \Box_{\rightarrow}\neg\phi \wedge \Box_{\leftarrow}\neg\phi) \wedge \Box_{\uparrow}(\neg\phi \wedge \Box_{\rightarrow}\neg\phi \wedge \Box_{\leftarrow}\neg\phi)$;
- $ver(\phi) \triangleq \phi \wedge \Box_{\downarrow}\phi \wedge \Box_{\uparrow}\phi \wedge \Box_{\rightarrow}(\neg\phi \wedge \Box_{\downarrow}\neg\phi \wedge \Box_{\uparrow}\neg\phi) \wedge \Box_{\leftarrow}(\neg\phi \wedge \Box_{\downarrow}\neg\phi \wedge \Box_{\uparrow}\neg\phi)$.

The formula $hor(\phi)$ holds at an interval $[d_0, d_1]$ if and only if ϕ holds at any $[d_2, d_1]$ where $d_2 \leq d_1$ and nowhere else. Geometrically, it represents a horizontal line on which ϕ is true, and only there. Likewise $ver(\phi)$ says that ϕ is true exactly at the points of some vertical line.

Theorem 23 *The above axiomatic system is sound and complete for the class of all non-strict structures.*

Theorem 24 *The following results hold:*

1. *A sound and complete axiomatic system for the class of discrete structures can be obtained from the system for the class of all non-strict structures by adding the following axiom:*

   (A-HS^{di}) *discrete.*

2. *A sound and complete axiomatic system for the class of linear structures can be obtained from the system for the class of all non-strict structures by replacing axiom (A-HS8) by the following axiom:*

   (A-HS^{lin}) $(\Diamond_{\uparrow}\Diamond_{\downarrow}p) \rightarrow (\Diamond_{\uparrow}p \vee p \vee \Diamond_{\downarrow}p)$, $(\Diamond_{\downarrow}\Diamond_{\uparrow}p) \rightarrow (\Diamond_{\uparrow}p \vee p \vee \Diamond_{\downarrow}p)$.

3. *A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of linear structures by adding the following axiom:*

   (A-HS$^{\mathbb{Q}}$) $\Diamond_{\rightarrow}\top \wedge \Diamond_{\downarrow}\top \wedge$ *dense.*

In conclusion, we notice that many results about the complexity of HS interpreted over particular classes of non-strict linear orderings are still missing, as well as the strict versions of HS have practically not been studied. For what concerns the fragments of HS, besides BE, D, B$\overline{\text{B}}$, and E$\overline{\text{E}}$, and those fragments with only modalities for the *meets* and *met by* relations (propositional neighborhood logics, cfr. Chapter 3), there is a huge number of possible fragments that still deserve to be studied.

2.4 Interval Logics with Binary Operators

2.4.1 The Chop Operator and the Class $\mathcal{PITL}$

Arguably, the most natural binary interval modality is the *chop* operator C. As proved in [59], such an operator is not definable in HS. Logics in the class $\mathcal{PITL}$ feature the operator C and the modal constant π, interpreted according to the non-strict semantics, and it is the class of propositional fragment of first-order Interval Temporal Logic (ITL) introduced by Moszkowski in [65] (cfr. 2.5.1). Discrete non-strict semantics is the standard for logics in this class, so we will omit the superscript $^{di+}$ for short.

PITL-formulas are defined as follows:

$$\phi ::= p \mid \pi \mid \neg\phi \mid \phi \wedge \psi \mid \phi C \psi.$$

The semantic clause for the *chop* in the non-strict semantics is:

(C) $\mathbf{M}^+, [d_0, d_1] \Vdash \phi C \psi$ if and only if there exists d_2 such that $d_0 \leq d_2 \leq d_1$, and $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$ and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$.

The modalities $\langle B \rangle$ and $\langle E \rangle$ are definable in PITL as follows:

- $\langle B \rangle \phi \triangleq \phi C \neg\pi$, and
- $\langle E \rangle \phi \triangleq \neg\pi C \phi$.

As a matter of fact, the study of PITL was originally confined to the class of discrete linear orderings with finite time, with the *chop* operator paired with a **next** operator, denoted by $\bigcirc$, instead of π. For any ϕ, $\bigcirc\phi$ holds at a given (discrete) interval $\sigma = s_1 s_2 \ldots s_n$, with $n > 1$, if ϕ holds at the interval $\sigma' = s_2 \ldots s_n$. It is immediate to see that, over discrete linear orderings, the modal constant π and the *next* operator are inter-changeable. On the one hand, we have that

- $\pi \triangleq \bigcirc\bot$;

on the other hand, it holds that

- $\bigcirc\phi \triangleq l1C\phi$.

The logic PITL is quite expressive, as the following result from [65] testifies.

Theorem 25 *The satisfiability problem for* PITL *interpreted over the class of non-strict discrete structures is undecidable.*

The proof of the above theorem is actually an adaptation of the proof of a theorem by Chandra et al. [14] showing the undecidability of satisfiability for a propositional process logic. Given two context-free grammars G_1 and G_2 one can build a PITL-formula that is satisfiable if and only if the intersection of the languages generated by the two grammars is nonempty. Since the latter problem is not decidable (see [47]), the claim follows.

Since PITL is strictly more expressive than BE over the class of discrete linear structures, the above result does not transfer to it. On the contrary, the undecidability of the satisfiability problem for PITL over dense structures (PITL^{de+}), as well as over all linear structures immediately follows from the undecidability of BE over such structures.

Corollary 26 *The satisfiability problem for* PITL*-formulas interpreted over the class of (non-strict) dense linear structures is undecidable.*

Corollary 27 *The satisfiability problem for* PITL *interpreted over the class of (non-strict) linear structures is undecidable.*

It is worth remarking that the propositional counterpart of the fragment of ITL that only includes the *chop* operator, as far as we know, has not been investigated yet.

Decidable variants of PITL, interpreted over finite or infinite discrete structures, have been obtained by imposing the so-called *locality projection principle* [65]. Such a locality constraint states that each propositional variable is true over an interval if and only if it is true at its first state. This allows one to collapse all the intervals starting at the same state into the single interval consisting of the first state only.

Let Local PITL (LPITL for short) be the logic obtained by imposing the locality projection principle to PITL. The syntax of LPITL coincides with that of PITL, while its semantic clauses are obtained from PITL by modifying the truth definition of propositional variables as follows:

(PL) $\mathbf{M}^+, [d_0, d_1] \Vdash p$ if $p \in V(d_0)$

where the valuation function V has been adapted to evaluate propositional variables over points instead of intervals. Various extensions of LPITL have been proposed in the literature. In [65], Moszkowski focused his attention on the extension of LPITL (over finite time) with quantification over propositional variables, and he proved the decidability of the resulting logic, denoted by QLPITL, by reducing its satisfiability problem to that of QPTL, namely, the point-based Quantified Propositional Temporal Logic, interpreted over discrete linear structures with an initial point (as a matter of fact, QLPITL is translated into QPTL over finite time whose decidability can be proved by a simple adaptation of the standard proof for QPTL over infinite time).

Theorem 28 QPTL *is at least as expressive as* QLPITL *interpreted over the class of (non-strict) discrete linear structures.*

As a consequence, since QPTL is (non-elementarily) decidable, we have the following result.

Corollary 29 *The satisfiability problem for the logic* QLPITL*, interpreted over the class of (non-strict) discrete linear structures is (non-elementarily) decidable.*

From Corollary 29, it immediately follows the (non-elementary) decidability of the logic LPITL. A lower bound for the satisfiability problem for LPITL, and thus for any extension of it, has been given by Kozen (the proof of such a result can be found in [65]).

Theorem 30 *Satisfiability for* LPITL *is non-elementary.*

In a number of papers [65, 67, 68, 69, 70], Moszkowski explored the extension of LPITL with the so-called *chop-star* modality, denoted by *. For any ϕ, ϕ^* holds over a given (discrete) interval if and only if the interval can be chopped into zero or more parts such that ϕ holds over each of them. The resulting logic, that we denote by LPITL*, is interpreted over either finite or infinite discrete linear structures. A sound and complete axiomatic system for LPITL* with finite time is given in [70]; consider the following axioms:

(A-LPITL*1) enough propositional tautologies;

(A-LPITL*2) $(\phi C\psi)C\xi \leftrightarrow \phi C(\psi C\xi)$;

(A-LPITL*3) $(\phi \vee \psi)C\xi \rightarrow (\phi C\xi) \vee (\psi C\xi)$;

(A-LPITL*4) $\xi C(\phi \vee \psi) \rightarrow (\xi C\phi) \vee (\xi C\psi)$;

(A-LPITL*5) $\pi C\phi \leftrightarrow \phi$;

(A-LPITL*6) $\phi C\pi \leftrightarrow \phi$;

(A-LPITL*7) $p \rightarrow \neg(\neg pC\top)$, with $p \in \mathcal{AP}$;

(A-LPITL*8) $\neg(\neg(\phi \rightarrow \psi)C\top) \wedge \neg(\top C\neg(\xi \rightarrow \chi)) \rightarrow (\phi C\xi) \rightarrow (\psi C\chi)$;

(A-LPITL*9) $\bigcirc\phi \rightarrow \neg \bigcirc \neg\phi$;

(A-LPITL*10) $\phi \wedge \neg(\top C\neg(\phi \rightarrow \neg \bigcirc \neg\phi)) \rightarrow \neg(\top C\neg\phi)$;

(A-LPITL*11) $\phi^* \leftrightarrow \pi \vee (\phi \wedge \bigcirc\top)C\phi^*$,

together with Modus Ponens and the following inference rules:

$$\frac{\phi}{\neg(\top C\neg\phi)}, \quad \frac{\phi}{\neg(\neg\phi C\top)}.$$

Theorem 31 *The above axiomatic system is sound and complete for the class of (non-strict) discrete linear structures.*

All axioms have a fairly natural interpretation. In particular, locality is basically dealt with by Axiom A-LPITL*7.

As a matter of fact, the chop-star operator is a special case of a more general operator, called the *projection* operator. Such a binary operator, denoted by *proj*, yields general repetitive behaviour: for any given pair of formulas ϕ, ψ, $\phi\ proj\ \psi$ holds over an interval if such an interval can be partitioned into a series of sub-intervals each of which satisfies ϕ and ψ (called the *projected formula*) holds over the new interval formed from the end points of these sub-intervals. Let us denote by LPITL_{proj} the extension of LPITL with the projection operator *proj*. By taking advantage from such an operator, LPITL_{proj} can express meaningful iteration constructs, such as *for* and *while* loops. Indeed, the fact that the length of the current interval is exactly n can be expressed by $len(n)$, defined as follows:

- $len(n) \triangleq \bigcirc^n \top \wedge \bigcirc^{n+1} \bot$.

So, the *for* and the *while* loops can be expressed, as we show here:

- $for\ n\ times\ do\ p \triangleq p\ proj\ len(n)$;
- $while\ p\ do\ q \triangleq (p \wedge q)^* \wedge \neg((\top C(\neg len(n)\ \vee\ p))C\top)$.

Furthermore, the chop-star operator can be easily defined in terms of projection operator as follows:

- $\phi^* \triangleq \phi\ proj\ \top$.

LPITL_{proj} was originally proposed by Moszkowski in [65] and later systematically investigated by Bowman and Thompson [9, 10]. In particular, a tableau-based decision procedure and a sound and complete axiomatic system for LPITL_{proj}, interpreted over finite discrete structures, is given in [10].

The core of the tableau method is the definition of suitable normal forms for all operators of the logic. These normal forms provide inductive definitions of the operators. Then a tableau decision procedure to check satisfiability of LPITL_{proj} formulas is established, in the style of [98]. Although the method has been developed at the propositional level, the authors advocate its validity also for first-order LPITL_{proj}.

The normal form for LPITL_{proj} formulas has the following general format:

$$(\pi \wedge \phi_e)\ \vee\ \bigvee_i (\phi_i \wedge \bigcirc \phi'_i)$$

where ϕ_e and ϕ_i are point formulas, that is, formulas that are evaluated at single points, and ϕ'_i is an arbitrary LPITL_{proj} formula. The first disjunct states when a formula is satisfied over a point interval, while the second one states the possible ways in which a formula can be satisfied over a strict interval, namely, a point formula must hold at the initial point and then an arbitrary formula must hold over the remainder of the interval. It is worth noting that this normal form embodies a recipe for evaluating LPITL_{proj} formulas: the first disjunct is the base case, while the second disjunct is the inductive step. Bowman and Thomson showed that any LPITL_{proj} formula can be equivalently transformed into this normal form.

In [10], Bowman and Thomson also provided a sound and complete axiomatic system for LPITL_{proj}, interpreted over discrete linear structures. Let $\phi, \psi,\ \xi$ be arbitrary formulas and $p \in \mathcal{AP}$. The proposed system includes the following axioms:

(A-LPITL1) enough propositional tautologies;

(A-LPITL2) $\neg\pi \leftrightarrow \bigcirc\top$;

(A-LPITL3) $\bigcirc\phi \rightarrow \neg\bigcirc\neg\phi$;

(A-LPITL4) $\bigcirc(\phi \rightarrow \psi) \rightarrow \bigcirc\phi \rightarrow \bigcirc\psi$;

(A-LPITL5) $(\bigcirc\phi)C\psi \leftrightarrow \bigcirc(\phi C\psi)$;

(A-LPITL6) $(\phi \vee \psi)C\xi \leftrightarrow \phi C\xi \vee \phi C\xi$;

(A-LPITL7) $\phi C(\psi \vee \xi) \leftrightarrow \phi C\psi \vee \phi C\xi$;

(A-LPITL8) $\phi C(\psi C\xi) \leftrightarrow (\phi C\psi)C\xi$;

(A-LPITL9) $(p \wedge \phi)C\phi \leftrightarrow p \wedge (\phi C\psi)$, with $p \in \mathcal{AP}$;

(A-LPITL10) $\pi C\phi \leftrightarrow \phi C\pi \leftrightarrow \phi$;

(A-LPITL11) $\phi\ proj\ \pi \leftrightarrow \pi$;

(A-LPITL12) $\phi\ proj\ (\psi \vee \xi) \leftrightarrow (\phi\ proj\ \psi) \vee (\phi\ proj\ \xi)$;

(A-LPITL13) $\phi\ proj\ (p \wedge \psi) \leftrightarrow p \wedge (\phi\ proj\ \psi)$;

(A-LPITL14) $\phi\ proj\ \bigcirc\psi \leftrightarrow (\phi \wedge \neg\pi)C(\phi\ proj\ \psi)$.

The inference rules, besides Modus Ponens and $\bigcirc$-generalization, include the following rule:

$$\frac{\phi \rightarrow \bigcirc^k\phi}{\neg\phi}.$$

Theorem 32 *The above axiomatic system is sound and complete for the class of (non-strict) discrete structures.*

Finally, Kono [51] presents a tableau-based decision procedure for QLPITL with *projection*, which has been successfully implemented. The method generates a deterministic state diagram as a verification result. Although the associated axiomatic system is probably unsound (see [70]), Kono's work actually inspired Bowman and Thompson's one.

2.4.2 The Class $\mathcal{CDT}$

The most expressive propositional interval logic over (non-strict) linear orderings proposed in the literature is Venema's CDT [95]. In the original paper, CDT was restricted to linear orderings, so we introduce a different notation (BCDT^+) for the logic in the class $\mathcal{CDT}$ interpreted over the class of all (non-strict) partial orderings. In the present section we restrict ourselves to consider the logic CDT interpreted over all linear orderings, and over some classes of linear orderings with particular properties. The main result about BCDT^+ is the tableau method, which is presented in Chapter 4.

Logics in the class $\mathcal{CDT}$ contain the three binary operators C, D, and T, together with the modal constant π. Formulas are generated by the following abstract grammar:

$$\phi ::= \pi \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \phi C \psi \mid \phi D \psi \mid \phi T \psi.$$

The semantic clauses for D and T over non-strict structures are:

(D) $\mathbf{M}^+, [d_0, d_1] \Vdash \phi D \psi$ if there exists d_2 such that $d_2 \leq d_0$, $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$, and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$;

(T) $\mathbf{M}^+, [d_0, d_1] \Vdash \phi T \psi$ if there exists d_2 such that $d_1 \leq d_2$, $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$, and $\mathbf{M}^+, [d_0, d_2] \Vdash \psi$.

As for the expressive power, Venema compared CDT's ability of defining binary operators with that of the fragment $\text{FO}_3[<](x_i, x_j)$ of first-order logic over linear orderings with at most three variables, say x_1, x_2, and x_3, among which at most x_i, x_j, with $i, j \in \{1, 2, 3\}$, are free [95]. He proves the following result.

Theorem 33 *Every binary modal operator definable in $FO_3[<](x_i, x_j)$ has an equivalent in* CDT*, and vice versa.*

As for the relationships with the other propositional interval logics, interpreted over linear orderings, CDT is strictly more expressive than PITL, since the latter is not able to access any interval which is not a sub-interval of the current interval. Moreover, it is immediate to show that CDT subsumes HS:

- $\overset{\rightarrow}{\Diamond}\phi = (\neg\pi) C \phi$;
- $\overset{\leftarrow}{\Diamond}\phi = (\neg\pi) D \phi$;
- $\overset{\uparrow}{\Diamond}\phi = (\neg\pi) T \phi$;
- $\overset{\downarrow}{\Diamond}\phi = \phi C (\neg\pi)$.

A sound and complete axiomatic system for CDT over (non-strict) linear structures has been defined by Venema in [95]. Let us define $hor(\phi)$ as in the case of HS. The axiomatic system for CDT includes the following axioms, and their inverses (obtained by exchanging the arguments of all C occurrences, and replacing each occurrence of T by D and vice versa):

(A-CDT1) enough propositional tautologies;

(A-CDT2a) $(\phi \vee \psi) C \xi \leftrightarrow \phi C \xi \vee \psi C \xi$;

(A-CDT2b) $(\phi \vee \psi) T \xi \leftrightarrow \phi T \xi \vee \psi T \xi$;

(A-CDT2c) $\phi T(\psi \vee \xi) \leftrightarrow \phi T\psi \vee \phi T\xi$;

(A-CDT3a) $\neg(\phi T\psi)C\phi \rightarrow \neg\psi$;

(A-CDT3b) $\neg(\phi T\psi)D\psi \rightarrow \neg\phi$;

(A-CDT3c) $\phi T\neg(\psi C\phi) \rightarrow \neg\psi$;

(A-CDT4) $\neg\pi C\top \leftrightarrow \neg\pi$;

(A-CDT5a) $\pi C\phi \leftrightarrow \phi$;

(A-CDT5b) $\pi T\phi \leftrightarrow \phi$;

(A-CDT5c) $\phi T\pi \rightarrow \phi$;

(A-CDT6) $[(\pi \wedge \phi)C\top \wedge ((\pi \wedge \psi)C\top)C\top] \rightarrow (\pi \wedge \psi)C\top$;

(A-CDT6a) $(\phi C\psi)C\xi \leftrightarrow \phi C(\psi C\xi)$;

(A-CDT6b) $\phi T(\psi C\xi) \leftrightarrow (\psi C(\phi T\xi) \vee (\xi T\phi)T\psi)$;

(A-CDT6c) $\psi C(\phi T\xi) \rightarrow \phi T(\psi C\xi)$;

(A-CDT7d) $(\phi T\psi)C\xi \rightarrow ((\xi D\phi)T\psi \vee \psi C(\phi D\xi))$;

and the following derivation rules: Modus Ponens, Generalization:

$$\frac{\phi}{\neg(\neg\phi C\psi)}, \quad \frac{\phi}{\neg(\neg\varphi T\psi)}, \quad \frac{\phi}{\neg(\psi T\neg\phi)}, \text{ and their inverses,}$$

and the Consistency rule: if $p \in \mathcal{AP}$ and p does not occur in ϕ, then

$$\frac{\mathrm{hor}(p) \rightarrow \phi}{\phi}.$$

As for the above axiom A-CDT5c, a brief discussion is needed. In its original formulation it was

(A-CDT5c)′ $\phi T\pi \leftrightarrow \phi$.

In such a version, the axiom is not valid. Indeed, suppose that the current interval $[d_0, d_1]$ satisfies ϕ: there is no reason for a point d_2 such that $d_1 \leq d_2$ and $[d_0, d_2]$ satisfies π (which would imply that $d_0 = d_2$) to exist. So, here we report a corrected version of the axiom[1].

Theorem 34 *The above axiomatic system is sound and complete for the class of (non-strict) linear orderings.*

[1]We discovered that (non-critical) mistake during the tests of the implementation [85] of the tableau method for propositional interval temporal logics which can be found in the paper [35], and described in Chapter 4.

Theorem 35 *The following results hold:*

1. *A sound and complete axiomatic system for the class of (non-strict) dense linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:*

   (A-CDT^{de}) $\neg\pi \rightarrow (\neg\pi C \neg\pi)$.

2. *A sound and complete axiomatic system for the class of (non-strict) discrete linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:*

   (A-CDT^{di}) $\pi \vee (l1C\top) \wedge (\top Cl1)$;

3. *A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:*

   (A-CDT$^{\mathbb{Q}}$) $(\neg\pi \rightarrow (\neg\pi C \neg\pi)) \wedge (\neg\pi T\top) \wedge (\neg\pi D\top)$.

In [95], Venema also developed a sound and complete natural deduction system for CDT, similar to the natural deduction system for relation algebras earlier developed by Maddux [57].

Finally, as a consequence from previous results for HS and PITL, the satisfiability (resp. validity) for CDT is not decidable over almost all interesting classes of linear orderings, including all, dense, discrete, etc. Again, the strict versions of CDT and BCDT$^+$ have not been explicitly studied yet, but it is natural to expect that similar results apply there, too. The logic BCDT$^+$ has actually been developed with the specific aim to give a general tableau method for a number of propositional interval logics; we will be addressing this topic in Chapter 4.

2.5 First-Order Interval Logics and Duration Calculi

Research on interval temporal logics in computer science was originally motivated by problems in the field of specification and verification of hardware protocols, rather than by abstract philosophical or logical issues. Not surprisingly, it focused on first-order, rather than propositional, interval logics. In this section, we summarize some of the most-important developments in first-order interval logics and duration calculi, referring the interested reader to [70] and [44], respectively, for more details.

2.5.1 The Logic ITL

First-order ITL, interpreted over discrete linear orderings with finite time intervals (again, here we omit he superscript ^{di}), was originally developed by Halpern, Manna, and Moszkowski in [65, 41]. The language of ITL includes terms, predicates, Boolean connectives, first-order quantifiers, and the temporal modalities C and $\bigcirc$. Terms are built on variables, constants, and function symbols in the usual way. Constants and function symbols are classified as *global* (or *rigid*), when their interpretation does not vary with time, and *temporal* (or *flexible*), when their interpretation may change over time. Terms are usually denoted by $\theta_1, \ldots, \theta_n$. Predicate symbols are also partitioned into global and temporal ones. They are denoted by $p^i, q^j, \ldots$, where p^i is a predicate of arity i, q^j is a predicate of arity j, and so on. The abstract syntax of ITL formulas is:

$$\phi ::= \theta \mid p^n(\theta_1, \ldots, \theta_n) \mid \exists x \phi | \ \neg\phi \mid \phi \wedge \psi \mid \bigcirc \phi \mid \phi C \psi.$$

The semantics of ITL-formulas is a combination of the standard semantics of a first-order temporal logic with the semantics of PITL. An account of possible uses and applications is e.g. [66].

In [27] Dutertre studies the fragment of ITL which we will denote here by C^{fo}, involving only the *chop* operator. First, C^{fo} is considered over abstract, Kripke-style models $\mathbf{M}^+ = \langle W, R, I \rangle$, where W is a set of worlds (abstract intervals), R is a ternary relation corresponding to Venema's A, and I is a first-order interpretation. Further, Dutertre considers a more concrete semantics, over interval structures with associated 'length' measure represented by a special temporal variable l which takes values in a commutative group $\langle D, +, -, 0 \rangle$. The language is assumed to have the flexible constant l, and the rigid symbols 0 and $+$, respectively interpreted as the neutral element and the addition in $\langle D, +, 0 \rangle$. The semantics of C^{fo}-formulas is a combination of the semantics of ITL (without *next*), and the interpretation of l in a model $\mathbf{M}^+$ for an interval $[d_0, d_1]$ is $d_1 - d_0$.

As for the expressive power of C^{fo}, note that by means of l one can easily define the modal constant

- $\pi \triangleq (l = 0)$.

So, the HS modalities corresponding to *starts* and *fiinishes* are also definable in the language, thus, by the results of the Section 2.3.2, this means that C^{fo} is at least as expressive as PITL. The undecidability of this logic easily follows.

Dutertre has provided an axiomatic system for C^{fo}, the soundness and completeness proof for which can be found in [27]. In addition to the standard axioms of first-order classical logic, incl. the axioms of identity, and the axioms describing the properties for the temporal domain $\mathbb{D}$, Dutertre's systems involves the following specific axioms for C^{fo}:

(A-ITL1) $(\phi C \psi) \wedge \neg(\phi C \xi) \rightarrow (\phi C(\psi \wedge \neg\xi)$;

(A-ITL2) $(\phi C \psi) \wedge \neg(\xi C \psi) \rightarrow (\phi \wedge \neg\xi) C \psi$;

(A-ITL3) $((\phi C \psi); \xi) \leftrightarrow (\phi C(\psi C \xi))$;

(A-ITL4) $(\phi C \psi) \rightarrow \phi$ if ϕ is a rigid formula;

(A-ITL5) $(\phi C \psi) \rightarrow \psi$ if ψ is a rigid formula;

(A-ITL6) $((\exists x)\phi C \psi) \rightarrow (\exists x)(\phi C \psi)$ if x is not free in ψ;

(A-ITL7) $(\phi C(\exists x)\psi) \rightarrow (\exists x)(\phi C \psi)$ if x is not free in ϕ;

(A-ITL8) $((l = x)C\phi) \rightarrow \neg((l = x)C\phi)$;

(A-ITL9) $(\phi C(l = x)) \rightarrow \neg(\neg\phi C(l = x))$;

(A-ITL10) $(l = x + y) \leftrightarrow ((l = x)C(l = y))$;

(A-ITL11) $\phi \rightarrow (\phi C(l = 0))$;

(A-ITL12) $\phi \rightarrow ((l = 0)C\phi)$.

The inference rules are Modus Ponens, Generalization, Necessitation, and the following Monotonicity rule:

$$\frac{\phi \rightarrow \psi}{\phi C\xi \rightarrow \psi C\xi},$$

and the symmetric one. It should be noted that certain restrictions apply to the instantiation with flexible terms in quantified formulas.

As in the propositional case, variants of ITL obtained by imposing the locality constraint have been explored in the literature. In particular, sound and complete axiomatic systems for local variants of ITL (LITL for short) have been developed in [27, 28, 69].

Also, an interesting variant of ITL is the Signed Interval Logic (SIL) introduced by Rasmussen, whose semantics is based on *signed intervals*, i.e. intervals provided with a direction (forward or backward). For SIL it has been developed a sound and complete axiomatic system, a natural deduction system, and a sequent calculus. See [77, 78, 79].

2.5.2 The Logic NL

The logic ITL has an intrinsic limitation: its modalities do not allow one to 'look' outside the current interval (modalities with this characteristic are called *contracting* modalities). To overcome such a limitation, Zhou and Hansen [102] proposed the first-order logic of *left* and *right* neighbourhood modalities, called *neighbourhood logic* (NL for short), whose propositional fragment will be studied in Chapter 3.

First-order syntactic features are as in the ITL case. Right and left neighbourhood modalities are denoted by $\Diamond_r$ and $\Diamond_l$, respectively. The abstract syntax of NL formulas is:

$$\phi ::= \theta \mid p^n(\theta_1, \ldots, \theta_n) \mid \exists x\phi \mid \ \neg\phi \mid \phi \wedge \psi \mid \Diamond_l\phi \mid \Diamond_r\phi,$$

where terms $(\theta_1, \ldots, \theta_n)$ are defined as in ITL.

As in the propositional case, the neighborhood modalities are interpreted in non-strict structures by means of the following clauses:

($\Diamond_r$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_r \phi$ if there exists d_2 such that $d_1 \leq d_2$ and $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$;

($\Diamond_l$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_l \phi$ if there exists d_2 such that $d_2 \leq d_0$ and $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$.

The rest of the semantics of NL is defined exactly as in the ITL case. While practically meant to be the ordered additive group of the real numbers, the temporal domain is abstractly specified by means of a set of first-order axioms defining the so-called $\mathcal{A}$-*models* [100].

The first-order neighborhood logic NL is quite expressive. In particular, it allows one to express the *chop* modality as follows:

- $\phi C \psi \triangleq \exists x, y(l = x + y) \wedge \Diamond_l \Diamond_r((l = x) \wedge \phi \wedge \Diamond_r((l = y) \wedge \psi))$,

as well as any of the modalities corresponding to Allen's relations. Consequently, NL can virtually express all interesting properties of the underlying linear ordering, such as discreteness, density, etc.

Here we give an axiomatic system for NL, due to Barua, Roy, and Zhou [5], where the soundness and completeness proofs can be found. In what follows, the symbol $\Diamond$ stands either for $\Diamond_l$ and $\Diamond_r$, while $\overline{\Diamond}$ stands for $\Diamond_r$ (resp., $\Diamond_l$) when $\Diamond$ stands for $\Diamond_l$ (resp., $\Diamond_r$). The axiomatic system consists of the following axioms:

(A-NL1) $\Diamond\phi \rightarrow \phi$, where ϕ is a global formula;

(A-NL2) $l \geq 0$;

(A-NL3) $x \geq 0 \rightarrow \Diamond(l = x)$;

(A-NL4) $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi \vee \Diamond\psi$;

(A-NL5) $\Diamond\exists x\phi \rightarrow \exists x\Diamond\phi$;

(A-NL6) $\Diamond((l = x) \wedge \phi) \rightarrow \Box((l = x) \rightarrow \phi)$;

(A-NL7) $\Diamond\overline{\Diamond}\phi \rightarrow \Box\overline{\Diamond}\phi$;

(A-NL8) $(l = x) \rightarrow (\phi \leftrightarrow \overline{\Diamond}\Diamond((l = x) \wedge \phi)$;

(A-NL9) $((x \geq 0) \wedge (y \geq 0)) \rightarrow (\Diamond((l = x) \wedge \Diamond((l = y) \wedge \Diamond\phi)) \leftrightarrow \Diamond((l = x+y) \wedge \Diamond\phi))$,

plus the axioms for the domain $\mathbb{D}$ (axioms for $=, +, \leq$, and $-$), and the usual axioms for first-order logic. The same restrictions that have been made for the ITL concerning the instantiation of quantified formulas still apply here. The inference rules are, as usual, Modus Ponens, Necessitation, Generalization, and the following rule for Monotonicity:

$$\frac{\phi \rightarrow \psi}{\Diamond\phi \rightarrow \Diamond\psi}.$$

In [6], NL has been extended to a 'two-dimensional' version, called NL^2, where two modalities $\Diamond_u$ and $\Diamond_d$ have been added and interpreted as 'up' and 'down' neighbourhoods. NL^2 can be used to specify super-dense computations, taking vertical time as virtual time, and horizontal time as real time.

2.5.3 Duration Calculi

Duration Calculus (DC for short) is an interval temporal logic endowed with the additional notion of *state*. Each state is denoted by means of a state expression, and it is characterized by a *duration*. The duration of a state is (the length of) the time period during which the system remains in the state. DC has been successfully applied to the specification and verification of real-time systems. For instance, it has been used to express the behaviour of communicating processes sharing a processor and to specify their scheduler, as well as to specify the requirements of a gas burner [88].

DC has originally been developed as an extension of Moszkowski's ITL, and thus denoted by DC/ITL. Since the seminal work by Zhou, Hoare, and Ravn [102], various meaningful fragments of DC/ITL have been isolated and analyzed. Recently, an alternative Duration Calculus, based on the logic NL, and thus denoted by DC/NL, has been proposed by Roy in [84]. As a matter of fact, most results for DC/ITL and its fragments transfer to DC/NL and its fragments. In the following we introduce the basic notions and we summarize the main results about DC/ITL. Further details can be found in [44].

The calculus DC/ITL.

Zhou, Hoare, and Ravn's calculus DC/ITL is grounded on Moszkowski's ITL interpreted over the class of non-strict interval structures based on $\mathbb{R}$. Its only interval modality is *chop*. Its distinctive feature is the notion of state. States are represented by means of a new syntactic category, called **state expression**, which is defined as follows: the constants 0 and 1 are state expressions, a state variable X is a state expression, and, for any pair of state expression S and T, $\neg S$ and $S \vee T$ are state expressions (the other Boolean connectives are defined in the usual way). Furthermore, given a state expression S, the duration of S is denoted by $\int S$. DC/ITL terms are defined as in ITL, provided that temporal variables are replaced by state expressions. DC/ITL formulas are generated by the following abstract syntax:

$$\phi ::= p^n(r_1, \ldots, r_n) \mid \top \mid \neg\phi \mid \phi \vee \psi \mid \phi C \psi \mid \forall x \phi$$

where $r_1, \ldots, r_n$ are terms, p^n is a n-ary (global) predicate, C is the *chop* modality, and x is a global variable.

Any state (expression) S is associated with a total function $S : \mathbb{R} \mapsto \{0, 1\}$, which has a finite number of discontinuity points only. For any time point t, the state expression interpretation $\mathcal{I}$ is defined as follows:

- $\mathcal{I}[0](t) = 0$;
- $\mathcal{I}[1](t) = 1$;
- $\mathcal{I}[S](t) = S(t)$;
- $\mathcal{I}[\neg S](t) = 1 - \mathcal{I}[S](t)$;

- $\mathcal{I}[S \vee T](t) = 1$ if $\mathcal{I}[S](t) = 1$ or $\mathcal{I}[T](t) = 1$; 0 otherwise.

The semantics of a duration $\int S$ in a given (non-strict) model, with respect to an interval $[d_0, d_1]$, is the Riemann definite integral, that is, $\int_{d_0}^{d_1} \mathcal{I}[S](t)dt$. The semantics of the other syntactic constructs is given as in ITL case.

A number of useful abbreviations can be defined in DC/ITL. In particular, $\lceil S \rceil$ stands for: "S holds almost everywhere over a strict interval", and it is defined as follows:

- $\lceil S \rceil \triangleq (\int S = \int 1) \wedge \neg(\int 1 = \int 0)$;

$\int 1$ is usually abbreviated by l, and it can be viewed as the length of the current interval; finally, $\lceil\ \rceil$, which holds over point-intervals, can be defined as $l = 0$.

The satisfiability problem for both first-order DC/ITL (full DC/ITL) and its fragment devoid of first-order quantification (Propositional DC/ITL) has been shown to be undecidable. First-order DC/ITL, provided with, at least, the functional symbol $+$ and the predicate symbol $=$, with the usual interpretation, has been completely axiomatized in [46]. The axiomatic system includes the following specific axioms:

(A-DC1) $\int 0 = 0$;

(A-DC2) $\int S \geq 0$;

(A-DC3) $\int S + \int T = \int (S \vee T) + \int (S \wedge T)$;

(A-DC4) $((\int S = x) C (\int S = y)) \leftrightarrow (\int S = x + y)$;

(A-DC5) $\int S = \int T$ provided that $S \leftrightarrow T$ holds in propositional logic

and the following inference rule (provided that $S_1 \ldots S_n$ are state expressions, and that $\bigvee_{i=1}^{n} \leftrightarrow 1$):

$$\frac{H(\lceil\ \rceil),\ H(\phi)\ \rightarrow\ H(\phi \vee \bigvee_{i=1}^{n}(\phi C \lceil S_i \rceil)}{H(\top)},$$

in conjunction with its inverse (obtained by exchanging the ordering of the formulas in every *chop*), where $H(\phi)$ represents the formula obtained from $H(X)$ by replacing every occurrence of X in H with ϕ.

Various interesting fragments of DC have been investigated by Zhou, Hansen, and Sestof in [101]. First, they consider the possibility of interpreting DC formulas over different classes of structures. In particular, the fragment of DC *interpreted over* $\mathbb{N}$ is the set of DC formulas interpreted over $\mathbb{R}$ evaluated with respect of $\mathbb{N}$-intervals, that is, intervals whose endpoints are in $\mathbb{N}$. The fragment of DC *interpreted over* $\mathbb{Q}$ is similarly defined. Then, the authors took into consideration some syntactic subfragments of the above calculi and studied the decidability/undecidability of their satisfiability problem. It turned out that the fragments of propositional DC whose formulas are built up from primitive formulas of the type $\lceil S \rceil$ only have a decidable satisfiability problem when interpreted over $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$. By adding to the set of primitive formulas those of the form $l = k$, the problem remains decidable over $\mathbb{N}$, but it becomes undecidable over the other classes of structures. The same fragment at the first-order level is undecidable in all the considered cases. Finally, the fragment of propositional DC whose formulas are built up from primitive formulas of the type $\int S = \int T$ only is also undecidable.

As for the complexity of the satisfiability problem, in [75] Rabinovich reported a result by Sestoft (personal communication) stating that the satisfiability problem for the fragment of DC whose formulas are built up from primitive formulas of the type $\lceil S \rceil$ only, interpreted over $\mathbb{N}$, has a non-elementary complexity. Rabinovich showed that the satisfiability problem for the same fragment, interpreted over $\mathbb{R}$, is non-elementarily decidable too, by providing a linear time reduction from the equivalence problem for star-free expressions to the validity problem for the considered fragment of DC.

In [15], Checuti-Sperandio and Fariñas del Cerro isolated another fragment of propositional DC by imposing suitable syntactic restrictions. Formulas of such a fragment are generated by the following abstract syntax:

$$\phi ::= \top \mid \bot \mid lPk \mid I = 0 \mid I = l \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi C \psi,$$

where k is a constant, $P \in \{<, \leq, =, \geq, >\}$, and I is $\int S$, for a given state S. The resulting logic is shown to be expressive enough to capture Allen's Interval Algebra. The authors developed a sound, complete, and terminating tableau system for the logic, thus showing that its satisfiability problem is decidable. The tableau system is a mixed procedure, combining standard tableau techniques with temporal constraint network resolution algorithms.

The calculus DC/NL.

Finally, the classical DC and the first-order neighbourhood logic (NL) have been combined into the (clearly, undecidable) DC/NL which has been completely axiomatized by merging the axiomatic systems for DC and NL. The fragment of DC/NL obtained by restricting the formulas to be built up only from primitive formulas of the type $\lceil S \rceil$ has been proved to be decidable, while the extension of the latter with primitive formulas of the type $l = k$ is undecidable, as already mentioned.

Other variations of DC include the Propositional and First-Order Mean Value calculus, which have been studied by several authors including Pandya [73], and Zhou and Xiaoshan [103].

3

The Class of Propositional Neighborhood Logics

"The scientist is not the one who gives true answers:
is just the one who asks true questions."
Claude L. Strauss

Given that even at the propositional level, interval logics turned out to be so difficult to deal with from the computational point of view, one has to find some way to easily reason with them and to obtain better 'behaved' fragments. In Chapter 4 we deal with tableau methods for propositional interval logics. In Chapter 5 we choose to consider structures that do not allow all the intervals. In this chapter, we consider propositional neighborhood interval logics, that is, we consider a smaller set of unary modalities. Hereafter in this chapter we will be restricting ourselves to linear structures, and thus we omit the superscript ^{lin}.

Logics in $\mathcal{PNL}$ can express meaningful timing properties, without being excessively expressive to an extent easily leading to high undecidability. They feature two modalities, which correspond to $\langle A \rangle$ and its transpose $\langle \overline{A} \rangle$ in HS, and respectively $\Diamond_r$ and $\Diamond_l$ in NL, intuitively capturing a *right* neighboring interval and a *left* neighboring interval. As a matter of fact, these neighborhood modalities correspond to Allen's *meets* and *met by* relations. The class of the strict propositional neighborhood logics is denoted by $\mathcal{PNL}^-$, while the class of the non-strict propositional neighborhood logics is denoted by $\mathcal{PNL}^+$. While the logics in $\mathcal{PNL}^+$are built on the propositional fragment of NL, those in $\mathcal{PNL}^-$can rather be considered as based on the A$\overline{\text{A}}$-fragment of HS, because, although the semantics of HS admits non-strict intervals, and is non-strict in that sense, the modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ only refer to strict intervals, so, concerning the fragment A$\overline{\text{A}}$, it is essentially strict.

Unlike classical logic and most modal and temporal logics where the first-order axiomatic systems are obtained by extending their propositional fragments with relevant axioms for the quantifiers, the first-order NL was axiomatized first, without its propositional fragment having been identified. It now turns out that the latter was hidden into the originally introduced first-order axiomatic system, the propositional axioms of which, taken alone, are substantially incomplete. In particular, a curious feature of NL is that while it is finitely axiomatized, its propositional fragment involves an infinite axiom scheme. The strict analogue, however, is a finitely axiomatized subsystem of the latter.

As an aside, notice that the decidability/undecidability of satisfiability for logics in $\mathcal{PNL}$ is still an open problem.

3.1 Syntax and Semantics

The language for the (sub-)class $\mathcal{PNL}^+$ of **non-strict propositional neighborhood logics** contains a set of propositional variables $\mathcal{AP}$, the propositional logical connectives $\neg$ and $\rightarrow$, and the modalities $\Box_r$ and $\Box_l$, the dual operators of which will be denoted by $\Diamond_r$ and $\Diamond_l$, respectively. The remaining classical propositional connectives can be considered as abbreviations. The formulas of $\mathcal{PNL}^+$ are recursively defined as follows:

$$\phi = p \mid \neg\phi \mid \phi\wedge\psi \mid \Box_r\phi \mid \Box_l\phi.$$

The language for the (sub-)class $\mathcal{PNL}^-$ of **strict propositional neighborhood logics** differs from the non-strict one only in the notation for the modalities, now denoted by $[A]$ and $[\overline{A}]$, with dual operators $\langle A\rangle$ and $\langle\overline{A}\rangle$, respectively. The formulas of $\mathcal{PNL}^-$ are defined as follows:

$$\phi = p \mid \neg\phi \mid \phi\wedge\psi \mid [A]\phi \mid [\overline{A}]\phi.$$

We use different notations for the modalities in the two languages only to reflect their historical links and to make it easier to distinguish between the two semantics from the syntax. Clearly, there is a straightforward translation between the two languages.

The semantics of propositional neighborhood logics is given in terms of linear structures. We recall the formal semantics of the modal operators:

($\langle A\rangle$) $\mathbf{M}^-, [d_0, d_1] \Vdash \langle A\rangle\phi$ iff there is a d_2 such that $d_1 < d_2$ and $\mathbf{M}^-, [d_1, d_2] \Vdash \phi$;

($\langle\overline{A}\rangle$) $\mathbf{M}^-, [d_0, d_1] \Vdash \langle\overline{A}\rangle\phi$ iff there is a d_2 such that $d_2 < d_0$ and $\mathbf{M}^-, [d_2, d_0] \Vdash \phi$;

($\Diamond_r$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_r\phi$ iff there is a d_2 such that $d_1 \leq d_2$ and $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$;

($\Diamond_l$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_r\phi$ iff there is a d_2 such that $d_2 \leq d_0$ and $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$.

3.2 Some Propositional Neighborhood Logics

Consider the following formulas:

(A-SNF^{ur}) $[A]p\rightarrow\langle A\rangle p$ (or, equivalently, $\langle A\rangle\top$);

(A-SNF^{der}) $(\langle A\rangle\langle A\rangle p\rightarrow\langle A\rangle\langle A\rangle\langle A\rangle p) \wedge (\langle A\rangle[A]p\rightarrow\langle A\rangle\langle A\rangle[A]p)$;

(A-SNF^{aux}) $\langle\overline{A}\rangle\top\rightarrow\langle\overline{A}\rangle\langle\overline{A}\rangle\top$;

(A-SNF^{dir}) $([A]\bot \rightarrow [\overline{A}]([A][A]\bot \vee \langle A\rangle(\langle A\rangle\top \wedge [A][A]\bot)) \wedge (\langle A\rangle\top \wedge [A](p \wedge [\overline{A}]\neg p \wedge [A]p) \rightarrow [\overline{A}][\overline{A}]\langle A\rangle(\langle A\rangle\neg p \wedge [A][A]p))$;

(A-SNF$^{\mathbf{c}}$) $\langle A\rangle\langle A\rangle[\overline{A}]p \wedge \langle A\rangle[A]\ \neg[\overline{A}]p \rightarrow \langle A\rangle(\langle A\rangle[\overline{A}]\ [\overline{A}]p \wedge [A]\langle A\rangle\neg[\overline{A}]p)$.

Moreover, we define:

- **A-SNF^{di}** $\triangleq$ A-SNF^{dir} $\wedge$ A-SNF^{dil}, where A-SNF^{dil} is the inverse of A-SNF^{dir};
- **A-SNF^{u}** $\triangleq$ A-SNF^{ur} $\wedge$ A-SNF^{ul}, where A-SNF^{ul} is the inverse of A-SNF^{ur};
- **A-SNF^{de}** $\triangleq$ A-SNF^{der} $\wedge$ A-SNF^{del}, where A-SNF^{del} is the inverse of A-SNF^{der}.

Proposition 36 *In the strict semantics:*

1. *the class of all discrete structures is defined by the formula A-SNF^{di};*
2. *the class of all unbounded structures is defined by the formula A-SNF^{u};*
3. *the class of all dense structures, extended with the 2-element linear ordering*[1]*, is defined by the formula A-SNF^{de} or, alternatively, by the formula A-SNF$^{der}\wedge$ A-SNF^{aux};*
4. *the class of all Dedekind complete structures is defined by the formula A-SNF^{c};*
5. *the class of all unbounded and dense structures is defined by the formulas A-SNF^{ur}, A-SNF^{der}, and their inverses A-SNF^{ul}, A-SNF^{del};*
6. *the class of all unbounded and discrete structures is defined by the formulas A-SNF^{ur}, A-SNF^{dir}, and their inverses A-SNF^{ul}, A-SNF^{dil};*
7. *the class of all unbounded and Dedekind complete structures is defined by the formulas A-SNF^{ur} and its inverse A-SNF^{ul}, and A-SNF^{c}.*

Proof.

We prove in detail the first point, and we sketch the rest.

1. We have to show the following: (i) if the formula A-SNF^{di} is valid, then the underlying linear ordering is discrete (i.e., every non-first point has an immediate predecessor, and every non-last point has an immediate successor), and (ii) if the formula A-SNF^{di} is not valid on a class of structures, then at least one of such structures is not discrete. We consider only the formula A-SNF^{dir}, and, in this proof, we refer to discreteness meaning only that every non-first point has an immediate predecessor; the proof for the inverse formula is similar.

   For the point (i), we consider a linear ordering $\mathbb{D} = \langle D, <\rangle$ such that each point having a predecessor has an immediate one, and we show that the formula A-SNF^{dir} is valid. Let $[d_0, d_1]$ be an interval; then we have two cases, namely, the one where d_1 has no successors, and the one where d_1 has at least one successor. In the first case we have that:

[1] The 2-element linear ordering cannot be separated in the language of PNL^{-}.

(a) $[A]\bot$ is true on $[d_0, d_1]$, so $[\overline{A}]([A][A]\bot \vee \langle A\rangle(\langle A\rangle\top \wedge ([A][A]\bot))$ must be true on $[d_0, d_1]$. This means that we have to show that for all d_2 such that $d_2 < d_0$, $[d_2, d_0] \Vdash [A][A]\bot \vee \langle A\rangle(\langle A\rangle\top \wedge ([A][A]\bot)$. If in between d_0 and d_1 there exist only one point, then the first disjunct is verified. Otherwise, by hypothesis, there exists an immediate predecessor of d_1, say d_1^*. Consider the interval $[d_0, d_1^*]$; it holds that $[d_0, d_1^*]$ satisfies $\langle A\rangle\top$ and also $[A][A]\bot$, and so we are done (notice that, if there is no such point d_2, the formula is trivially satisfied);

(b) $\langle A\rangle\top$ is not true on $[d_0, d_1]$, so $\langle A\rangle\top \wedge [A](p \wedge [\overline{A}]\neg p \wedge [A]p) \rightarrow [\overline{A}][\overline{A}]\langle A\rangle (\langle A\rangle\neg p \wedge [\overline{A}][\overline{A}]p)$ is (trivially) verified.

In the second case we have:

(a) $[A]\bot$ is false on $[d_0, d_1]$, so $([A]\bot \rightarrow [\overline{A}]([A][A]\bot \vee \langle A\rangle(\langle A\rangle\top \wedge ([A][A]\bot)))$ is (trivially) verified;

(b) $\langle A\rangle\top$ is true on $[d_0, d_1]$, so we assume the truth of the antecedent of the second part of A-SNF^{dir}, meaning that any intervals ending in d_1 satisfies $\neg p$, and any intervals beginning in d_1 or after satisfies p, and we show that $[\overline{A}][\overline{A}]\langle A\rangle(\langle A\rangle\neg p \wedge [A][A]p))$ is true on $[d_0, d_1]$. Indeed, for all $d_2 < d_3 < d_0$, $[d_2, d_3] \Vdash \langle A\rangle(\langle A\rangle\neg p \wedge [A][A]p)$, since, by hypothesis, there exists an immediate predecessor of d_1, let say d_1^*. and the interval $[d_3, d_1^*]$ satisfies both $\langle A\rangle\neg p$ and $[A][A]p$. If there are no such d_2, d_3, the formula (trivially) holds.

For the point (ii), suppose that A-SNF^{dir} is not valid on a class of structures. So, we show that there exists a linear ordering $\mathbb{D} = \langle D, <\rangle$, a model $\mathbf{M}^-$ based on it, and an interval $[d_0, d_1]$ such that $\mathbf{M}^-, [d_0, d_1] \Vdash \neg$A-SNF^{dir}. Again, two cases are possible, depending on $\mathbf{M}^-, [d_0, d_1]$ satisfying the first or the second disjunct of the negation of A-SNF^{dir}. First, suppose that $\mathbf{M}^-, [d_0, d_1] \Vdash [A]\bot \wedge \langle\overline{A}\rangle(\langle A\rangle\langle A\rangle\top \wedge [A]([A]\bot \vee \langle A\rangle\langle A\rangle\top))$. This means that d_1 has no successors, that there exists at least one point between d_0 and d_1 and that, for every point d_2 between d_0 and d_1, there exists a point between d_2 and d_1; so, d_1 is an 'accumulation' point, and $\mathbb{D} = \langle D, <\rangle$ is not discrete. Conversely, if d_1 has at least one successor, then suppose that $[d_0, d_1]$ satisfies the second disjunct of $\neg$A-SNF^{dir}. So, being p true for every interval beginning in d_1 or after, the formula forces the existence of a point d_4 between d_0 and d_1 for $\langle\overline{A}\rangle\langle\overline{A}\rangle[A]([A]p \vee \langle A\rangle\langle A\rangle\neg p)$ to be true on $[d_2, d_3]$ (where $d_2 < d_3 < d_0$), and, consequently, the existence of a point between d_4 and d_1, and so on. This implies that, again, d_1 is an 'accumulation' point. Hence, the thesis;

2. Straightforward;

3. The formula A-SNF^{der} says that every interval with a left neighbor can be split into two sub-intervals. In addition, A-SNF^{aux} guarantees that if there are at least 2 intervals (i.e., at least 3 points), then the left-most interval, if there is one, can be split into two sub-intervals, too;

4. The formula A-SNF^c says that every non-empty and bounded above set of points has a least upper bound.

■

Proposition 37 *The above-defined logics satisfy the following relations:*

1. *For every* $\lambda_1, \lambda_2 \in \{u, de, di, c, ude, udi, uc\}$*,* $\mathrm{PNL}^{\lambda_1 -} \subsetneq \mathrm{PNL}^{\lambda_2 -}$ *if and only if the class of linear orders characterized by the condition* λ_2 *is strictly contained in the class of linear orders characterized by the condition* λ_1*;*
2. $\mathrm{PNL}^{ude-} \subsetneq \mathrm{PNL}^{+}$*, where the inclusion is in terms of the obvious translation between the two languages;*
3. $\mathrm{PNL}^{+} = \mathrm{PNL}^{u+} = \mathrm{PNL}^{de+} = \mathrm{PNL}^{ude+} = \mathrm{PNL}^{di+} = \mathrm{PNL}^{udi+}$.

Proof.
Sketch:

1. First, $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{u-}$ because the formula A-SNF^u $\in \mathrm{PNL}^{u-} - \mathrm{PNL}^{-}$. Likewise, $\mathrm{PNL}^{de-} \subsetneq \mathrm{PNL}^{ude-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{de-}$ because the formula A-SNF^{de} $\in \mathrm{PNL}^{de-} - \mathrm{PNL}^{-}$, since it is valid in every strict and dense neighborhood structure, but e.g. not in the one based on $\mathbb{Z}$. Likewise, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{ude-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{di-}$ because the formula A-SNF^{di} $\in \mathrm{PNL}^{di-} - \mathrm{PNL}^{-}$, since it is valid in every strict and discrete neighborhood structure, but not in the one based on $\mathbb{Q}$. Likewise, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{udi-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{c-}$ because the formula A-SNF^c $\in \mathrm{PNL}^{c-} - \mathrm{PNL}^{-}$, since it is valid in every Dedekind-complete strict neighborhood structure, but not in the one based on $\mathbb{Q}$. Finally, we have that $\mathrm{PNL}^{di-} \subsetneq \mathrm{PNL}^{udi-}$, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{uc-}$, and $\mathrm{PNL}^{c-} \subsetneq \mathrm{PNL}^{uc-}$.
2. Every PNL^{+}-formula satisfiable in a model $\mathbf{M}^{+}$ over non-strict neighborhood structure is satisfiable in a dense and unbounded strict one. Indeed, replacing every point in $\mathbf{M}^{+}$ by a copy of $\mathbb{Q}$ produces a dense and unbounded strict model $\mathbf{M}^{-*}$ such that $\mathbf{M}^{+}$ is a p-morphic copy of $\mathbf{M}^{-*}$.
3. Essentially the same construction works for the equalities $\mathrm{PNL}^{+} = \mathrm{PNL}^{u+} = \mathrm{PNL}^{de+} = \mathrm{PNL}^{ude+}$, but now we take the non-strict version of $\mathbf{M}^{*}$. For the equality $\mathrm{PNL}^{+} = \mathrm{PNL}^{udi+}$, we can similarly replace every point in $\mathbf{M}$ by a copy of $\mathbb{Z}$, and thus produce an unbounded and discrete non-strict model which maps p-morphically onto $\mathbf{M}$. ■

It is worth noting that the logic PNL^{udi-} does not yet characterize the interval structure of the integers, because the formula

$$\langle A\rangle p \wedge [A](p \rightarrow \langle A\rangle p) \wedge [A][A](p \rightarrow \langle A\rangle p) \rightarrow [A]\langle A\rangle\langle A\rangle p$$

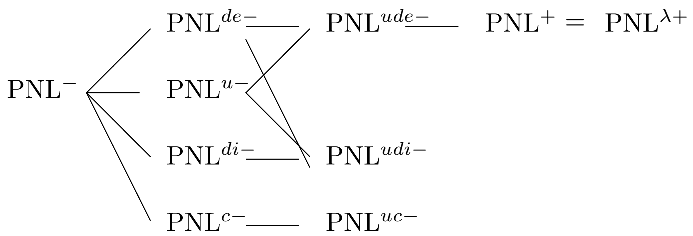


Figure 3.1: (Relative) expressive power of logics in $\mathcal{PNL}$.

is valid in the integers, but not in PNL^{udi-} since it fails in a PNL^{udi-}-model based on $\mathbb{Z}+\mathbb{Z}$.

The above proposition shows that there is a collapse of the expressiveness in the non-strict semantics, while the strict one is at least as expressive as the point-based temporal logic over linear orders. The situation is graphically depicted Figure 3.1, where $\lambda \in \{u, de, di, ude, udi\}$.

3.3 Expressing Timing Properties in $\mathcal{PNL}$

Here we give some simple examples of properties that can be expressed in $\mathcal{PNL}$. First of all, notice that PNL^{-}, besides distinguishing among different properties of the underlying linear order, is powerful enough to express the **difference** operator:

- $[\neq](q) \triangleq [\overline{A}][\overline{A}][A]q \wedge [\overline{A}][A][A]q \wedge [A][A][\overline{A}]q \wedge [A][\overline{A}][\overline{A}]q$,

and consequently to simulate **nominals**:

- $n(q) \triangleq q \wedge [\neq](\neg q)$,

that is, to express the fact that q holds in the current interval and nowhere else. Therefore, every universal property of strict interval structures can be expressed in PNL^{-}.

The following more practical examples are borrowed from typical application domains in Artificial Intelligence. As a first example, consider the case of a robot that, in order to accomplish a given goal, must pick a finite set of objects $a_1, a_2, \ldots, a_n$ in whatever order. Moreover, assume that the robot cannot pick up more than one object at a time. Such a scenario can be modeled as follows. Let the propositional variable p_{a_i}, with $1 \leq i \leq n$, denote the action "the robot is picking up the objects a_i" and the propositional variable $h_{a_{i_1},\ldots,a_{i_k}}$, with $a_{i_j} \in \{a_1, \ldots, a_n\}$ and $1 \leq k \leq n$, denote the state "the robot holds the objects $a_{i_1}, \ldots, a_{i_k}$". The constraint that picking up and holding each object is a necessary pre-condition of any situation in which the robot simultaneously holds all objects can be expressed in PNL^{+} as follows:

$$h_{a_1,\ldots,a_n} \to \Diamond_l\Diamond_l(p_{a_1} \wedge \Diamond_r h_{a_1}) \wedge \ldots \wedge \Diamond_l\Diamond_l(p_{a_n} \wedge \Diamond_r h_{a_n})$$

Notice that such a formulation does not constrain "picking up" actions to be instantaneous. However, such a condition can be easily expressed in $\mathrm{PNL}^{\pi+}$:

$$h_{a_1,\ldots,a_n} \to \Diamond_l\Diamond_l(p_{a_1} \wedge \pi \wedge \Diamond_r h_{a_1}) \wedge \ldots \wedge \Diamond_l\Diamond_l(p_{a_n} \wedge \pi \wedge \Diamond_r h_{a_n}).$$

As another example, we note that both PNL^{+} and PNL^{-} allows one to define an interval version of the *until* operator by means of the formulas

$$\phi\Diamond_u\psi \equiv \Diamond_r(\phi\wedge\Diamond_r\psi) \;\text{ and }\; \phi\langle U\rangle\psi \equiv \langle A\rangle(\phi\wedge\langle A\rangle\psi),$$

respectively. Such an operator can be used to express conditions of the form "The flight from Milano to Johannesburg initiates a period of time during which the traveler is in Johannesburg" as follows:

$$\textsf{Milano-to-Johannesburg}\langle U\rangle\textsf{Stay-in-Johannesburg}.$$

Moreover, in $\mathrm{PNL}^{\pi+}$ one can express the constraint that "the non-instantaneous period of time during which the light is on is initiated (resp. terminated) by an instantaneous action of switch on (resp. switch off)" as follows:

$$\textsf{Switch-On} \wedge \pi \wedge ((\textsf{Light-On} \wedge \neg\pi)\Diamond_u(\textsf{Switch-Off} \wedge \pi)).$$

As a matter of fact, the proposed interval version of the until operator suffers from some limitations. In particular, to obtain a decomposable version of it we should force homogeneity either implicitly (via the assumption of the homogeneity principle [3]) or explicitly (by means of sub-interval operators). Besides the well-known fields of planning and natural language processing, successful applications of interval temporal logics can be found in the areas of digital system design and verification [65] and of model validation phase support [74]. As for Moszkowski's ITL, the logic $\mathrm{PNL}^{di\pi+}$ can be exploited to express various interesting statements about digital systems. As an example, one can constrain "the output q of a device to strictly follow the input p" (being p and q two non-instantaneous states of the device) as follows:

$$(\neg\pi \wedge p) \;\to\; (\neg\pi\Diamond_u(\neg\pi \wedge q)).$$

Other useful statements about digital systems can be captured by exploiting the difference operator. As for the model validation task, interval temporal logics have been used to keep significantly low the number of states to be checked in HSTS Planner, a model-based planning system of the Remote Agent autonomous system architecture [74].

Finally, as a very simple case study, we take a medical context, borrowing an example from [17]. In our specific case, we deal with information related to the management of stroke, which is the loss or alteration of some function of the body due to an insufficient supply of blood to some parts of the brain [48]. The insufficient supply can be related to different pathological situations: obstruction of one or many blood vessel (ischemic stroke), or rupture of a weakened blood vessel (hemorrhagic stroke). Besides ischemic and hemorrhagic strokes, minor (warning) strokes (TIA: Transient Ischemic Attack) can happen: these are transient pathologies and tend to resolve themselves, even though they must be immediately considered, being indicators of possible future ischemic strokes. Among risk factors (i.e.: factors which increase the risk of stroke) we mention cigarette smoke, high blood pressure, heart disease, and the above-mentioned TIA. Common medical treatments include anticoagulation (for ischemic stroke and their prevention), and antihypertensive (for hemorrhagic stroke) therapies [48]. A central observation here is that typical sentences that can be of interest in a automatic treatment of such information are both *qualitative* and *quantitative*: for the first one of these classes, propositional interval logics such as those in the class of $\mathcal{PNL}$ can be successfully applied. As an example, suppose that a temporal database for a patient's medical history contains data of the form that we give below, for a given temporal linear domain:

1. "from the moment m_0 to the moment m_1 the patient an anticoagulation-therapy was administered";
2. "from the moment m_2 to the moment m_3 and from the moment m_4 to the moment m_5, the patient had a sever headache";
3. "from the moment m_6 to the moment m_7 the patient had aphasia";
4. ...

Opportune (unique) propositional letters can be used to model such a situation. So, suppose that the anti-coagulation therapy is denoted by at_1 (the subscript indicates the specific patient under observation), that h_1 denotes headache, and that ap_1 denoted aphasia. Also, suppose that whenever a propositional variable is not set to true, we impose it to be false. Now, it is not difficult to see that typical (qualitative) assertions in this context can be formalized in the language of $\mathcal{PNL}^-$. For example, the phrase "did the patient have a stroke (s_1) immediately after an anticoagulation therapy?" can be formalized as follows:

$$\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle(at_1 \wedge \langle A\rangle s_1),$$

or "did the patient have aphasia in between two cycles of antihypertensive (ah_1) therapy?", by:

$$\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle(as_1 \wedge \langle\overline{A}\rangle\langle\overline{A}\rangle ah_1 \wedge \langle A\rangle\langle A\rangle ah_1),$$

or, finally, "did the patient have a stroke under an anticoagulation therapy cycle?", by:

$$\langle A\rangle\langle A\rangle\langle\overline{A}\rangle\langle\overline{A}\rangle(at_1 \wedge n(q) \rightarrow \langle\overline{A}\rangle\langle A\rangle\langle A\rangle(s_1 \wedge \neg\langle\overline{A}\rangle\langle\overline{A}\rangle q)).$$

where q is an unused propositional variable.

3.4 Axiomatic Systems for $\mathcal{PNL}^+$

In this section and in the next one we will be using the notions given in Chapter 1 concerning neighborhood frames and representation theorems.

3.4.1 An Axiomatic System for PNL$^+$.

We propose the following axioms for PNL$^+$, where the inverse of a formula is obtained by interchanging $\Box_r$ and $\Box_l$:

(A-NT) Enough propositional tautologies;

(A-NK) The K axioms for $\Box_r$ and $\Box_l$;

(A-NNF0) $\Box_r p \rightarrow \Diamond_r p$, and its inverse;

(A-NNF1) $p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF2) $\Diamond_r \Diamond_l p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF3) $\Box_r \Diamond_l p \rightarrow \Diamond_l \Diamond_r \Diamond_r p \vee \Diamond_l \Diamond_l \Diamond_r p$, and its inverse;

(A-NNF4) $\Diamond_r \Diamond_r \Diamond_r p \rightarrow \Diamond_r \Diamond_r p$, and its inverse;

(A-NNF$_\infty$) $\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n \rightarrow \Diamond_r(\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n)$, and its inverse, for each $n \geq 1$.

The rules of inference are, as usual, Modus Ponens, Uniform Substitution, and $\Box_r$ and $\Box_l$ Generalization.

As an aside, notice that PNL$^+$ presents some interesting features. First, it is not finitely axiomatizable, while, as we shall see later, both PNL$^-$ and PNL$^{\pi+}$ have a finite, sound and complete axiomatic system. Second, PNL$^+$ has no finite model property with respect to standard models: indeed, it is not hard to show that the formula

$$p \wedge \Diamond_r \neg p \wedge \Box_r \Box_r (p \rightarrow \Diamond_r \neg p) \wedge \Box_r \Box_r (\neg p \rightarrow \Diamond_r p)$$

is not satisfiable in any finite standard model.

Proposition 38 *A neighborhood frame* $\mathbf{F}^+ = \langle \mathbb{I}, R, L \rangle$ *is an interval neighborhood frame if and only if the axioms A-NNF1,...,A-NNF4 are valid in* $\mathbf{F}^+$.

Proof.

It is simple to check that the axioms A-NNF1, ..., A-NNF4 modally define the semantic conditions NF1-NF4 in the non-strict semantics. ∎

We show that the given axiomatic system for PNL$^+$ is sound and complete.

Lemma 39 *The following formulas and their inverses are derivable in* PNL$^+$*:*

1. $\Diamond_r p \rightarrow \Box_r \Box_l \Diamond_r p$;

2. $\Diamond_r\Diamond_l\Diamond_r p\rightarrow\Diamond_r p$;

3. $\Diamond_l\Diamond_r p\rightarrow\Diamond_r\Diamond_l\Diamond_l p\vee\Diamond_r\Diamond_r\Diamond_l p$.

Proof.
For 1, use A-NNF1 and A-NNF2. For 2, observe that $\mathrm{PNL}^+\vdash\ \Diamond_r\Diamond_l\Diamond_r p\rightarrow\Diamond_r\Box_l\Diamond_r p$ by Axiom A-NNF2 (and Axiom A-NNF0), hence $\mathrm{PNL}^+\vdash\ \Diamond_r\Diamond_l\Diamond_r p\rightarrow\Diamond_r p$ by Axiom A-NNF1. Finally, 3 follows from A-NNF2 and A-NNF3. ■

Lemma 40 *Let $\mathbf{M}^{+^*}=\langle\mathbb{I}^*,R^*,L^*,V^*\rangle$ be any generated sub-model of the canonical model for* PNL^+ *and let $w\in\mathbb{I}^*$. Then there is $w_b\in\mathbb{I}^*$ such that $\{\phi\mid\Box_l\phi\in w\}\cup\{\Diamond_l\psi\mid\Diamond_l\psi\in w\}\cup\{\Box_l\xi\mid\Box_l\xi\in w\}\subseteq w_b$, and $w_e\in\mathbb{I}^*$ such that $\{\phi\mid\Box_r\phi\in w\}\cup\{\Diamond_r\psi\mid\Diamond_r\psi\in w\}\cup\{\Box_r\xi\mid\Box_r\xi\in w\}\subseteq w_e$.*

Proof.
It suffices to show that the set $\Gamma=\{\phi\mid\Box_l\phi\in w\}\cup\{\Diamond_l\psi\mid\Diamond_l\psi\in w\}\cup\{\Box_l\xi\mid\Box_l\xi\in w\}$ is PNL^+-consistent. Suppose otherwise. Then for some ϕ such that $\Box_l\phi\in w$, $\Box_l\xi\in w$, and $\{\Diamond_l\psi_1,\ldots\Diamond_l\psi_n\}\subseteq w$, the set $\{\phi,\Diamond_l\psi_1,\ldots\Diamond_l\psi_n,\Box_l\xi\}$ is PNL^+-inconsistent, i.e., $\mathrm{PNL}^+\vdash\ \phi\rightarrow\neg(\Box_l\xi\wedge\Diamond_l\psi_1\wedge\ldots\wedge\Diamond_l\psi_n)$. Hence $\mathrm{PNL}^+\vdash\ \Box_l\phi\rightarrow\Box_l\neg(\Box_l\xi\wedge\Diamond_l\psi_1\wedge\ldots\wedge\Diamond_l\psi_n)$. Thus $\Box_l\neg(\Box_l\xi\wedge\Diamond_l\psi_1\wedge\ldots\wedge\Diamond_l\psi_n)\in w$, i.e., $\neg\Diamond_r(\Box_l\xi\wedge\Diamond_l\psi_1\wedge\ldots\Diamond_l\psi_n)\in w$. On the other hand, $\Diamond_l(\Box_l\xi\wedge\Diamond_l\psi_1\wedge\ldots\wedge\Diamond_l\psi_n)\in w$ by A-NF$_\infty$, which is a contradiction. Thus, Γ is contained in a maximal PNL^+-consistent set w_b in $\mathbb{I}^*$. The existence of w_e is proved likewise. ■

Theorem 41 *The logic* PNL^+ *is sound and complete for the class of all non-strict interval neighborhood structures.*

Proof.
Soundness is straightforward. Note that the truth of most axioms, including the axiom scheme A-NNF$_\infty$, hinges on the inclusion of point intervals.

For the completeness, we take any PNL^+-consistent formula ϕ. It is satisfied at the root w of some generated sub-model $\mathbf{M}^+$ of the canonical model for PNL^+. Regarding that generated sub-model as a first-order structure of the language with $=$, R, L, and unary predicates corresponding to the atomic propositions occurring in ϕ, we take (using the Downwards Löwenheim-Skolem theorem) a countable elementary substructure $\mathbf{M}^{+^*}$ of $\mathbf{M}^+$ containing w. Let $\mathbf{M}^{+^*}=\langle\mathbf{F}^{+^*},V^*\rangle$, where $\mathbf{F}^{+^*}=\langle\mathbb{I}^*,R^*,L^*\rangle$. The elements of $\mathbb{I}^*$ will henceforth be called 'intervals'. Notice that $\mathbf{M}^{+^*},w\Vdash\phi$ since truth of an interval formula at a given interval of a given PNL^+-model is a first-order property. Furthermore, Lemma 40 implies the truth of the first-order formulas

$$\forall x(\exists y(xLy\wedge\forall t(xLt\leftrightarrow yLt)))$$

and

$$\forall x(\exists z(xRz\wedge\forall t(xRt\leftrightarrow zRt)))$$

in $\mathbf{M}^+$ and hence in $\mathbf{M}^{+^*}$. Thus, with every interval v, $\mathbf{M}^{+^*}$ contains intervals v_b and v_e satisfying the conditions of Lemma 40. Notice also that the 'point intervals' in $\mathbf{M}^{+^*}$ are distinguished by being both R^* -reflexive and L^*-reflexive. (In fact, one reflexivity implies the other since R^* and L^* are mutually inverse.)

Now, let w_b and w_e be as in Lemma 40. We are going to build step-by-step an interval neighborhood structure, mapping p-morphically over $\mathbf{F}^{+^*}$. We will inductively define a chain of interval neighborhood structures $\mathbf{F}_0^+ \subseteq \ldots \mathbf{F}_n^+ \subseteq \ldots$, where $\mathbf{F}_n^+ = \langle \mathbb{I}(\mathbb{D})_n, R_n, L_n \rangle$, and a sequence of mappings $f_n : \mathbf{F}_n^+ \mapsto \mathbf{F}^{+^*}$, for $n = 0, 1, 2, \ldots$, satisfying the conditions

(i) $yR_n z \rightarrow f_n(y)R^* f_n(z)$, and
(ii) $yL_n z \rightarrow f_n(y)L^* f_n(z)$,

as follows. Let $\mathbb{D}_0 = \{d_0, d_1\}$, with $d_0 < d_1$. R_0 and L_0 are standard right neighbor and left neighbor relations on $\mathbb{I}(\mathbb{D})_0^+$. $f_0([d_0, d_1]) = w$, $f_0([d_0, d_0]) = w_b$, and $f_0([d_1, d_1]) = w_e$. Clearly, the function f_0 preserves the right and left neighbor relations.

Suppose now that $\mathbf{F}_n^+$ and f_n are defined and satisfy the conditions (i) and (ii). Let $\mathbb{D}_n = \{d_0, \ldots, d_n\}$, where $d_0 < \ldots < d_n$. In general, f_n is not a p-morphism from $\mathbf{F}_n^+$ to $\mathbf{F}^{+^*}$ because there are *p-morphism defects* in $\mathbf{F}_n^+$ which we will have to repair during the construction, viz.: the image under f_n of an interval $[d_k, d_m]$ in $\mathbf{F}_n^+$ has a right neighbor (resp., a left neighbor) v in $\mathbf{F}^{+^*}$, which is 'missing' in $\mathbf{F}_n^+$, i.e., v is not an f_n-image of any interval from $\mathbb{I}(\mathbb{D})_n^+$, related likewise to $[d_k, d_m]$. Let all possible defects, i.e., pairs of neighboring intervals from $\mathbf{F}^{+^*}$ (which are countably many since $\mathbf{F}^{+^*}$ is countable), each repeated countably many times, be listed in a sequence $\mathcal{D} = \{\delta_n\}_{n<\omega}$, and let δ be the first one in the sequence, which has not been dealt with yet, and which occurs in $\mathbf{F}_n^+$. We are going to expand $\mathbf{F}_n^+$ to $\mathbf{F}_{n+1}^+$ in such a way that the defect δ will be fixed.

Suppose that δ relates the (image of the) interval $[d_k, d_m]$ from $\mathbf{F}_n^+$ and, say, a right neighbor v of $f_n([d_k, d_m])$ in $\mathbf{F}^{+^*}$, which is not an image of any interval from $\mathbf{F}_n^+$. (In particular, that means that $f_n([d_k, d_m]) \neq v$.) We then extend $\mathbf{F}_n^+$ to $\mathbf{F}_{n+1}^+$ with a new point d_h and f_n to f_{n+1} so that $f_{n+1}([d_m, d_h]) = v$. We must still find an appropriate place of d_h in the linear ordering $\mathbb{D}_n$ and define f_{n+1} over all other intervals with an endpoint d_h in a way which preserves the neighborhood relations. Note that $f_n([d_k, d_m])R^*R^*L^*v$, hence $f_n([d_0, d_m])R^*R^*R^*L^*v$, and so $f_n([d_0, d_m])R^*R^*L^*v$ by axiom A-NNF4. Let d_{m+i} be the greatest element of $\mathbf{F}_n^+$ such that $f_n([d_0, d_{m+i}])\ R^*R^*L^*v$. Then, for each $j = 0, \ldots, m+i$, $f_n([d_0, d_j])\ R^*R^*L^*$ $f_n([d_0, x_{m+i}])$, so $f_n([d_0, d_j])\ R^*R^*L^*R^*R^*L^*v$, and hence $f_n([d_0, d_j])\ R^*R^*L^*v$ by Lemma 39 (part 2) and axiom A-NNF4. Therefore, for each $j = 0, \ldots, m+i$, there is $w_j \in \mathbf{F}^{+^*}$ such that $f_n([d_0, d_j])R^*w_j$ and $w_jR^*L^*v$.

We now place d_h between d_{m+i} and d_{m+i+1} (if $m+1 \leq n$, otherwise we place d_h to the right of d_n) and extend f_n over all new intervals as follows. First, we put $f_{n+1}([d_m, d_h]) = v$. Then, for each $j = 1, \ldots, m+i$, $j \neq m$, we define $f_{n+1}([d_j, d_h]) = w_j$. For $j > m+i$, it is not the case that $f_n([d_m, d_j])R^*R^*L^*v$ (otherwise, $f_n([d_0, d_j])$ $R^*\ R^*\ L^*v$). On the other hand, $f_n([d_m,\ d_j])L^*\ R^*v$ because $f_n([d_m, d_j])\ L^*f_n([d_k,$ $d_m])$ and $f_n([d_k,\ d_m])\ R^*v$ by assumption. Then, by Lemma 39 (part 3), $f_n([d_m,$

$d_j]) \; R^* \; L^*L^*v$. Therefore, there exists $w_j \in \mathbf{F}^{+^*}$ such that $f_n([d_m \;, d_j]) \; R^* \; L^*w_j$ and w_jL^*v. We define $f_{n+1}([d_h, d_j]) = w_j$. Finally, choose $v_e \in \mathbf{F}^{+^*}$ satisfying the condition of Lemma 40 and put $f_{n+1}([d_h, d_h]) = v_e$. It is straightforward to check that conditions (i) and (ii) still hold for $\mathbf{F}^+_{n+1}$. For example, if $d_j < d_h < d_l$, then $[d_j, d_h]R_{n+1}[d_h, d_h]$, and thus $f_{n+1}([d_j, d_h])R^* \; L^*v$ and $f_{n+1}([d_h, d_l])L^*v$. Hence $f_{n+1}([d_j, d_h]) \; R^*L^*R^*f_{n+1} \; ([d_h, d_l])$, and therefore $f_{n+1}([d_j, d_h]) \; R^* \; f_{n+1} \; ([d_h, \, d_l])$. This completes the inductive procedure.

Now, we define

$$\mathbb{D}_\omega = \bigcup_{n<\omega} \mathbb{D}_n, L_\omega = \bigcup_{n<\omega} L_n, R_\omega = \bigcup_{n<\omega} R_n, f_\omega = \bigcup_{n<\omega} f_n$$

and

$$\mathbf{F}^+_\omega = \langle \mathbb{I}(\mathbb{D})^+_\omega, R_\omega, L_\omega \rangle.$$

Finally, we define a valuation V_ω in $\mathbf{F}^+_\omega$ according to V^* in $\mathbf{F}^{+^*}$, viz. for all $p \in \mathcal{AP}$, $V_\omega(p) = \{i \in \mathbb{I}(\mathbb{D})^+_\omega \mid f_\omega(i) \in V^*(p)\}$. Let $\mathbf{M}^+_\omega = \langle \mathbf{F}^+_\omega, V_\omega \rangle$. Then $f_\omega : \mathbf{M}^+_\omega \mapsto \mathbf{M}^{+^*}$ is a surjective p-morphism, hence $\mathbf{M}^+_\omega, [d_0, d_1] \Vdash \phi$. ■

3.4.2 An axiomatic system for PNL$^{\pi+}$.

We extend the axiomatic system for PNL$^+$ to PNL$^{\pi+}$ by adding the following axioms:

(A-π1) $\Diamond_l \pi \wedge \Diamond_r \pi$;

(A-π2) $\Diamond_r(\pi \wedge p) \rightarrow \Box_r(\pi \rightarrow p)$ and its inverse;

(A-π3) $\Diamond_r p \wedge \Box_r q \rightarrow \Diamond_r(\pi \wedge \Diamond_r p \wedge \Box_r q)$ and its inverse.

By induction on n, one can show that all formulas $\Diamond_r(\pi \wedge p_1) \wedge ... \wedge \Diamond_r(\pi \wedge p_n) \rightarrow \Diamond_r(\pi \wedge p_1 \wedge ... \wedge p_n)$ and their inverses are derivable in PNL$^{\pi+}$, and thus that $\Box_r q \wedge \Diamond_r p_1 \wedge ... \wedge \Diamond_r p_n \rightarrow \Diamond_r(\pi \wedge \Box_r q \wedge \Diamond_r p_1 \wedge ... \wedge \Diamond_r p_n)$ is derivable as well. Therefore, the infinite scheme A-NNF$_\infty$ becomes derivable, hence redundant, in PNL$^{\pi+}$.

The completeness proof for PNL$^+$ is readily adaptable to PNL$^{\pi+}$.

3.5 Axiomatic Systems for $\mathcal{PNL}^-$

3.5.1 An Axiomatic System for PNL$^-$.

Except for the scheme A-NF$_\infty$, which is no longer valid, the axioms for PNL$^-$ are very similar to those for PNL$^+$ (accordingly modified to reflect the fact that point-intervals are now excluded), where $\Diamond_r, \Diamond_l$ are replaced by $\langle A \rangle, \langle \overline{A} \rangle$, and $\Box_r, \Box_l$ accordingly by $[A], [\overline{A}]$. We propose the following system for PNL$^-$:

(A-ST) Enough propositional tautologies;

(A-SK) The K axioms for $[A]$ and $[\overline{A}]$;

(A-SNF1) $p \to [A]\langle\overline{A}\rangle p$ and its inverse;

(A-SNF2) $\langle A\rangle\langle\overline{A}\rangle p \to [A]\langle\overline{A}\rangle p$ and its inverse;

(A-SNF3) $(\langle\overline{A}\rangle\langle\overline{A}\rangle\top \wedge \langle A\rangle\langle\overline{A}\rangle p) \to p \vee \langle\overline{A}\rangle\langle A\rangle\langle A\rangle p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\langle A\rangle p$ and its inverse;

(A-SNF4) $\langle A\rangle\langle A\rangle\langle A\rangle p \to \langle A\rangle\langle A\rangle p$ and its inverse.

Proposition 42 *A neighborhood frame* $\mathbf{F} = \langle \mathbb{I}, R, L\rangle$ *is an interval neighborhood frame if and only if the axioms A-SNF1, ..., A-SNF4 are valid in* $\mathbf{F}$.

Proof.
As in Proposition 38. ■

Notice that the axioms cannot guarantee strictness of the neighborhood frame as irreflexivity is not definable in the language of PNL$^-$.

Theorem 43 *The logic* PNL$^-$ *is sound and complete for the class of all strict interval neighborhood structures.*

Proof.
We closely follow the technique applied in the proof of Theorem 41. Again, the soundness is straightforward. For the completeness, we take any PNL$^-$-consistent formula ϕ. It is satisfied at the root w of some generated sub-model of the canonical model for PNL$^-$. We then pick a countable elementary sub-model $\mathbf{M}^{-*} = \langle \mathbf{F}^{-*}, V^*\rangle$ which contains w and satisfies ϕ there. Let $\mathbf{F}^{-*} = \langle \mathbb{I}^*, R^*, L^*\rangle$. Note that $\mathbf{F}^*$ is a weakly connected interval neighborhood frame in which the axioms A-SNF1, ..., A-SNF4 are valid since they are canonical (being of Sahlqvist type, up to tautological equivalence) and first-order definable. We then build step-by-step a model over a strict interval neighborhood structure, which maps p-morphically over $\mathbf{M}^{-*}$ very much like in the proof of Theorem 41, but easier, because we need not worry about point-intervals. ■

3.5.2 Axiomatic Systems for Extensions of PNL$^-$.

Theorem 44 *The following completeness results hold:*

1. *The axiomatic system for* PNL$^-$ *extended with A-SNF^u is sound and complete for* PNL^{u-}*;*

2. *The axiomatic system for* PNL$^-$ *extended with A-SNF^{de}, is sound and complete for* PNL^{de-}*;*

3. *The axiomatic system for* PNL$^-$ *extended with A-SNF^{di} is sound and complete for* PNL^{di-}*;*

4. *The axiomatic system for* PNL$^-$ *combining* PNL^{u-} *and* PNL^{de-} *is sound and complete for* PNL^{ude-}*;*

5. *The axiomatic system for* PNL^{-} *combining* PNL^{u-} *and* PNL^{di-} *is sound and complete for* PNL^{udi-}.

Proof.

All proofs are adaptations of the one for PNL^{-}, because the respective axioms are canonical and define semantic conditions which either are reflected by p-morphisms (unboundedness) or can be forced during the step-by step construction to hold in the limit structure (density or discreteness).

4

Semantic Tableau for Propositional Interval Logics

[...] indeed if I myself were just to pick up this book today without having spent the past twenty years thinking about its contents, I have little doubt that I too would not believe many of the things it says.[...]
Stephen Wolfram ("A New Kind of Science")

In this chapter we explore a tableau method for propositional interval logics. As we have anticipated in previous chapters, we choose to develop a tableau method for the basic logic BCDT$^+$ (for its syntax and semantics see 2.4.2). This method can be adapted to the strict version BCDT$^-$, and it can be accordingly restricted to the logics in the classes $\mathcal{CDT}, \mathcal{HS}$, and $\mathcal{PNL}$, interpreted over various classes of structures. It is worth reminding that a first implementation of the method is described in [85].

We start with a brief comparison between the tableau method proposed here and other existing methods for point-based and interval-based modal and temporal logics (see [98, 29, 52]). As a preliminary remark, we note that most tableau methods for modal and temporal logics are terminating tableaux for *decidable* logics, and thus they yield decision procedures. Tableau methods for modal and (point-based) temporal logics can be classified as *explicit* or *implicit* (see [18]). Unlike implicit tableaux, explicit ones maintain the accessibility relation by means of some sort of external device. In implicit tableaux [30, 80], the accessibility relation is built-in into the rules. In particular, in linear and branching time point-based temporal logics the tableau represents a model of the satisfiable formulas (a time-line or a tree, respectively). The non-standard finite model property can then be exploited to show that the resulting tableau methods are actually decision procedures (they do not lead to infinite computations). Explicit tableau methods have been developed for several modal logics. They capture the accessibility relation by means of labeled formulas, and they provide suitable notions of closed branches and tableaux. Whenever the logic is decidable, its properties can be exploited to turn the tableau method into a decision procedure. In this respect, the tableau method for BCDT$^+$, while sharing basic features with explicit tableaux for modal logics, comes closer to the classical, possibly non-terminating tableau method for first-order logic, which only provides a semi-decision procedure for non-satisfiability. It also presents some similarities with the explicit tableau method developed for the *guarded fragment* of first-order logic (see [39]).

To the best of our knowledge, there exist very few other tableau methods for interval temporal logics (and duration calculi) in the literature. As we have already seen, a tableau-based decision procedure for an extension of Local QPITL, which, besides the *chop* operator and the modal constant π, has a projection operator *proj*, has been proposed by Kono [51] and later refined by Bowman and Thompson [10]. In [15], Chetcuti-Sperandio and Fariñas del Cerro focus on a decidable fragment of Duration Calculus (DC) which encompasses a proper subset of DC operators, namely, $\wedge$, $\vee$, and C. The tableau construction for the resulting logic combines application of the rules of classical tableaux with that of a suitable constraint resolution algorithm and it essentially depends on the assumption of *bounded* variability of the state variables. Finally, tableau systems for the propositional and first-order Linear Temporal Logic, which employ a mechanism for labeling formulas with temporal constraints somewhat similar to ours, have been developed respectively in [86] and [13]. The main differences between these tableau methods and ours are:

1. They are specifically designed to deal with integer time structures (i.e., linear and discrete) while ours is quite generic;

2. The LTL is essentially point-based, and intervals only play a secondary role in it (viz., a formula it true on an interval if and only if it is true at every point in it), while in our systems intervals are primary semantic objects on which the truth definitions are entirely based;

3. The closedness of a tableau in the above described systems is defined in terms of unsatisfiability of the associated set of temporal constraints, while in our system it is entirely syntactic.

4.1 The Method

First, some basic terminology. A **finite tree** is a finite directed connected graph in which every node, apart from one (the **root**), has exactly one incoming arc. A **successor** of a node $\mathbf{n}$ is a node $\mathbf{n}'$ such that there is an edge from $\mathbf{n}$ to $\mathbf{n}'$. A **leaf** is a node with no successors; a **path** is a sequence of nodes $\mathbf{n_0}, \ldots, \mathbf{n_k}$ such that, for all $i = 0, \ldots, k-1$, $\mathbf{n_{i+1}}$ is a successor of $\mathbf{n_i}$; a **branch** is a path from the root to a leaf. The **height** of a node $\mathbf{n}$ is the maximum length (number of edge) of a path from $\mathbf{n}$ to a leaf. If $\mathbf{n}, \mathbf{n}'$ belong to the same branch and the height of $\mathbf{n}$ is less than or equal to the height of $\mathbf{n}'$, we write $\mathbf{n} \prec \mathbf{n}'$.

Let $\mathbb{C} = \langle C, < \rangle$ be a finite partial order. A **labeled formula**, with label in $\mathbb{C}$, is a pair $(\phi, [c_i, c_j])$, where $\phi \in \text{BCDT}^+$ and $[c_i, c_j] \in \mathbb{I}(\mathbb{C})^+$.

For a node $\mathbf{n}$ in a tree, the **decoration** $\nu(\mathbf{n})$ is a triple $((\phi, [c_i, c_j]), \mathbb{C}, u_{\mathbf{n}})$, where $\mathbb{C}$ is a finite partial order, $(\phi, [c_i, c_j])$ is a labeled formula, with label in $\mathbb{C}$, and $u_{\mathbf{n}}$ is a **local flag function** which associates the values 0 or 1 with every branch B

containing $\mathbf{n}$. Intuitively, the value 0 for a node n with respect to a branch B means that n can be expanded on B. For the sake of simplicity, we will often assume the interval $[c_i, c_j]$ to consist of the elements $c_i < c_{i+1} < \cdots < c_j$, and sometimes, with a little abuse of notation, we will write $\mathbb{C} = \{c_i < c_k, c_m < c_j, \ldots\}$. A **decorated tree** is a tree in which every node has a decoration $\nu(\mathbf{n})$. For every decorated tree, we define a **global flag function** u acting on pairs *(node, branch through that node)* as $u(\mathbf{n}, B) = u_{\mathbf{n}}(B)$. Sometimes, for convenience, we will include in the decoration of the nodes the global flag function instead of the local ones. For any branch B in a decorated tree, we denote by $\mathbb{C}_B$ the ordered set in the decoration of the leaf B, and for any node $\mathbf{n}$ in a decorated tree, we denote by $\Phi(\mathbf{n})$ the formula in its decoration. If B is a branch, then $B \cdot \mathbf{n}$ denotes the result of the expansion of B with the node $\mathbf{n}$ (addition of an edge connecting the leaf of B to $\mathbf{n}$). Similarly, $B \cdot \mathbf{n_1} \mid \ldots \mid \mathbf{n_k}$ denotes the result of the expansion of B with k immediate successor nodes $\mathbf{n_1}, \ldots, \mathbf{n_k}$ (which produces k branches extending B). A tableau for BCDT$^+$ will be defined as a special decorated tree. We note again that $\mathbb{C}$ remains finite throughout the construction of the tableau.

Definition 45 *Given a decorated tree $\mathcal{T}$, a branch B in $\mathcal{T}$, and a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\phi, [c_i, c_j]), \mathbb{C}, u)$, with $u(\mathbf{n}\ , B) = 0$, the **branch-expansion rule** for B and $\mathbf{n}$ is defined as follows (in all the considered cases, $u(\mathbf{n'}\ , B') = 0$ for all new pairs $(\mathbf{n'}\ , B')$ of nodes and branches):*

- *If $\phi = \neg\neg\psi$, then expand the branch to $B \cdot \mathbf{n_0}$, with $\nu(\mathbf{n_0}) = ((\psi, [c_i, c_j]), \mathbb{C}_B, u)$;*

- *If $\phi = \psi_0 \wedge \psi_1$, then expand the branch to $B \cdot \mathbf{n_0} \cdot \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$;*

- *If $\phi = \neg(\psi_0 \wedge \psi_1)$, then expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$;*

- *If $\phi = \neg(\psi_0 C \psi_1)$ and c is the least element of $\mathbb{C}_B$, with $c_i \leq c \leq c_j$, which has not been used yet to expand the node $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c, c_j]), \mathbb{C}_B, u)$;*

- *If $\phi = \neg(\psi_0 D \psi_1)$, c is a minimal element of $\mathbb{C}_B$ such that $c \leq c_i$, and there exists $c' \in [c, c_i]$ which has not been used yet to expand the node $\mathbf{n}$ on B, then take the least such $c' \in [c, c_i]$ and expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0})$ $= ((\neg\psi_0, [c', c_i]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c', c_j]), \mathbb{C}_B, u)$;*

- *If $\phi = \neg(\psi_0 T \psi_1)$, c is a maximal element of $\mathbb{C}_B$ such that $c_j \leq c$, and there exists $c' \in [c_j, c]$ which has not been used yet to expand the node $\mathbf{n}$ on B, then take the greatest such $c' \in [c_j, c]$ and expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, so that $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_j, c']), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c']), \mathbb{C}_B, u)$;*

- *If $\phi = (\psi_0 C \psi_1)$, then expand the branch to $B \cdot (\mathbf{n_i} \cdot \mathbf{m_i})|\ldots|(\mathbf{n_j} \cdot \mathbf{m_j})|(\mathbf{n'_i} \cdot \mathbf{m'_i})|\ldots|(\mathbf{n'_{j-1}} \cdot \mathbf{m'_{j-1}})$, where:*

1. *for all* $c_k \in [c_i, c_j]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_i, c_k]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_j]), \mathbb{C}_B, u)$*;*
2. *for all* $i \leq k \leq j-1$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c *between* c_k *and* c_{k+1} *in* $[c_i, c_j]$, $\nu(\mathbf{n'_k}) = ((\psi_0, [c_i, c]), \mathbb{C}_k, u)$*, and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c, c_j]), \mathbb{C}_k, u)$*;*

- *If* $\phi = (\psi_0 D \psi_1)$*, then repeatedly expand the current branch, once for each minimal element* c *(where* $[c, c_i] = \{c = c_0 < c_1 < \cdots c_i\}$*), by adding the decorated sub-tree* $(\mathbf{n_0} \cdot \mathbf{m_0})|\ldots|(\mathbf{n_i} \cdot \mathbf{m_i})|(\mathbf{n'_1} \cdot \mathbf{m'_1})|\ldots|(\mathbf{n'_i} \cdot \mathbf{m'_i})|(\mathbf{n''_0} \cdot \mathbf{m''_0})|\ldots|(\mathbf{n''_i} \cdot \mathbf{m''_i})$ *to its leaf, where:*

  1. *for all* c_k *such that* $c_k \in [c, c_i]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_k, c_j]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_i]), \mathbb{C}_B, u)$*;*
  2. *for all* $0 < k \leq i$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *immediately before* c_k *in* $[c, c_i]$*, and* $\nu(\mathbf{n'_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$ *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$*;*
  3. *for all* $0 \leq k \leq i$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *in* $\mathbb{C}_B$*, with* $c' < c_k$*, which is incomparable with all existing predecessors of* c_k, $\nu(\mathbf{n''_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$*, and* $\nu(\mathbf{m''_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$*;*

- *If* $\phi = (\psi_0 T \psi_1)$*, then repeatedly expand the current branch, once for each maximal element* c *(where* $[c_j, c] = \{c_j < c_{j+1} < \cdots c_n = c\}$*), by adding the decorated sub-tree* $(\mathbf{n_j} \cdot \mathbf{m_j})|\ldots|(\mathbf{n_n} \cdot \mathbf{m_n})|(\mathbf{n'_j} \cdot \mathbf{m'_j})|\ldots|(\mathbf{n'_{n-1}} \cdot \mathbf{m'_{n-1}})|(\mathbf{n''_j} \cdot \mathbf{m''_j})|\ldots|(\mathbf{n''_n} \cdot \mathbf{m''_n})$ *to its leaf, where:*

  1. *for all* c_k *such that* $c_k \in [c_j, c]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_j, c_k]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_i, c_k]), \mathbb{C}_B, u)$*;*
  2. *for all* $j \leq k < n$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *immediately after* c_k *in* $[c_j, c]$*, and* $\nu(\mathbf{n'_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$ *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$*;*
  3. *for all* $j \leq k \leq n$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *in* $\mathbb{C}_B$*, with* $c_k < c'$*, which is incomparable with all existing successors of* c_k, $\nu(\mathbf{n''_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$*, and* $\nu(\mathbf{m''_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$*.*

Finally, for any node $\mathbf{m}$ $(\neq \mathbf{n})$ *in* B *and any branch* B' *extending* B*, let* $u(\mathbf{m}, B')$ *be equal to* $u(\mathbf{m}, B)$*, and for any branch* B' *extending* B, $u(\mathbf{n}, B') = 1$*, unless* $\phi = \neg(\psi_0 C \psi_1)$, $\phi = \neg(\psi_0 D \psi_1)$*, or* $\phi = \neg(\psi_0 T \psi_1)$ *(in such cases* $u(\mathbf{n}, B') = 0$*).*

Let us briefly explain the expansion rules for $\psi_0 C \psi_1$ and $\neg(\psi_0 C \psi_1)$ (similar considerations hold for the other temporal operators). The rule for the (existential) formula $\psi_0 C \psi_1$ deals with the two possible cases: either there exists c_k in $\mathbb{C}_B$ such that $c_i \leq c_k \leq c_j$ and ψ_0 holds over $[c_i, c_k]$ and ψ_1 holds over $[c_k, c_j]$ or such an element c_k must be added. The (universal) formula $\neg(\psi_0 C \psi_1)$ states that, for all $c_i \leq c \leq c_j$, ψ_0 does not hold over $[c_j, c]$ or ψ_1 does not hold over $[c, c_j]$. As a matter of fact, the expansion rule imposes such a condition for a single element c in $\mathbb{C}_B$ (the least element which has not been used yet), and it does not change the flag (which remains equal to 0). In this way, all elements will be eventually taken into consideration, including those elements in between c_i and c_j that will be added to $\mathbb{C}_B$ in some subsequent steps of the tableau construction.

Let us define now the notions of open and closed branch. We say that a node $\mathbf{n}$ in a decorated tree $\mathcal{T}$ is **available on a branch** B to which it belongs if and only if $u(\mathbf{n}, B) = 0$. The branch-expansion rule is **applicable** to a node $\mathbf{n}$ on a branch B if the node is available on B and the application of the rule generates at least one successor node with a new labeled formula. This second condition is needed to avoid looping of the application of the rule on formulas $\neg(\psi_0 C \psi_1), \neg(\psi_0 D \psi_1)$, and $\neg(\psi_0 T \psi_1)$.

Definition 46 *A branch B is **closed** if some of the following conditions holds: (i) there are two nodes $\mathbf{n}, \mathbf{n}' \in B$ such that $\nu(\mathbf{n}) = ((\psi, [c_i, c_j]), \mathbb{C}, u)$ and $\nu(\mathbf{n}') = ((\neg\psi, [c_i, c_j]), \mathbb{C}', u)$ for some formula ψ and $c_i, c_j \in \mathbb{C} \cap \mathbb{C}'$; (ii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i \neq c_j$; or (iii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\neg\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i = c_j$. If none of the above conditions hold, the branch is **open**.*

Definition 47 *The **branch-expansion strategy** for a branch B in a decorated tree $\mathcal{T}$ is defined as follows:*

1. *Apply the branch-expansion rule to a branch B only if it is open;*

2. *If B is open, apply the branch-expansion rule to the closest to the root available node in B for which the branch-expansion rule is applicable.*

Definition 48 *A **tableau** for a given formula $\phi \in \mathrm{BCDT}^+$ is any finite decorated tree $\mathcal{T}$ obtained by expanding the three-node decorated tree built up from an empty-decoration root and two leaves with decorations $((\phi, [c_b, c_e]), \{c_b < c_e\}, u)$ and $((\phi, [c_b, c_b]), \{c_b\}, u)$, where the value of u is 0, through successive applications of the branch-expansion strategy to the existing branches.*

Proposition 49 *If $\phi \in \mathrm{BCDT}^+$, $\mathcal{T}$ is a tableau for ϕ, $\mathbf{n} \in \mathcal{T}$, and $\mathbb{C}$ is the ordered set occurring in the decoration of a node $\mathbf{n} \in \mathcal{T}$, then $\mathbb{C}$ has linear intervals.*

Proof.

By induction on the cardinality of $\mathbb{C}$; suppose that $\mathbf{n} \in B$, where B is a branch of the tableau.

If $|\mathbb{C}| = 1$ (resp., $|\mathbb{C}| = 2$), then $\mathbf{n}$ is such that $\nu(n) = ((\psi, [c_b, c_b]), \{c_b\}, u)$ (resp., $((\psi, [c_b, c_e]), \{c_b, c_e\}, u)$) for some formula ψ. Since by construction $c_b < c_e$, the claim holds.

Suppose that $|\mathbb{C}| = k+1$, $k > 1$. This means that on B there is a node $\mathbf{n}'$ such that $\mathbf{n} \prec \mathbf{n}$, $\nu(\mathbf{n}') = ((\psi, [c_i, c_j]), \mathbb{C}', u')$, $|C| = k$, and ψ is either $\xi_0 C \xi_1$, $\xi_0 D \xi_1$ or $\xi_0 T \xi_1$.

By inductive hypothesis, $\mathbb{C}'$ has linear intervals. If $\phi = \xi_0 C \xi_1$, then a new element c is introduced in between c_i and c_j in one of the possible ways maintaining the linear ordering of the interval $[c_i, c_j]$. So $\mathbb{C}$ has linear intervals. If $\phi = \xi_0 D \xi_1$, then a new element c is introduced before c_i. There are two cases: (i), c is introduced in between c' and c_j (where c' is a minimal element) in one of the possible ways maintaining the linear ordering of the interval $[c, c_i]$, or (ii) c is introduced before some c' (where $c' \leq c_i$) and is incomparable with any minimal element if $\mathbb{C}$. In either case, $\mathbb{C}'$ has linear intervals. The remaining case is similar. ■

Definition 50 *A tableau for* BCDT$^+$ *is* ***closed*** *if and only if every branch in it is closed, otherwise it is* ***open****.*

As an example, consider the unsatisfiable BCDT$^+$ formula $\phi = pT\psi$, where $\psi = \neg(\top Cp)$. Here we show some steps of the construction of a closed tableau for that formula:

1. Starting by the **root**, we produce two children $\mathbf{n_0} = ((pT\psi, [c_0, c_1]), \{c_0 < c_1\}, 0)$ and $\mathbf{n_1} = ((pT\psi, [c_0, c_0]), \{c_0\}, 0)$;

2. Following the left branch, $\mathbf{n_0}$ produces two immediate successors $\mathbf{n_2} : ((\psi, [c_0, c_1]) , \{c_0 < c_1\} , 0)$ and $\mathbf{n_4} = ((\psi, [c_0, c_2]), \{c_0 < c_1 < c_2\}, 0)$, each one of them has an immediate successor, $\mathbf{n_3} = ((p, [c_1, c_1]), \{c_0 < c_1\}, 0)$ and, respectively, $\mathbf{n_5} = ((p, [c_1, c_2]), \{c_0 < c_1 < c_2\}, 0)$;

3. The node $\mathbf{n_3}$ produces two immediate successors $\mathbf{n_6} = (\bot, [c_0, c_0], \{c_0 < c_1\}, 0)$ $\mathbf{n_7} = (\neg p, [c_0, c_1], \{c_0 < c_1\}, 0)$;

4. $\mathbf{n_6}$ expands into $\mathbf{n_8} = (\bot, [c_0, c_1], \{c_0 < c_1\}, 0)$ and $\mathbf{n_9} = (\neg p, [c_1, c_1], \{c_0 < c_1\}, 0)$.

At this stage, we see that the branch $\mathbf{root}, \mathbf{n_0}, \mathbf{n_2}, \mathbf{n_3}, \mathbf{n_6}, \mathbf{n_8}$ is closed. It is not difficult to see that also the remaining branches are closed, so the initial formula ϕ is not satisfiable, as expected.

4.2 Soundness and Completeness

Definition 51 *Given a set S of labeled formulas with labels in an interval structure $\mathbb{C}$, we say that S is* ***satisfiable over*** *$\mathbb{C}$ if there exists a non-strict model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V \rangle$ such that $\mathbb{D} = \langle D, < \rangle$ is an extension of $\langle C, < \rangle$, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$ for all $(\psi, [c_i, c_j]) \in S$.*

If S contains only one labeled formula, the notion of satisfiability of a (labeled) formula over $\mathbb{C}$ is equivalent to the standard notion of satisfiability.

Theorem 52 *If $\phi \in$*BCDT$^+$ *and a tableau $\mathcal{T}$ for ϕ is closed, then ϕ is not satisfiable.*

Proof.

We will prove by induction on the height h of a node $\mathbf{n}$ in the tableau $\mathcal{T}$ the following claim: *if every branch including* $\mathbf{n}$ *is closed, then the set* $S(\mathbf{n})$ *of all labeled formulas in the decorations of the nodes between* $\mathbf{n}$ *and the root is not satisfiable over* $\mathbb{C}$*, where* $\mathbb{C}$ *is the interval structure in the decoration of* $\mathbf{n}$.

If $h = 0$, then $\mathbf{n}$ is a leaf and the unique branch B containing $\mathbf{n}$ is closed. Then, either $S(\mathbf{n})$ contains both the labeled formulas $(\psi, [c_k, c_l])$ and $(\neg\psi, [c_k, c_l])$ for some BCDT$^+$-formula ψ and $c_k, c_l \in \mathbb{C}$, or the labeled formula $(\pi, [c_k, c_l])$ and $c_k \neq c_l$, or the labeled formula $(\neg\pi, [c_k, c_l])$ and $c_k = c_l$. Take any model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V\rangle$ where $\langle D, <\rangle$ is an extension of $\langle C, <\rangle$. In the first case, clearly $\mathbf{M}^+, [c_k, c_l] \Vdash \psi$ if and only if $\mathbf{M}^+, [c_k, c_l] \nVdash \neg\psi$. In the second (resp., third) case, $\mathbf{M}^+, [c_k, c_l] \Vdash \pi$ (resp., $\neg\pi$) if and only if $c_k = c_l$ (resp., $c_k \neq c_l$). Hence, $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

Suppose $h > 0$. Then either $\mathbf{n}$ has been generated as one of the successors, *but not the last one*, when applying the branch-expansion rule in $\wedge$, C, D, T, $\neg C, \neg D$, or $\neg T$ cases, or the branch-expansion rule has been applied to some labeled formula $(\psi, [c_i, c_j]) \in S(\mathbf{n}) - \{\Phi(\mathbf{n})\}$ to extend the branch at $\mathbf{n}$. We deal with the latter case. The former can be dealt with in the same way. Let $C = \{c_1, \ldots, c_n\}$, be the domain of the interval structure from the decoration of $\mathbf{n}$. Notice that every branch passing through any successor of $\mathbf{n}$ must be closed, so the inductive hypothesis applies to all successors of $\mathbf{n}$. We consider the possible cases for the branch-expansion rule applied at $\mathbf{n}$:

- Let $\psi = \neg\neg\xi$. Then there exists $\mathbf{n_0}$ such that $\nu(\mathbf{n_0}) = ((\xi, [c_i, c_j]), \mathbb{C}, u)$ and $\mathbf{n_0}$ is a successor of $\mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_0}$ is closed. By the inductive hypothesis, $S(\mathbf{n_0})$ is not satisfiable over $\mathbb{C}$ (since $\mathbf{n_0} \prec \mathbf{n}$). Since ξ_0 and $\neg\neg\xi_0$ are equivalent, $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$;

- Let $\psi = \xi_0 \wedge \xi_1$. Then there are two nodes $\mathbf{n_0} \in B$ and $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_0}) = ((\xi_0, [c_i, c_j]), \mathbb{C}, u)$, $\nu(\mathbf{n_1}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u)$, and, without loss of generality, $\mathbf{n_0}$ is the successor of $\mathbf{n}$ and $\mathbf{n_1}$ is the successor of $\mathbf{n_0}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_1}$ is closed. By the inductive hypothesis, $S(\mathbf{n_1})$ is not satisfiable over $\mathbb{C}$ since $\mathbf{n_1} \prec \mathbf{n}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must, in particular, satisfy $(\xi_0 \wedge \xi_1, [c_i, c_j])$, and hence $(\xi_0, [c_i, c_j])$ and $(\xi_1, [c_i, c_j])$, it follows that $S(\mathbf{n})$, $S(\mathbf{n_0})$, and $S(\mathbf{n_1})$ are equi-satisfiable over $\mathbb{C}$. Therefore, $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$;

- Let $\psi = \neg(\xi_1 \wedge \xi_2)$. Then there exist two successor nodes $\mathbf{n_0}$ and $\mathbf{n_1}$ of $\mathbf{n}$ such that $\nu(\mathbf{n_0}) = ((\xi_0, [c_i, c_j]), \mathbb{C}, u_0)$, $\nu(\mathbf{n_1}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u_1)$, $\mathbf{n_0}, \mathbf{n_1} \prec \mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_0}$ and every branch containing $\mathbf{n_1}$ is closed. By the inductive hypothesis $S(\mathbf{n_0})$ and $S(\mathbf{n_1})$ are not satisfiable over $\mathbb{C}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must also satisfy $(\xi_0, [c_i, c_j])$ or $(\xi_1, [c_i, c_j])$, it follows that $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$;

- Let $\psi = \neg(\xi_0 C \xi_0)$. Suppose that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$. Then, since $(\neg(\xi_0 C \xi_1), [c_i, c_j]) \in S(\mathbf{n})$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V \rangle$ such that $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \neg(\xi_0 C \xi_1)$. So, for every c_k such that $c_i \leq c_k \leq c_j$, we have that $\mathbf{M}^+, [c_i, c_k] \Vdash \neg\xi_0$ or $\mathbf{M}^+, [c_k, c_j] \Vdash \neg\xi_1$. By construction, the two immediate successors of $\mathbf{n}$ are $\mathbf{n_1}$ and $\mathbf{n_2}$ such that, for an element c_k with $c_i \leq c_k \leq c_j$, $(\neg\xi_0, [c_i, c_k])$ is in the decoration of $\mathbf{n_0}$ and $(\neg\xi_1, [c_k, c_j])$ is in the decoration of $\mathbf{n_1}$. By inductive hypothesis, since $\mathbf{n_1}, \mathbf{n_2} \prec \mathbf{n}$, $S(\mathbf{n_1})$ and $S(\mathbf{n_2})$ are not satisfiable over $\mathbb{C}$. Thus, such a model $\mathbf{M}^+$ cannot exist, and $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$;

- The cases $\psi = \neg(\xi_0 D \xi_1)$ and $\psi = \neg(\xi_0 T \xi_1)$ are analogous;

- Let $\psi = \xi_0 C \xi_1$. Assuming that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V \rangle$, where $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$, such that $\mathbf{M}^+, [c_i, c_j] \Vdash \theta$ for all $(\theta, [c_i, c_j]) \in S(\mathbf{n})$. In particular, $\mathbf{M}^+, [c_i, d] \Vdash \xi_0$ and $\mathbf{M}^+, [d, c_j] \Vdash \xi_1$ for some $c_i \leq d \leq c_j$. Consider two cases:

  1. If $d \in \mathbb{C}$, then $d = c_m$ for some $c_i \leq c_m \leq c_j$. But among the successors of $\mathbf{n}$ there are two nodes $\mathbf{n_m}, \mathbf{m_m}$ where $\nu(\mathbf{n_m}) = ((\xi_0,\ [c_i, c_m]), \mathbb{C}, u)$ and $\nu(\mathbf{m_m}) = ((\xi_1,\ [c_m, c_j]), \mathbb{C}, u)$, and since $\mathbf{n_m}, \mathbf{m_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n_m} \prec \mathbf{m_m}$), by the inductive hypothesis $S(\mathbf{n_m})$ $= S(\mathbf{n}) \cup \{(\xi_0,\ [c_i, c_m]), (\xi_1, [c_m, c_j])\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction;
  2. If $d \notin \mathbb{C}$, then there is an m such that $i \leq m \leq j-1$ and $c_m < d < c_{m+1}$. Hence, there are two successors $\mathbf{n'_m}, \mathbf{m'_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n'_m})$ $= ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$, $\nu(\mathbf{m'_m}) = ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n'_m}, \mathbf{m'_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n'_m} \prec \mathbf{m'_m}$), by the inductive hypothesis $S(\mathbf{n'_m}) = S(\mathbf{n}) \cup \{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction;

  Thus, in either case $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$;

- Let $\psi = \xi_0 D \xi_1$. Assuming that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, V \rangle$, where $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$, such that $\mathbf{M}^+, [c_i, c_j] \Vdash \theta$ for all $(\theta, [c_i, c_j]) \in S(\mathbf{n})$. In particular, $\mathbf{M}^+, [d, c_i] \Vdash \xi_0$ and $\mathbf{M}^+, [d, c_j] \Vdash \xi_1$ for some $d \leq c_i$. Consider 3 cases:

  1. If $d \in \mathbb{C}$, then $d = c_m$ for some $c_m \leq c_i$. But between the successors of $\mathbf{n}$ there are two nodes $\mathbf{n_m}, \mathbf{m_m}$ where $\nu(\mathbf{n_m}) = ((\xi_0, [c_m, c_i]), \mathbb{C}, u)$ and $\nu(\mathbf{m_m}) = ((\xi_1, [c_m, c_j]), \mathbb{C}, u)$, and since $\mathbf{n_m}, \mathbf{m_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n_m} \prec \mathbf{m_m}$), by the inductive hypothesis $S(\mathbf{n_m}) = S(\mathbf{n})$ $\cup\{(\xi_0, [c_m, c_i]), (\xi_1, [c_m, c_j])\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction;
  2. If $d \notin \mathbb{C}$ and there is a minimal element $c \in \mathbb{C}$ and an index m such that $c_m, c_{m+1} \in [c, c_i]$ and $c_m < d < c_{m+1}$, then there are two successors $\mathbf{n'_m}, \mathbf{m'_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n'_m}) = ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$ and $\nu(\mathbf{m'_m})$

$= ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n'_m}, \mathbf{m'_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n'_m} \prec \mathbf{m'_m}$), by the inductive hypothesis $S(\mathbf{n'_m}) = S(\mathbf{n})$ $\cup\{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction;

3. If $d \notin \mathbb{C}$ and there is an index m such that $c_{m+1} \in [c, c_i]$, $d < c_{m+1}$, and d is not comparable with all predecessors of c_{m+1}, then, again, there are two successor nodes $\mathbf{n''_m}, \mathbf{m''_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n''_m}) = ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$ and $\nu(\mathbf{m''_m}) = ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n''_m}$, $\mathbf{m''_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n''_m} \prec \mathbf{m''_m}$), by the inductive hypothesis $S(\mathbf{n''_m})$ $= S(\mathbf{n})$ $\cup\{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction;

Thus, in either case $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$;

- The case of $\psi = \xi_0 T \xi_1$ is similar.

It remains to note that if the formula ϕ is satisfiable, then the singleton sets $\{(\phi, [c_b, c_e])\}$ or $\{(\phi, [c_b, c_b])\}$, belonging to the decorations of the two successors of the root of $\mathcal{T}$, must be satisfiable over $\{c_b < c_e\}$ or $\{c_b\}$. Therefore, ϕ is not satisfiable.
■

Definition 53 *If $\mathcal{T}_0$ is the three-node tableau built up from a root with void decoration and two leaves decorated respectively by $((\phi, [c_b, c_e]),\ \{c_b < c_e\},\ 0)$ and $((\phi, [c_b, c_b])$ $, \{c_b\},\ 0)$ for a given* BCDT$^+$*-formula ϕ, the* ***limit tableau*** *$\overline{\mathcal{T}}$ for ϕ is the (possibly infinite) decorated tree obtained as follows. First, for all i, $\mathcal{T}_{i+1}$ is the tableau obtained by the simultaneous application of the branch-expansion strategy to every branch in $\mathcal{T}_i$. Then, we ignore all flags from the decorations of the nodes in every $\mathcal{T}_i$. Thus, we obtain a chain by inclusion of decorated trees: $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \ldots$, and we define $\overline{\mathcal{T}} = \bigcup_{i=0}^{\infty} \mathcal{T}_i$.*

Notice that the chain above may stabilize at some $\mathcal{T}_i$ if it closes, or if the branch-expansion rule is not applicable to any of its branches. If $\overline{\mathcal{T}}$ is a limit tableau, we associate with each branch B in $\overline{\mathcal{T}}$ the interval structure $\mathbb{C}_B = \bigcup_{i=0}^{\infty} \mathbb{C}_{B_i}$, where, for all i, $\mathbb{C}_{B_i}$ is the interval structure from the decoration of the leaf of the (sub-)branch B_i of B in $\mathcal{T}_i$. The definitions of closed and open branches readily apply to $\overline{\mathcal{T}}$.

Definition 54 *A branch in a (limit) tableau is* ***saturated*** *if there are no nodes on that branch to which the branch-expansion rule is applicable on the branch. A (limit) tableau is* ***saturated*** *if every open branch in it is saturated.*

Now we will show that the set of all labeled formulas on an open branch in a limit tableau has the saturation properties of a Hintikka set in first-order logic.

Lemma 55 *Every limit tableau is saturated.*

Proof.

Given a node $\mathbf{n}$ in a limit tableau $\overline{\mathcal{T}}$, we denote by $d(\mathbf{n})$ the distance (number of edges) between $\mathbf{n}$ and the root of $\overline{\mathcal{T}}$. Now, given a branch B in $\overline{\mathcal{T}}$, we will prove by induction on $d(\mathbf{n})$ that after every step of the expansion of that branch at which the branch-expansion rule becomes applicable to $\mathbf{n}$ (because $\mathbf{n}$ has just been introduced, or because a new point has been introduced in the interval structure on B) that rule is subsequently applied on B to that node.

Suppose the inductive hypothesis holds for all nodes with distance to the root less than l. Let $d(\mathbf{n}) = l$ and the branch-expansion rule has become applicable to $\mathbf{n}$. If there are no nodes between the root (incl. the root) and $\mathbf{n}$ (excl. $\mathbf{n}$) to which the branch-expansion rule is applicable at that moment, the next application of the branch-expansion rule on B is to $\mathbf{n}$. Otherwise, consider the closest-to-$\mathbf{n}$-node $\mathbf{n}^*$ between the root and $\mathbf{n}$ to which the branch-expansion rule is applicable or will become applicable on B at least once thereafter. (Such a node exists because there are only finitely many nodes between $\mathbf{n}$ and the root.) Since $d(\mathbf{n}^*) < d(\mathbf{n})$, by the inductive hypothesis the branch-expansion rule has been subsequently applied to $\mathbf{n}^*$. Then the next application of the branch-expansion rule on B must have been to $\mathbf{n}$ and that completes the induction. Now, assuming that a branch in a limit tableau is not saturated, consider the closest to the root node $\mathbf{n}$ on that branch B to which the branch-expansion rule is applicable on that branch. If $\Phi(\mathbf{n})$ is none of the cases $\neg C, \neg D$, and $\neg T$, then the branch-expansion rule has become applicable to $\mathbf{n}$ at the step when $\mathbf{n}$ is introduced, and by the claim above, it has been subsequently applied, at which moment the node has become unavailable thereafter, which contradicts the assumption. Suppose that $\Phi(\mathbf{n}) =\neg(\psi_0 C \psi_1)$. Then an application of the rule on B would create two successors with labels $(\neg\psi_0, [c_i, c])$ and $(\neg\psi_1, [c, c_j])$, at least one of them new on B. But c_i, c_j, c have already been introduced at some (finite) step of the construction of B and at the first step when the three of them, as well as $\mathbf{n}$, have appeared on the branch, the branch-expansion rule has become applicable to $\mathbf{n}$, hence is has been subsequently applied on B and that application must have introduced the labels $(\psi_0, [c_i, c])$ and $(\psi_1, [c, c_j])$ on B, which again contradicts the assumption. The same holds if $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$ or $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$. ■

Corollary 56 *Let ϕ be a* BCDT$^+$*-formula and $\overline{\mathcal{T}}$ be the limit tableau for ϕ. For every open branch B in $\overline{\mathcal{T}}$, the following closure properties hold.*

- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg\neg\psi, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\psi, [c_i, c_j]), \mathbb{C}, u_0)$;*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\psi_0 \wedge \psi_1, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\psi_0, [c_i, c_j]), \mathbb{C}, u_0)$ and a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$;*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg(\psi_0 \wedge \psi_1), [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}, u_0)$ or a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$;*

- *If there is a node* $\mathbf{n} \in B$ *such that* $\nu(\mathbf{n}) = ((\psi_0 C \psi_1, [c_i, c_j]), \mathbb{C}, u)$*, then, for some* $c \in \mathbb{C}_B$ *such that* $c_i \leq c \leq c_j$ *there are two nodes* $\mathbf{n}', \mathbf{m}' \in B$ *such that* $\nu(\mathbf{n}') = ((\psi_0, [c_i, c]), \mathbb{C}', u')$ *and* $\nu(\mathbf{m}') = ((\psi_1, [c, c_j]), \mathbb{C}', u')$*;*

- *Similarly for every node* $\mathbf{n}$ *with* $\Phi(\mathbf{n}) = \psi_0 D \psi_1$ *or* $\Phi(\mathbf{n}) = \psi_0 T \psi_1$*;*

- *If there is a node* $\mathbf{n} \in B$ *such that* $\nu(\mathbf{n}) = ((\neg(\psi_0 C \psi_1), [c_i, c_j]), \mathbb{C}, u)$*, then for all* $c \in \mathbb{C}_B$ *such that* $c_i \leq c \leq c_j$*, there is a node* $\mathbf{n}' \in B$ *such that* $\nu(\mathbf{n}') = ((\neg\psi_0, [c_i, c]), \mathbb{C}', u')$ *or a node* $\mathbf{m}' \in B$ *such that* $\nu(\mathbf{m}') = ((\neg\psi_1, [c, c_j]), \mathbb{C}', u')$*;*

- *Similarly for every node* $\mathbf{n}$ *with* $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$ *or* $\Phi(\mathbf{m}) = \neg(\psi_0 T \psi_1)$*.*

Lemma 57 *If the limit tableau for some formula* $\phi \in$BCDT$^+$ *is closed, then some finite tableau for* ϕ *is closed.*

Proof.

Suppose the limit tableau for ϕ is closed. Then every branch closes at some finite step of the construction and then remains finite. Since the branch-expansion rule always produces finitely many successors, every finite tableau is finitely branching, and hence so is the limit tableau. Then, by König's lemma, the limit tableau, being a finitely branching tree with no infinite branches, must be finite, hence its construction stabilizes at some finite stage. At that stage a closed tableau for ϕ is constructed. ■

Theorem 58 *Let* $\phi \in$BCDT$^+$ *be a valid formula. Then there is a closed tableau for* $\neg\phi$*.*

Proof.

We will show that the limit tableau $\overline{\mathcal{T}}$ for $\neg\phi$ is closed, whence the claim follows by the previous lemma.

By contraposition, suppose that $\overline{\mathcal{T}}$ has an open branch B. Let $\mathbb{C}_B$ be the interval structure associated with B and $S(B)$ be the set of all labeled formulas on B. Consider the model $\mathbf{M}^+ = \langle \mathbb{C}_B, \mathbb{I}(\mathbb{C}_B)^+, V \rangle$ where, for every $[c_i, c_j] \in \mathbb{I}(\mathbb{C}_B)^+$ and $p \in \mathcal{AP}$, $p \in V([c_i, c_j])$ iff $(p, [c_i, c_j]) \in \Phi(B)$. We show by induction on ψ that, for every $(\psi, [c_i, c_j]) \in S(B)$, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

We reason by induction on the complexity of ψ:

- Let $\psi = \pi$ (resp., $\psi = \neg\pi$). Since $(\pi, [c_i, c_j]) \in S(B)$ (resp., $(\neg\pi, [c_i, c_j]) \in S(B)$) and B is open, then $c_i \neq c_j$ (resp., $c_i = c_j$). Hence $\mathbf{M}^+, [c_i, c_j] \Vdash \pi$ (resp., $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\pi$);

- Let $\psi = p$ or $\psi = \neg p$ where $p \in \mathcal{AP}$. Then the claim follows by definition, because if $(\neg p, [c_i, c_j]) \in S(B)$ then $(p, [c_i, c_j]) \notin S(B)$ since B is open;

- Let $\psi = \neg\neg\xi$. Then by Corollary 56, $(\xi, [c_i, c_j]) \in S(B)$, and by inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \xi$. So $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$;

- Let $\psi = \xi_0 \wedge \xi_1$. Then by Corollary 56, $(\xi_0, [c_i, c_j]) \in S(B)$ and $(\xi_1, [c_i, c_j]) \in S(B)$. By inductive hypothesis, $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_0$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$;

- Let $\psi = \neg(\xi_0 \wedge \xi_1)$. Then by Corollary 56, $(\neg\xi_0, [c_i, c_j]) \in S(B)$ or $(\neg\xi_1, [c_i, c_j]) \in S(B)$. By inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_0$ or $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$;

- Let $\psi = \xi_0 C \xi_1$. Then by Corollary 56, $(\xi_0, [c_i, c]) \in S(B)$ and $(\xi_1, [c, c_i]) \in S(B)$ for some $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$. Thus, by inductive hypothesis, $\mathbf{M}^+, [c_i, c] \Vdash \xi_0$ and $\mathbf{M}^+, [c, c_j] \Vdash \xi_1$, and thus $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$;

- Similarly for $\psi = \xi_0 D \xi_1$ and $\psi = \xi_0 T \xi_1$;

- Let $\psi = \neg(\xi_0 C \xi_1)$. Then by Corollary 56, for all $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$, $(\neg\xi_0, [c_i, c]) \in S(B)$ and $(\neg\xi_1, [c, c_j]) \in S(B)$. Hence, by the inductive hypothesis, $\mathbf{M}^+, [c_i, c] \Vdash \neg\xi_0$ and $\mathbf{M}^+, [c, c_j] \Vdash \neg\xi_1$, for all $c_i \leq c \leq c_j$. Thus, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$;

- Similarly for $\psi = \neg(\xi_0 D \xi_1)$ and $\psi = \neg(\xi_0 T \xi_1)$.

This completes the induction. In particular, we obtain that $\neg\phi$ is satisfied in $\mathbf{M}^+$, which is in contradiction with the assumption that ϕ is valid. ■

4.3 Reduction of the Tableau Method to $\mathcal{PNL}$

As we have seen, the logic BCDT$^+$ is a generalization of CDT for branching structures. This means that from the tableau method for BCDT$^+$ it can be tailored a tableau method for any (non-strict) propositional logic which can be viewed as a subsystem of CDT or BCDT$^+$. We show this fact for the logic PNL$^+$ [38].

It is not difficult to see that the modalities $\Diamond_r, \Diamond_l$ and their dual ones can be defined in BCDT$^+$. Indeed, we have:

- $\Box_r \phi \triangleq \neg(\neg\phi T \top)$;
- $\Box_l \phi \triangleq \neg(\neg\phi D \top)$.

In the same way, the tableau can be adapted to the case of PNL$^+$. First, the ordering in the labels of the nodes must be linear; second, the expanding rule for the modalities become as follows:

- If $\phi = \Box_r \psi$ and c is the least element of $\mathbb{C}_B$, with $c_j \leq c$, which has not been used yet to expand $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_1}$ with $\nu(\mathbf{n_1}) = ((\psi, [c_j, c]), \mathbb{C}_B, u)$.

- If $\phi = \Box_l\psi$ and c is the greatest element of $\mathbb{C}_B$, with $c \leq c_i$, which has not been used yet to expand $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_1}$ with $\nu(\mathbf{n_1})$ $= ((\psi, [c, c_i]), \mathbb{C}_B, u)$.

- If $\phi = \neg\Box_r\psi$, then expand the branch to $B \cdot \mathbf{n_j} \mid \ldots \mid \mathbf{n_n} \mid \mathbf{n'_j} \mid \ldots \mid \mathbf{n'_n}$, where

  1. for all $j \leq k \leq n$, $\nu(\mathbf{n_k}) = ((\neg\psi, [c_j, c_k]), \mathbb{C}_B, u)$, and
  2. for all $j \leq k \leq n$, $\nu(\mathbf{n'_k}) = ((\neg\psi, [c_j, c]), \mathbb{C}_k, u)$, where, for $j \leq k \leq n-1$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c between c_k and c_{k+1} in $\mathbb{C}_B$, and, for $k = n$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c after c_n in $\mathbb{C}_B$.

- if $\phi = \neg\Box_l\psi$, then expand the branch to $B \cdot \mathbf{n_1} \mid \ldots \mid \mathbf{n_i} \mid \mathbf{n'_1} \mid \ldots \mid \mathbf{n'_i}$, where:

  1. for all $1 \leq k \leq i$, $\nu(\mathbf{n_k}) = ((\neg\psi, [c_k, c_i]), \mathbb{C}_B, u)$, and
  2. for all $1 \leq k \leq i$, $\nu(\mathbf{n'_k}) = ((\neg\psi, [c, c_i]), \mathbb{C}_k, u)$, where, for $2 \leq k \leq i$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c between c_{k-1} and c_k in $\mathbb{C}_B$, and, for $k = 1$, $\mathbb{C}_1$ is the linear ordering obtained by inserting a new element c before c_1 in $\mathbb{C}_B$.

Clearly, the same can be done for PNL$^-$.

5

The Class of Split Logics

"To define recursion, we must first define recursion."
Anonymous

The problem of finding decidable interval logics has been raised by many authors. In this chapter we study a logic called Split Logic (SL), interpreted over particular classes of structures. Such structures are identified by means of a peculiar characteristic: every interval can be 'chopped' in at most one way. As we will see, the resulting logic is still very expressive, allowing one to write useful timing properties. Decidability of split logics has been proved by a correspondence with the first-order fragments of the monadic theories of time granularity proposed in the literature. Furthermore, we will establish the completeness of SL with respect to the guarded fragment of these theories. The major contribution consists in finding decidable interval logics without resorting to any "projection" principle. The main limitation of split structures is that a given interval can be divided at most into two consecutive intervals. But since such structures have no notion of measure, this limitation seems not to be too much significative. Indeed, we will show that split logics are able to express many of the examples present in the (propositional) interval logics literature.

5.1 Preliminaries

As anticipated in Chapter 2, here we define interval structures in a more general way. From now on, an interval structures is a pair $\langle \mathbb{D}, \mathbb{H}(\mathbb{D}) \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linear ordering (in this chapter we will exclude non-linear orderings), and $\mathbb{H}(\mathbb{D})$ is a subset of $\mathbb{I}(\mathbb{D})$. Also, for convenience, in this chapter we assume the strict semantics. We are interested in two particular unary relations between intervals, namely (see Section 1.2):

- the *subinterval* relation (for simplicity, we recall that such relation is defined as follows: $[d_0, d_1] \sqsubset [d_2, d_3]$ if and only if $d_2 \leq d_0 \wedge d_1 < d_3$ or $d_2 < d_0 \wedge d_1 \leq d_3$);
- the $\{m, b\}$ relation, defined as follows: $[d_0, d_1] \prec [d_2, d_3]$ iff $d_1 \leq d_2$ (this is the union of Allen's relations *meets* and *before*).

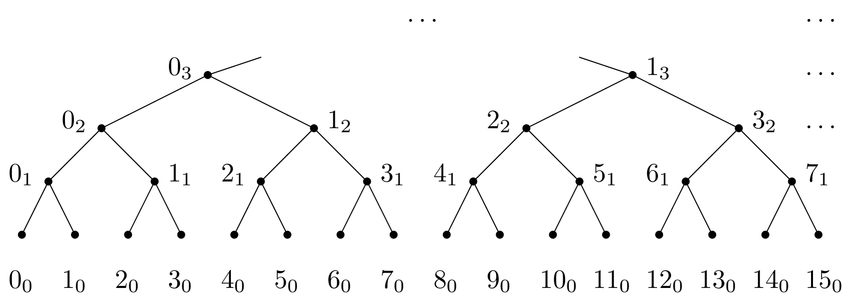


Figure 5.1: A 2-refinable upward unbounded layered structure.

Moreover, we will be considering the ternary relation $A(i, j, k)$, where i, j, k are intervals. We will be particularly interested in logics interpreted on structures based on $\mathbb{N}$ and $\mathbb{Q}^+$ with their usual orderings and with additional properties. The (new) properties we need are defined hereafter; an interval structure is said to be:

- **split** if every interval $[d_0, d_1]$ in $\mathbb{H}(\mathbb{D})$ is such that there are at most two intervals $[d_0, d_2]$ and $[d_2, d_1]$ with $d_0 \leq d_2 \leq d_1$;
- **atomic** if for every interval $[d_0, d_1]$ there exists an interval $[d_2, d_3]\sqsubset[d_0, d_1]$ such that no $[d_4, d_5]\sqsubset[d_2, d_3]$ exists. The interval $[d_2, d_3]$ is called an **atom** or an **atomic interval**;
- **well founded** if for no $[d_0, d_1]$ an infinite descending sequence $[d_0, d_1]\sqsupset [d_{i_1}, d_{j_1}] \sqsupset [d_{i_2}, d_{j_2}] \sqsupset \cdots$ exists[1];
- with **maximal intervals** if, for every $[d_0, d_1] \in \mathbb{H}(\mathbb{D})$, there is $[d_2, d_3] \in \mathbb{H}(\mathbb{D})$ such that $[d_0, d_1]\sqsubset[d_2, d_3]$ and there is no $[d_4, d_5] \in \mathbb{H}(\mathbb{D})$ such that it holds $[d_2, d_3]\sqsubset [d_4, d_5]$. The interval $[d_2, d_3]$ is called a **maximal interval**;

Notice that split structures do not contain *all* intervals; to use Venema's terminology, in these particular structures, every interval can be *chopped* in at most one way.

5.1.1 Monadic logics and layered structures

In this section we introduce classical monadic logics and interpret them over layered structures for time granularity.

Definition 59 *Let $\tau = c_1, \ldots, c_r, u_1, \ldots, u_s, b_1, \ldots, b_t$ be a finite alphabet of symbols, where $c_1, \ldots, c_r$ (resp. $u_1, \ldots, u_s$, $b_1, \ldots, b_t$) are constant symbols (resp. unary relational symbols, binary relational symbols) and let $\mathcal{P}$ be an alphabet of unary relational symbols. The second-order language with equality* $\mathrm{MSO}[\tau \cup \mathcal{P}]$ *is built up as follows:* **atomic formulas** *are of the forms $x = y$, $x = c_i$, with $1 \leq i \leq r$, $u_i(x)$, with*

[1]Notice that every well-founded interval structure is atomic.

$1 \leq i \leq s$, $b_i(x, y)$, with $1 \leq i \leq t$, $x \in X$, and $P(x)$, where x, y are individual variables, X is a set variable, and $P \in \mathcal{P}$; **formulas** *are built up from atomic formulas by means of the Boolean connectives $\neg$ and $\wedge$, and the quantifier $\exists$ ranging over both individual and set variables.*

In the following, we will write $MSO_{\mathcal{P}}[\tau]$ for $MSO[\tau \cup \mathcal{P}]$ and we will write $MSO[\tau]$ when $\mathcal{P}$ is meant to be the empty set. We interpret the above language over the following layered structures.

n-layered structures. Let $n, k \geq 1$. For every $i \geq 0$, let $T^i = \{j_i \mid j \geq 0\}$ and let $\mathcal{U}_n = \bigcup_{0 \leq i < n} T^i$. A k-refinable n**-layered structure** (n-LS) is a triplet $\langle \mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, that intuitively represents an infinite sequence of complete k-ary trees of height $n-1$, each one rooted at a point of T^0. The sets $\{T^i\}_{0 \leq i < n}$ are the layers of the trees, $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_{n-1}) = \perp$ and $\downarrow_i (a_b) = c_d$ if and only if $b < n-1$, $d = b+1$, and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{U}_n$ given by the *preorder* (root-left-right) visit of the nodes (for elements belonging to the same tree) and by the total linear ordering of trees (for elements belonging to different trees). A **path** over an n-LS is a subset of the domain whose elements can be written as a sequence $x_0, x_1, \ldots x_m$, with $m \leq n-1$, such that, for every $i = 1, \ldots m$, there exists $0 \leq j < k$ such that $x_i = \downarrow_j (x_{i-1})$. A **chain** is any subset of a path. A $\mathcal{P}$-labeled k-refinable n-layered structure is a tuple $\langle \mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P \in \mathcal{P}} \rangle$, where, $\downarrow_i$ and $<$ are defined as above and, for every $P \in \mathcal{P}$, $P \subseteq \mathcal{U}_n$ is the set of points in $\mathcal{U}_n$ labeled with symbol P. MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. MSO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$) is interpreted over k-refinable n-LSs $\langle \mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, < \rangle$ (resp. $\mathcal{P}$-labeled k-refinable n-LSs $\langle \mathcal{U}_n, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P \in \mathcal{P}} \rangle$) in the obvious way. The decidability of the theory MSO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ over n-LSs has been proved in [63] by reducing it to the monadic second-order theory of one successor, which is known to be (nonelementarily) decidable [89].

Theorem 60 MSO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over n-LSs is (nonelementarily) decidable.*

Upward unbounded layered structures. Let $\mathcal{U} = \bigcup_{i \geq 0} T^i$. A k-refinable **upward unbounded layered structure** (UULS) is a triplet $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, that intuitively represents a complete k-ary infinite tree generated from the leaves (cf. Figure 6.1). The sets $\{T^i\}_{i \geq 0}$ are the layers of the tree, $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_0) = \perp$ and $\downarrow_i (a_b) = c_d$ if and only if $b > 0$, $b = d+1$, and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{U}$ given by the *inorder* (left-root-right) visit of the treelike structure. A **path** over an UULS is a subset of the domain whose elements can be written as a possibly infinite sequence $x_0, x_1, \ldots$ such that, for every $i \geq 1$, there exists $0 \leq j < k$ such that $x_{i-1} = \downarrow_j (x_i)$. A **chain** is any subset of a path. Notice that every pair of infinite paths over an UULS may differ on a finite prefix only. A $\mathcal{P}$-labeled k-refinable UULS is obtained by augmenting an UULS with a set $P \subseteq \mathcal{U}$, for any $P \in \mathcal{P}$ (the elements of the structure labeled by P). MSO$[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. MSO$_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$) is interpreted over k-refinable UULSs $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$ (resp. $\mathcal{P}$-labeled k-refinable UULSs $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P \in \mathcal{P}} \rangle$). The decidability of the second-order theory of UULSs has been proved in [61] by a reduction to a decidable proper extension of the monadic second-order theory of one successor [62].

Theorem 61 $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over UULSs is (nonelementarily) decidable.*

Downward unbounded layered structures. Let $\mathcal{U} = \bigcup_{i \geq 0} T^i$. A k-refinable **downward unbounded layered structure** (DULS) is a triplet $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$, that intuitively represents an infinite sequence of complete k-ary infinite trees, each one rooted at a point of T^0 (cf. Figure 6.2). The sets $\{T^i\}_{i \geq 0}$ are the layers of the trees, $\downarrow_i$, with $i = 0, \ldots, k-1$, is a projection function such that $\downarrow_i (a_b) = c_d$ if and only if $d = b + 1$ and $c = a \cdot k + i$, and $<$ is the total ordering of $\mathcal{U}$ given by the *preorder* (root-left-right) visit of the nodes (for elements belonging to the same tree) and by the total linear ordering of trees (for elements belonging to different trees). A **path** over a DULS is a subset of the domain whose elements can be written as a possibly infinite sequence $x_0, x_1, \ldots$ such that, for every $i \geq 1$, there exists $0 \leq j < k$ such that $x_i = \downarrow_j (x_{i-1})$. A **chain** is any subset of a path. A $\mathcal{P}$-labeled k-refinable DULS is obtained by augmenting a DULS with a set $P \subseteq \mathcal{U}$, for any $P \in \mathcal{P}$ (the elements of the structure labeled by P). $\mathrm{MSO}[<, (\downarrow_i)_{i=0}^{k-1}]$ (resp. $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$) is interpreted over k-refinable DULSs $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, < \rangle$ (resp. $\mathcal{P}$-labeled k-refinable DULSs $\langle \mathcal{U}, (\downarrow_i)_{i=0}^{k-1}, <, (P)_{P \in \mathcal{P}} \rangle$). The decidability of the second-order theory of DULSs has been proved in [61] by reducing it to the monadic second-order theory of k successors, which is known to be (nonelementarily) decidable [89].

Theorem 62 $\mathrm{MSO}_{\mathcal{P}}[<, (\downarrow_i)_{i=0}^{k-1}]$ *over DULSs is (nonelementarily) decidable.*

In this paper we are interested in the first-order fragments of those theories, namely, $\mathrm{MFO}[<_1, <_2, \{\downarrow_i\}_{0 \leq i < k}]$ interpreted over k-refinable n-LSs and DULSs, and $\mathrm{MFO}[<_2, \{\downarrow_i\}_{0 \leq i < k}]$ interpreted over k-refinable UULSs. The symbols in the square brackets are (pre)interpreted as follows: for $0 \leq i < k$, $\downarrow_i (x, y)$ is a binary relation (actually, it is a functional relation; so, we will often write $\downarrow_i (x) = y$ instead of $\downarrow_i (x, y)$) such that y is the i-th point in the refinement of x; $<_1$ is a strict partial order such that $x <_1 y$ when x is in a tree preceding the tree containing y; $x <_2 y$ holds when y is a descendant of x.

5.2 Split Logics: Syntax and Semantics

The language of SL is based on a modal similarity type with four unary operators $\langle D \rangle$, $\langle \overline{D} \rangle$, $\langle F \rangle$, and $\langle \overline{F} \rangle$, and three binary operators C, D, and T. The former are borrowed from HS, while the latter are the 'irreflexive' variants of the operators of CDT [95]. The well-formed formulae, denoted by $\varphi, \phi, \ldots$, of SL are given by the following grammar:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg \varphi \mid \langle D \rangle \varphi \mid \langle \overline{D} \rangle \varphi \mid \langle F \rangle \varphi \mid \langle \overline{F} \rangle \varphi \mid \varphi C \varphi \mid \varphi D \varphi \mid \varphi T \varphi,$$

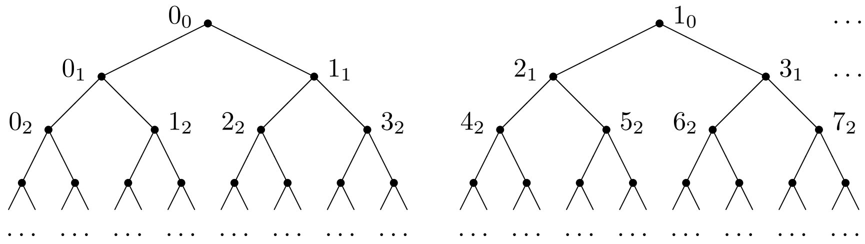


Figure 5.2: A 2-refinable downward unbounded layered structure.

where $p \in \mathcal{AP}$, and $\mathcal{AP}$ is a denumerable set of propositional letters. We also assume the existence of the two constants $\top$ and $\bot$ as usual. For a unary modality $\langle X \rangle$, we denote with $[X]\varphi$ the formula $\neg\langle X \rangle\neg\varphi$.

A split logic is interpreted over a split interval structures. In the following we will be considering three classes of structures, and we separately give interesting result for them. A **split-model** $\mathbf{M}^- = \langle \mathbb{D}, \mathbb{H}(\mathbb{D}), V \rangle$, where $\langle \mathbb{D}, \mathbb{H}(\mathbb{D}) \rangle$ is a bounded below, unbounded above split interval structure, is equipped with a **valuation function** $V : \mathcal{AP} \mapsto 2^{\mathbb{H}(\mathbb{D})}$. The semantics of well-formed formulas is given by the following clauses, for an interval model $\mathbf{M}^-$ and interval i:

- $\mathbf{M}^-, i \Vdash p$ if $i \in V(p)$, where $p \in \mathcal{AP}$;
- $\mathbf{M}^-, i \Vdash \neg\varphi$ if $\mathbf{M}^-, i \nVdash \varphi$;
- $\mathbf{M}^-, i \Vdash \varphi \wedge \psi$ if $\mathbf{M}^-, i \Vdash \varphi$ and $\mathbf{M}^-, i \Vdash \psi$;
- $\mathbf{M}^-, i \Vdash \langle D \rangle\varphi$ if for some j we have $j \sqsubset i$ and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \langle \overline{D} \rangle\varphi$ if for some j we have $i \sqsubset j$ and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \langle F \rangle\varphi$ if for some j we have $i \prec j$ and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \langle \overline{F} \rangle\varphi$ if for some j we have $j \prec i$ and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \varphi C \psi$ if for some j, k we have $A(j, k, i)$ and $\mathbf{M}^-, j \Vdash \varphi$ and $\mathbf{M}^-, k \Vdash \psi$;
- $\mathbf{M}^-, i \Vdash \varphi T \psi$ if for some j, k we have $A(i, j, k)$ and $\mathbf{M}^-, j \Vdash \varphi$ and $\mathbf{M}^-, k \Vdash \psi$;
- $\mathbf{M}^-, i \Vdash \varphi D \psi$ if for some j, k we have $A(j, i, k)$ and $\mathbf{M}^-, j \Vdash \varphi$ and $\mathbf{M}^-, k \Vdash \psi$.

Satisfiability and validity are defined in the standard way. It is also possible to introduce the modal constant π (which informally holds over all and only atomic intervals) as an abbreviation for the formula $\neg\langle D \rangle\top$. Notice that this variant of π do not hold over point-interval (which are excluded in our interpretations), but it does over intervals which do not contain other intervals.

Among all possible structures for interpreting a split logic, we are particularly interested in three cases:

1. for a fixed $n \geq 1$, let $\mathrm{INT_n}(\mathbb{N})$ be the set of all and only the intervals on $\langle \mathbb{N}, < \rangle$ that satisfy the following condition: for every $0 \leq i \leq n$, if $a = 2^i k$ with $k \in \mathbb{N}$, then $[a, a + 2^i] \in \mathrm{INT_n}(\mathbb{N})$. $\mathrm{INT_n}(\mathbb{N})$ is the set of all (strict) intervals on $\mathbb{N}$ which length is bounded by 2^n, and intervals with the same length are pairwise disjoint. The length of an interval $[a, b]$ is $len([a, b]) = b - a$. The split logic interpreted in the class of all interval structures isomorphic to $\langle \mathbb{N}, \mathrm{INT_n}(\mathbb{N})\rangle$ for some $n \geq 1$ is denoted by SL^n;

2. let $\mathrm{INT}_\infty(\mathbb{N})$ be the set of all and only the (strict) intervals over $\langle \mathbb{N}, < \rangle$ such that for every $i \geq 0$, if $a = 2^i k$, with $k \in \mathbb{N}$, then $[a, a + 2^i] \in \mathrm{INT}_\infty(\mathbb{N})$. $\mathrm{INT}_\infty(\mathbb{N})$ can be seen as the split domain obtained as the (infinite) union, for every n, of the split-domains $\mathrm{INT_n}(\mathbb{N})$. The split logic interpreted over the class of all split structures isomorphic to $\langle \mathbb{N}, \mathrm{INT_n}(\mathbb{N})\rangle$ will be denoted by SL^{uu};

3. given the linear ordering $\langle \mathbb{Q}^+, < \rangle$, let $\mathrm{INT}(\mathbb{Q}^+)$ be the set of intervals recursively built as follows:

   - for every integer $n \geq 0$, $[n, n + 1] \in \mathrm{INT}(\mathbb{Q}^+)$;
   - if $[x, y] \in \mathrm{INT}(\mathbb{Q}^+)$, then $[x, (x + y)/2], [(x + y)/2, y] \in \mathrm{INT}(\mathbb{Q}^+)$.

   The split logic interpreted over the class of split structures isomorphic to $\langle \mathbb{Q}^+, \mathrm{INT}(\mathbb{Q}^+)\rangle$ will be denoted by SL^{du}.

SL is expressive enough to capture useful timing properties. Conditions of the form "from now on, it will be true that any occurrence of stop is always preceded by an occurrence of start" (we found it in the context of the specification of a time-triggered protocol which allows a fixed number of stations to communicate via a shared bus) can be expressed as follows:

$$[F](\mathsf{stop} \rightarrow \langle \overline{F} \rangle \mathsf{start}).$$

As a second example, consider the following sentence: "During the run, the robot stopped to grasp the block", which can be expressed in SL as follows:

$$[F]\big(\mathsf{running} \wedge \langle D \rangle(\neg\mathsf{running} \wedge \mathsf{taking\text{-}block})\big).$$

Note that in other logics, where the projection principle of homogeneity is assumed, this formula would evaluate to false.

As for the relationships between SL operators, we know that the operators of CDT can be used to define the ones of HS [95], provided that both logics are interpreted over interval structures including "all" the intervals. For instance, the subinterval relation can be apparently captured with two applications of the *chop* operator, the first selecting a beginning or ending subinterval such that one endpoint matches an endpoint of the target interval and the second dividing the previously found subinterval at the other endpoint of the target interval. On the contrary, if they are interpreted over split structures, by chopping an interval one cannot get an arbitrary smaller subinterval, and hence one cannot go as deep in the inclusion chain as one would like. In Section 5.4 we will be discussing inter-definability of the operators in split logic.

5.3 Decidability Results for Some Split Logics

5.3.1 Discrete Frames and n-LSs

Given a first-order formula φ we denote by $\varphi\{x/t\}$ the substitution of the free occurrences of the variable x with the term t. Given an assignment σ mapping variables to elements of the domain, we denote by $\sigma[x_1/d_1, \ldots, x_n/d_n]$ the assignment which differs from σ, if at all, only because it assigns $d_1, \ldots, d_n$ to $x_1, \ldots, x_n$, respectively. We now provide a translation of SL-formulae into $\mathrm{MFO}[<_1, <_2, \downarrow_0, \downarrow_1]$ (cf. Section 5.1.1). Let $p \in \mathcal{AP}$ be a proposition letter and φ and ψ be SL-formulae. The translation τ is defined by the following inductive clauses:

- $\tau(p) = P(x)$, where $p \in AP$ and P is a unary predicate symbol;
- $\tau(\neg\varphi) = \neg(\tau(\varphi))$;
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$;
- $\tau(\langle D\rangle\varphi) = \exists y\, (x <_2 y \wedge \tau(\varphi)\{x/y\})$;
- $\tau(\langle \overline{D}\rangle\varphi) = \exists y\, (y <_2 x \wedge \tau(\varphi)\{x/y\})$;
- $\tau(\langle F\rangle\varphi) = \exists y\, \big((x <_1 y \wedge \tau(\varphi)\{x/y\}) \vee \exists z\, (\downarrow_0 (z) \leq_2 x \wedge \downarrow_1 (z) \leq_2 y \wedge \tau(\varphi)\{x/y\})\big)$;
- $\tau(\langle \overline{F}\rangle\varphi) = \exists y\, \big((y <_1 x \wedge \tau(\varphi)\{x/y\}) \vee \exists z\, (\downarrow_0 (z) \leq_2 y \wedge \downarrow_1 (z) \leq_2 x \wedge \tau(\varphi)\{x/y\})\big)$;
- $\tau(\varphi C \psi) = \exists y \exists z \big(\downarrow_0 (x) = y \wedge \downarrow_1 (x) = z \wedge \tau(\varphi)\{x/y\} \wedge \tau(\psi)\{x/z\}\big)$;
- $\tau(\varphi T \psi) = \exists y \exists z \big(\downarrow_0 (z) = x \wedge \downarrow_1 (z) = y \wedge \tau(\varphi)\{x/y\} \wedge \tau(\psi)\{x/z\}\big)$;
- $\tau(\varphi D \psi) = \exists y \exists z \big(\downarrow_0 (z) = y \wedge \downarrow_1 (z) = x \wedge \tau(\varphi)\{x/y\} \wedge \tau(\psi)\{x/z\}\big)$.

The idea is to map split models for SL^n onto 2-refinable $(n+1)$-layered structures in order to exploit decidability results for granular structures. Every point in a layered structure can be determined by a pair of coordinates s_t, where t is the t-th domain and s is the distance from the origin of that domain, also called the *displacement* of the point in the domain. The function $\delta\colon \mathrm{INT_n}(\mathbb{N}) \rightarrow \bigcup_{i=0}^{n} T^i$ defined, for every $[a, b] \in \mathrm{INT_n}(\mathbb{N})$, by $\delta([a, b]) = s_t$, where $s = a/len([a, b])$ and $t = n - \lg_2 len([a, b])$ is an effective bijection, as shown by the following result.

Proposition 63 *The function δ is a computable bijection.*

Proof.

The computability of δ is trivial since δ is the composition of computable functions. To prove that it is a bijection, consider the function $\delta^{-1}\colon \bigcup_{i=0}^{n} T^i \to \mathrm{INT_n}(\mathbb{N})$ defined by

$$\delta^{-1}(s_t) = \big[2^{n-t}s, 2^{n-t}(s+1)\big].$$

For every interval $[a,b] \in \mathrm{INT_n}(\mathbb{N})$ we have

$$\begin{aligned} & \delta^{-1}\big(\delta([a,b])\big) = \delta^{-1}\big((a/len([a,b]))_{n-\lg_2 len([a,b])}\big) = \\ = \quad & \Big[\frac{a}{len([a,b])}len([a,b]), len([a,b])\big(\frac{a}{len([a,b])}+1\big)\Big] = \\ = \quad & \big[a, a+len([a,b])\big] = [a,b], \end{aligned}$$

and for every $s_t \in \bigcup_{i=0}^{n} T^i$ we have

$$\begin{aligned} & \delta(\delta^{-1}(s_t)) = \delta\big([2^{n-l}s, 2^{n-t}(s+1)]\big) = \\ = \quad & \Big(\frac{2^{n-t}s}{2^{n-t}(s+1)-2^{n-t}d}\Big)_{n-\lg_2(2^{n-l}(s+1)-2^{n-t}s)} = \\ = \quad & s_t. \end{aligned}$$

■

We need the following technical lemmas and definitions.

Definition 64 *Let* $\mathbf{M}^- = \langle \mathrm{INT_n}(\mathbb{N}), V\rangle$ *be a split model and let* $(S,\mathcal{I})$ *be a* 2*-refinable* $(n+1)$-LS S *for* MFO$[<_1, <_2, \downarrow_0, \downarrow_1]$*-formulae paired with an interpretation* $\mathcal{I}$ *of predicate symbols. We say that* $(S,\mathcal{I})$ **corresponds** *to* $\mathbf{M}^-$ *if the following properties are satisfied:*

1. *the domain* $\bigcup_{i=0}^{n} T^i$ *is the mapping through* δ *of* $\mathrm{INT_n}(\mathbb{N})$*;*
2. *for every propositional letter* p*, the interpretation* $\mathcal{I}(\tau(P))$ *of the predicate* $\tau(P)$ *is* $\delta(V(p))$*, where, given a set* I *of intervals,* $\delta(I) = \{\, \delta(i) \mid i \in I \,\}$*.*

Lemma 65 *Given a split model* $\mathbf{M}^-$ *based on a structure isomorphic to* $\langle \mathbb{N}, \mathrm{INT_n}(\mathbb{N})\rangle$*, an interval* i *of* $\mathbf{M}^-$ *and a layered structure* $(S,\mathcal{I})$ *that corresponds to* $\mathbf{M}^-$*, the following equivalence holds:*

$$\mathbf{M}^-, i \Vdash \top C \top$$

if and only if two intervals j *and* k *exist such that*

$$(S,\mathcal{I}), \sigma[x/\delta(i), y/\delta(j), z/\delta(k)] \models \downarrow_0(x) = y \wedge \downarrow_1(x) = z.$$

Proof.

If $\mathbf{M}^-, i \Vdash \top C \top$ then two intervals j and k exist such that $A(j,k,i)$ holds. Let $j = [a,b]$, $k = [c,d]$ and $i = [e,f]$ and let $\delta(j) = (s_j)_{t_j}$, $\delta(k) = (s_k)_{t_k}$ and $\delta(i) = (s_i)_{t_i}$. If $A(j,k,i)$ holds, then $a = e$, $v = f$, $b = c$ and $len(j) = len(k)$. By the definition of δ, the last equation implies $t_j = t_k$. Therefore, $\delta(j)$ and $\delta(k)$ lie on the same layer. Observing that $b = a + len(j)$ we may write

$$d_k = \frac{c}{len(k)} = \frac{b}{len(j)} = \frac{a + len(j)}{len(j)} = \frac{a}{len(j)} + 1 = d_j + 1,$$

so that $\delta(k)$ is displaced by one point with respect to $\delta(j)$. The layer $\delta(i)$ lies on is

$$\begin{aligned} t_i = n - \lg_2 len(i) &= n - \lg_2(len(j) + len(k)) = \\ &= n - \lg_2(2\, len(j)) = \\ &= n - \lg_2 len(j) - 1 = t_j - 1, \end{aligned}$$

so $\delta(i)$ lies one layer above $\delta(j)$ (and one layer above $\delta(k)$ too, by the previous remark). The displacement of $\delta(i)$ is

$$d_i = \frac{e}{len(i)} = \frac{a}{2\, len(j)} = \frac{s_j}{2},$$

so it is half the displacement of $\delta(j)$. That proves that $\delta(i)$ is the father of $\delta(j)$ and $\delta(k)$ (in the view of n-LSs as trees).

Viceversa, let $(S, \mathcal{I}), \sigma[x/\delta(i), y/\delta(j), z/\delta(k)] \models \downarrow_0 (x) = y \wedge \downarrow_1 (x) = z$. From the semantics of the relational symbols $\downarrow_0, \downarrow_1$ we have that $\delta(j)$ and $\delta(k)$ are the left son and the right son of $\delta(i)$, respectively. Therefore:

1. $\delta(j)$ and $\delta(k)$ lie one layer beneath $\delta(i)$ (that is $t_i < t_j$ and $t_i < t_k$);
2. the displacement of $\delta(j)$ is twice the displacement of $\delta(i)$;
3. $\delta(k)$ is displaced by one point to the right of $\delta(j)$.

Let $j = [a,b]$, $k = [c,d]$ and $i = [e,f]$, $\delta(j) = (s_j)_{t_j}$, $\delta(k) = (s_k)_{t_k}$ and $\delta(i) = (s_i)_{t_i}$. The first point is expressed by

$$t_j = t_k = t_i + 1,$$

which implies

$$n - \lg_2 len(j) = n - \lg_2 len(k) = n - \lg_2 len(i) + 1.$$

Simplifying, we get

$$\lg_2 len(j) = \lg_2 len(k) = \lg_2 \frac{len(i)}{2},$$

that is

$$len(j) = len(k) = \frac{len(i)}{2}.$$

The second point is

$$s_j = 2\, s_i \Rightarrow \frac{s}{len(j)} = \frac{2e}{len(i)}.$$

But we have just proved that $len(j) = len(i)/2$, so we have $a = e$. The third point is formalized as follows:

$$s_k = s_j + 1 \Rightarrow \frac{u}{len(k)} = \frac{a}{len(j)} + 1 = \frac{a + len(j)}{len(j)} = \frac{b}{len(j)}.$$

Since $len(j) = len(k)$, we get $b = c$. Finally,

$$\begin{aligned} f &= e + len(i) = e + len(j) + len(k) = \\ &= a + len(j) + len(k) = b + len(k) = c + len(k) = d. \end{aligned}$$

The conjunction of the previous assertions allows us to conclude that $A(j, k, i)$ holds; hence, $\mathbf{M}^-, i \Vdash \top C \top$. ■

Lemma 66 *Given a split model $\mathbf{M}^-$ based on a structure isomorphic to $\langle \mathbb{N}, \mathrm{INT_n}(\mathbb{N})$ $\rangle$, an interval j of $\mathbf{M}^-$ and a layered structure $(S, \mathcal{I})$ that corresponds to $\mathbf{M}^-$, the following equivalence holds:*

$$\mathbf{M}^-, j \Vdash \langle D \rangle \top$$

if and only if an interval i exists such that

$$(S, \mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x,$$

and

$$\mathbf{M}^-, j \Vdash \langle \overline{F} \rangle \top$$

if and only if an interval i exists such that

$$(S, \mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models x <_1 y \vee \exists z\, (\downarrow_0 (z) \leq_2 x \wedge \downarrow_1 (z) \leq_2 y).$$

Proof.

For the sake of simplicity, in this proof we write S instead of $(S, \mathcal{I})$.

If $\mathbf{M}^-, j \Vdash \langle D \rangle \top$, then an interval i exists such that $i \sqsubset j$. Let $i = [a, b]$ and $j = \langle [c, d]$. The inclusion implies the relations $c \leq a$ and $b \leq d$; besides, at least one of them must be strict. We must show that $\delta(i)$ is a descendant of $\delta(j)$. Let $\delta(i) = (d_i)_{l_i}$ and $\delta(j) = (d_j)_{l_j}$. Since $len(i) < len(j)$, from the definition of δ we have that $l_j < l_i$. In order to prove that $\delta(i)$ and $\delta(j)$ are in the same path it is sufficient to show that

$$2^{t_i - t_j} s_j \leq s_i \leq 2^{t_i - t_j} (s_j + 1) - 1.$$

Substituting in the previous inequality leads to

$$\frac{len(j)}{len(i)} \cdot \frac{c}{len(j)} \leq \frac{a}{len(i)} \leq \frac{len(j)}{len(i)} \Big(\frac{c}{len(j)} + 1 \Big) - 1,$$

and simplifying

$$c \leq a \leq c + len(j) - len(i).$$

The left inequality is satisfied because $c \leq a$ by hypothesis. The right inequality follows from $c + len(j) = d$ and $len(i) = b - a$ and from the fact that $b \leq d$ by hypothesis.

Viceversa, let $S, \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x$. The inverse function δ^{-1} associates the intervals $[2^{n-t_i}s_i, 2^{n-t_i}(s_i+1)]$ and $[2^{n-t_j}s_j, 2^{n-t_j}(d_j+1)]$ to the considered nodes. By hypothesis, $\delta(i)$ and $\delta(j)$ are in the same path, with $\delta(j)$ above $\delta(i)$. So we have:

1. $t_j < t_i$;
2. $2^{t_i - t_j} s_j \leq s_i$;
3. $s_i \leq 2^{t_i - t_j}(s_j + 1) - 1$.

From (1) $len(i) < len(j)$ follows. From the inequality (2), by multiplying both sides by 2^{n-t_i}, we get $2^{n-t_j}s_j \leq 2^{n-t_i}s_i$. From (3), by summing one to both sides and multiplying by 2^{n-t_i} we get $2^{n-t_i}(s_i+1) \leq 2^{n-t_j}(s_j+1)$. Which allows us to conclude that $i \sqsubset j$ and therefore $M, j \Vdash \langle D \rangle \top$.

If $\mathbf{M}^-, j \Vdash \langle \overline{F} \rangle \top$, then there exists i such that $i \prec j$. If there is an interval k such that $i, j \sqsubset k$, then there exist $k' \sqsubset k$ and i', j' such that $A(i', j', k')$ holds and $i \sqsubset i'$ and $j \sqsubset j'$. By applying the just proved equivalence and Lemma 65, we get

$$\begin{aligned} S, \sigma[z/\delta(k'), v/\delta(i'), w/\delta(j')] &\models \downarrow_0 (z) = v \wedge \downarrow_1 (z) = w; \\ S, \sigma[x/\delta(i), v/\delta(i')] &\models v \leq_2 x; \\ S, \sigma[y/\delta(j), w/\delta(j')] &\models w \leq_2 y. \end{aligned}$$

After applying the semantic rules for conjunction and existential quantifier and simplifying the resulting formula we get

$$S, \sigma[x/\delta(i), y/\delta(j)] \models \exists z (\downarrow_0 (z) \leq_2 x \wedge \downarrow_1 (z) \leq_2 y).$$

If, on the contrary, there is no interval containing both i and j then there are two maximal intervals k, l such that $i \sqsubset k$ and $j \sqsubset l$ and, by the monotonicity of interval structures, $k \prec l$. Such intervals are mapped into two nodes in the first layer of the granular structure; the relation $<_1$ restricted to the first layer is a total ordering. One can easily check that the displacement of k is less than the displacement of l; hence, $S, \sigma[x/\delta(i), y/\delta(j)] \models x <_1 y$.

Viceversa, let $S, \sigma[x/\delta(i), y/\delta(j)] \models x <_1 y$. Then $\delta(i)$ and $\delta(j)$ are in two different trees. Let r, r' their roots, for which $r <_1 r'$ holds. For the matching intervals $k = \delta^{-1}(r)$ and $k' = \delta^{-1}(r')$ the relation $k \prec k'$ holds. Since $i \sqsubset k$ and $j \sqsubset k'$ necessarily, by the monotonicity of interval structures we get $i \prec j$ and so $M, j \Vdash \langle \overline{F} \rangle \top$. Finally, let $S, \sigma[x/\delta(i), y/\delta(j)] \models \exists z (\downarrow_0 (z) \leq_2 x \wedge \downarrow_1 (z) \leq_2 y)$. It is sufficient to go backwards through the previous proof the get the thesis. ■

We are ready to prove the following equi-satisfiability result.

Theorem 67 *Let $\varphi \in \mathrm{SL}^n$. Then, φ is satisfiable if and only if $\tau(\varphi)$ is satisfiable over 2−refinable $(n+1)$-LS.*

Proof.

Let $\mathbf{M}^-$ be a split model for SL^n. We show that $\mathbf{M}^-, i \Vdash \varphi$ if and only if $(S, \mathcal{I}), \sigma[x/\delta(i)] \models \tau(\varphi)$, where $(S, \mathcal{I})$ is a 2-refinable $(n+1)$-LS that corresponds to $\mathbf{M}^-$.

The proof proceeds by induction. For the sake of simplicity, we write S instead of $(S, \mathcal{I})$. The proof also shows that the translation is given into the subset of formulae containing exactly one free variable x, which intuitively denotes the current interval.

- $\mathbf{M}^-, i \Vdash p$ if and only if $i \in V(p)$ if and only if $\delta(i) \in \delta(V(p))$ if and only if (by Definition 64) $S, \sigma[x/\delta(i)] \models P(x)$; $\tau(p) \equiv P(x)$ clearly contains one free variable x.

- $\mathbf{M}^-, i \Vdash \neg\psi$ if and only if $\mathbf{M}^-, i \nVdash \psi$ if and only if, by induction, $S, \sigma[x/\delta(i)] \not\models \tau(\psi)$ if and only if $S, \sigma[x/\delta(i)] \models \neg\tau(\psi)$. Besides, by the inductive hypothesis, $\tau(\psi)$ contains only one free variable, namely x: therefore, also in $\neg\tau(\psi)$ the only variable occurring free is x;

- $\mathbf{M}^-, i \Vdash \psi \wedge \vartheta$ if and only if $\mathbf{M}^-, i \Vdash \psi$ and $\mathbf{M}^-, i \Vdash \vartheta$ if and only if, by induction, $S, \sigma[x/\delta(i)] \models \tau(\psi)$ and $S, \sigma[x/\delta(i)] \models \tau(\vartheta)$ if and only if $S, \sigma[x/\delta(i)] \models \tau(\psi) \wedge \tau(\vartheta)$. Besides, by the inductive hypothesis, in $\tau(\psi)$ and $\tau(\vartheta)$ the only variable occurring free is x: therefore, in their conjunction the only free variable is x;

- Suppose $\mathbf{M}^-, i \Vdash \langle D\rangle\psi$. Then there exists $j \sqsubset i$ such that $\mathbf{M}^-, j \Vdash \psi$ and, by the inductive hypothesis, $S, \sigma[x/\delta(j)] \models \tau(\psi)$ where x is the only free variable in $\tau(\psi)$. By the substitution lemma[2] we get $S, \sigma[y/\delta(j)] \models \tau(\psi)\{x/y\}$. Moreover, since $j \sqsubset i$ we have $\mathbf{M}^-, i \Vdash D\top$. By Lemma 66, $S, \sigma[x/\delta(i), y/\delta(j)] \models x <_2 y$; by applying the semantic rule for conjunction and observing that x does not occur free in $\tau(\psi)\{x/y\}$ (only y is free in it), we are able to conclude that $S, \sigma[x/\delta(i), y/\delta(j)] \models x <_2 y \wedge \tau(\psi)\{x/y\}$ and, by closing existentially, $S, \sigma[x/\delta(i)] \models \exists y \left(x <_2 y \wedge \tau(\psi)\{x/y\}\right)$. In the last formula the only free variable is x.

  As for the converse, suppose $S, \sigma[x/\delta(i)] \models \tau(\langle D\rangle\psi)$. Then there exists an element d in the domain of the layered structure such that $S, \sigma[x/\delta(i), y/d] \models x <_2 y \wedge \tau(\psi)\{x/y\}$. The element d is the mapping through δ of an interval j, so $S, \sigma[x/\delta(i), y/\delta(j)] \models x <_2 y$ and $S, \sigma[x/\delta(i), y/\delta(j)] \models \tau(\psi)\{x/y\}$. By the inductive hypothesis, y is the only free variable of the formula $\tau(\psi)\{x/y\}$ so the second truth relation is equivalent to $S, \sigma[y/\delta(j)] \models \tau(\psi)\{x/y\}$. By the substitution lemma and the inductive hypothesis, we have $\mathbf{M}^-, j \Vdash \psi$. On the other hand, by Lemma 66, the first truth relation implies $M, i \Vdash \langle D\rangle\top$, so $j \sqsubset i$. The semantic rule for $\langle D\rangle$ allows us to conclude $M, i \Vdash \langle D\rangle\psi$. The case of $\langle \overline{D}\rangle$ is similar, with i and j exchanged;

[2]We assume that if the variable to be replaced is not free for substitution in $\tau(\psi)$ then a renaming of the variables has been performed.

- Suppose that $\mathbf{M}^-, i \Vdash \langle \overline{F} \rangle \psi$. Then there exists $j \prec i$ such that $\mathbf{M}^-, j \Vdash \psi$. By induction, $S, \sigma[x/\delta(j)] \models \tau(\psi)$ (and x is the only free variable in $\tau(\psi)$) and by the substitution lemma $S, \sigma[y/\delta(j)] \models \tau(\psi)\{x/y\}$. Besides, $j \prec i$ implies $\mathbf{M}^-, i \Vdash \langle F \rangle \top$ and then, by Lemma 66, we have $S, \sigma[x/\delta(i), y/\delta(j)] \models y <_1 x \vee \exists z\,(\downarrow_0 (z) \leq_2 y \wedge \downarrow_1 (z) \leq_2 x)$. By applying the semantic rules for conjunction and existential quantifier we get $S, \sigma[x/\delta(i)] \models \exists y\,\big((y <_1 x \vee \exists z\,(\downarrow_0 (z) \leq_2 y \wedge \downarrow_1 (z) \leq_2 x)) \wedge \tau(\psi)\{x/y\}\big)$. An application of the De Morgan laws leads to the thesis. The only variable occurring free in the above formula is x.

  As for the converse, let $S, \sigma[x/\delta(i)] \models \tau(\langle \overline{F} \rangle \psi)$. Then there exists an element d in the domain such that $S, \sigma[x/\delta(i), y/d] \models (y <_1 x \vee \exists z\,(\downarrow_0 (z) \leq_2 y \wedge \downarrow_1 (z) \leq_2 x)) \wedge \tau(\psi)\{x/y\}$. On the other hand d is the image through δ of some interval j, so by Lemma 66 we have $\mathbf{M}^-, i \Vdash \langle \overline{F} \rangle \top$; hence, $j \prec i$ and by the inductive hypothesis (after noting that x does not occur free in $\tau(\psi)\{x/y\}$) we may conclude $\mathbf{M}^-, i \Vdash \langle \overline{F} \rangle \psi$. The case of $\langle F \rangle$ is similar, with the role of i and j exchanged;

- Suppose $\mathbf{M}^-, i \Vdash \psi C \vartheta$; then, there exist j, k such that $A(j,k,i)$ and $\mathbf{M}^-, j \Vdash \psi$ and $\mathbf{M}^-, k \Vdash \vartheta$. By induction, we have $S, \sigma[x/\delta(j)] \models \tau(\psi)$ and $S, \sigma[x/\delta(k)] \models \tau(\vartheta)$, with x (the only) free variable in $\tau(\psi)$ and $\tau(\vartheta)$. By the substitution lemma, $S, \sigma[y/\delta(j)] \models \tau(\psi)\{x/y\}$ and $S, \sigma[z/\delta(k)] \models \tau(\vartheta)\{x/z\}$. By the inductive hypothesis then, y is the only free variable in $\tau(\psi)$ and z is the only free variable in $\tau(\vartheta)$. On the other hand, since there exist j, k such that $C(j,k,i)$ holds, then $\mathbf{M}^-, i \Vdash \top C \top$ and, by Lemma 65, we have $S, \sigma[x/\delta(i), y/\delta(j), z/\delta(k)] \models \downarrow_0 (x) = y \wedge \downarrow_1 (x) = z$. By applying the semantic rule for conjunction we get $S, \sigma[x/\delta(i),\, y/\delta(j),\, z/\delta(k)] \models \downarrow_0 (x) = y \wedge \downarrow_1 (x) = z \wedge \tau(\psi)\{x/y\} \wedge \tau(\vartheta)\{x/z\}$. Finally, by introducing the existential quantifier we obtain $S, \sigma[x/\delta(i)] \models \exists y \exists z (\downarrow_0 (x) = y \wedge \downarrow_1 (x) = z \wedge \tau(\psi)\{x/y\} \wedge \tau(\vartheta)\{x/z\})$. In the above formula, x is the only free variable.

  Viceversa, suppose $S, \sigma[x/\delta(i)] \models \tau(\psi C \vartheta)$. Then there exist two elements d and d' in the domain such that $S, \sigma[x/\delta(i), y/d, z/d'] \models \downarrow_0 (x) = y \wedge \downarrow_1 (x) = z \wedge \tau(\psi)\{x/y\} \wedge \tau(\vartheta)\{x/z\}$. These two elements correspond through δ to two intervals j, k, so that $S, \sigma[x/\delta(i), y/\delta(j), z/\delta(k)] \models \downarrow_0 (x) = y \wedge \downarrow_1 (x) = z$ and $S, \sigma[x/\delta(i), y/\delta(j), z/\delta(k)] \models \tau(\psi)\{x/y\}$ and $S, \sigma[x/\delta(i), y/\delta(j),\, z/\delta(k)] \models \tau(\vartheta)\{x/z\}$. By the inductive hypothesis, the only free variable in $\tau(\psi)\{x/y\}$ is y and the only free variable in $\tau(\vartheta)\{x/z\}$ is z. Thus, the last two truth relations are equivalent to $S, \sigma[y/\delta(j)] \models \tau(\psi)\{x/y\}$ and $S, \sigma[z/\delta(k)] \models \tau(\vartheta)\{x/z\}$. By applying the substitution lemma and the inductive hypothesis we may conclude $M, j \Vdash \psi$ and $\mathbf{M}^-, k \Vdash \vartheta$. On the other hand, by Lemma 65, the first truth relation implies $\mathbf{M}^-, i \Vdash \top C \top$ and then $A(j,k,i)$ holds. By the semantic definition of the operator C we get $\mathbf{M}^-, i \Vdash \psi C \vartheta$. The proofs for T and D are similar, and left to the reader.

∎

Corollary 68 *The satisfiability problem in* SL^n *is decidable.*

5.3.2 Discrete Frames and UULSs

The class of split structures considered in Section 5.3.1 can be easily extended by relaxing the constraint of the existence of maximal intervals. It turns out that, in this case, we could consider UULSs as the granular counterpart of models for SL^{uu}

A bijective function $\delta\colon \mathrm{INT}_\infty(\mathbb{N}) \to \bigcup_{i\geq 0} T^i$ can be defined, for $[a,b] \in \mathrm{INT}_\infty(\mathbb{N})$, by $\delta([a,b]) = (a/len([a,b]))_{\lg_2 len([a,b])}$. In this definition of δ, we take into account the fact that the layers in UULSs are counted starting from the tree leaves.

Proposition 69 *The function δ is a computable bijection.*

Proof.
The computability of δ is trivial. As for bijectivity, let us consider the function $\delta\colon \bigcup_{i\geq 0} T^i \to \mathrm{INT}_\infty(\mathbb{N})$ defined by

$$\delta^{-1}(s_t) = [2^t s, 2^t(s+1)].$$

It is easily verified that $\delta^{-1}(\delta([a,b])) = [a,b]$ for every $[a,b] \in \mathrm{INT}_\infty(\mathbb{N})$ and $\delta^{-1}(\delta(s_t)) = s_t$ for every $s_t \in \bigcup_{i\geq 0} T^i$. ■

The correspondence between split models and layered structures of Definition 64 can be immediately adapted to UULSs. The translation of SL^{uu}-formulas in the language of $\mathrm{MFO}[<_2, \downarrow_0, \downarrow_1]$ can be given as in Section 5.3.1, with the exception of the cases relative to $\langle F\rangle$ and $\langle \overline{F}\rangle$ which change as follows:

- $\tau(\langle F\rangle\varphi) = \exists y \exists z\, (\downarrow_0(z) \leq_2 x \wedge \downarrow_1(z) \leq_2 y \wedge \tau(\varphi)\{x/y\});$
- $\tau(\langle \overline{F}\rangle\varphi) = \exists y \exists z\, (\downarrow_0(z) \leq_2 y \wedge \downarrow_1(z) \leq_2 x \wedge \tau(\varphi)\{x/y\}).$

Although its proof is slightly different (since the bijection δ and the translation τ have changed), the equivalence in Lemma 65 is still valid and (a variant of) Lemma 66 still holds.

Lemma 70 *Given a split model $\mathbf{M}^-$ which is based on a structure isomorphic to $\langle \mathbb{N}, \mathrm{INT}_\infty(\mathbb{N})\rangle$, an interval j of $\mathbf{M}^-$ and a layered structure $(S,\mathcal{I})$ that corresponds to $\mathbf{M}^-$ the following equivalences hold:*

$$\mathbf{M}^-, j \Vdash \langle D\rangle\top$$

if and only if there exists an interval i such that

$$(S,\mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x,$$

and

$$\mathbf{M}^-, j \Vdash \langle F\rangle\top$$

if and only if there exists an interval i such that

$$(S,\mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models \exists z\, (\downarrow_0(z) \leq_2 x \wedge \downarrow_1(z) \leq_2 y).$$

Proof.

In what follows, we write S instead of $(S, \mathcal{I})$.

If $\mathbf{M}^{-}, j \Vdash \langle D \rangle \top$, then there exists i such that $i \sqsubset j$. Let $\delta(i) = (s_i)_{t_i}$ and $\delta(j) = (s_j)_{t_j}$. We follow the same track as in the proof of Lemma 66. Since $len(i) < len(j)$, it follows from the definition of δ that $t_i < t_j$ and, since layers are numbered from below, that means that $\delta(j)$ lies at least one level above $\delta(i)$. The condition that $\delta(i)$ and $\delta(j)$ be on the same path becomes

$$2^{t_j - t_i} s_j \leq s_i \leq 2^{t_j - t_i}(s_j + 1) - 1.$$

Substituting and simplifying leads to the same result as in Lemma 66.

Viceversa, let $S, \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x$. The inverse function δ^{-1} associates the intervals $[2^{t_i} s_i, 2^{t_i}(s_i + 1)]$ and $[2^{t_j} s_j, 2^{t_j}(s_j + 1)]$ to the considered nodes. By hypothesis, $\delta(j)$ and $\delta(i)$ are on the same path, with $\delta(j)$ above $\delta(i)$. We have:

1. $t_i < t_j$;
2. $2^{t_j - t_i} s_j \leq s_i$;
3. $s_i \leq 2^{t_j - t_i}(s_j + 1) - 1$.

From (1) we have $len(i) < len(j)$. From the inequality (2), by multiplying both sides by 2^{t_i}, we get $2^{t_j} s_j \leq 2^{t_i} s_i$. From (3), by summing one to both sides and multiplying by 2^{t_i} we get $2^{t_i}(s_i + 1) \leq 2^{t_j}(s_j + 1)$. These three facts imply $i \sqsubset j$; so, $\mathbf{M}^{-}, j \Vdash \langle D \rangle \top$.

As for the *after* operator, we can still exploit the equivalence proved in Lemma 65. Given i, j, with $i \prec j$, there certainly exist k such that $i, j \sqsubset k$ and i', j' such that $i \sqsubset i'$ and $j \sqsubset j'$. Using these and the equivalence of Lemma 65 one can prove the thesis as in Lemma 66 (the only difference is that there is no need to consider a disjunction).

The converse is proved in a similar way, too. ■

Therefore, it is possible to state the following result, whose proof is like the proof for Theorem 67.

Theorem 71 *Let φ be a SL^{uu}-formula. Then φ is satisfiable if and only if $\tau(\varphi)$ is satisfiable over 2-refinable* UULS*s.*

Proof.

The proof is very much like the one for the Theorem 67, and it consists is showing that if $\mathbf{M}^{-}$ is a split model for SL^{uu}, and i is an interval of M, then $\mathbf{M}^{-}, i \Vdash \varphi$ iff $(S, \mathcal{I}), \sigma[x/\delta(i)] \models \tau(\varphi)$, where $(S, \mathcal{I})$ is a 2-refinable UULS that corresponds to $\mathbf{M}^{-}$. ■

Corollary 72 *The satisfiability problem for SL^{uu} formulas is decidable.*

5.3.3 Dense Frames and DULSs

For what concerns the logic SL^{du}, it turns out that DULSs are a natural counterpart. A bijection $\delta\colon \mathrm{INT}(\mathbb{Q}^+) \to \bigcup_{i\geq 0} T^i$ between split structures for SL^{du} and DULSs is the following: for every $[a,b] \in \mathrm{INT}(\mathbb{Q}^+)$ we define the function δ as $\delta([a,b]) = (a/len([a,b]))_{-\lg_2 len([a,b])}$.

Proposition 73 *The function δ is a computable bijection.*

Proof.

The computability of δ is trivial. As for the bijectivity, consider the (inverse) function $\delta^{-1}\colon \bigcup_{i\geq 0} T^i \to \mathrm{INT}(\mathbb{Q}^+)$ defined, for every $s_t \in \bigcup_{i\geq 0} T^i$, by

$$\delta^{-1}(s_t) = [2^{-t}d, 2^{-t}(s+1)].$$

One can easily check that $\delta^{-1}(\delta([a,b])) = [a,b]$ for all $[a,b] \in \mathrm{INT}(\mathbb{Q}^+)$ and that $\delta(\delta^{-1}(s_t)) = s_t$ for all $s_t \in \bigcup_{i\geq 0} T^i$. ■

Once again, the notion of correspondence between a split model and a DULS provided with a suitable interpretation of predicate symbols is straightforward from Definition 64. Moreover, the translation of SL^{du}-formulas is the same as in Section 5.3.1. Lemma 65 and Lemma 66 still hold, although, since the bijection has changed, their proofs are slightly different. In the Appendix, we sketch the proof of the equivalent of Lemma 66.

Lemma 74 *Given a split model $\mathbf{M}^-$ which is based on a structure isomorphic to $\langle \mathbb{Q}^+, \mathrm{INT}(\mathbb{Q}^+)\rangle$, an interval j of $\mathbf{M}^-$ and a layered structure $(S,\mathcal{I})$ that corresponds to $\mathbf{M}^-$, the following equivalences hold:*

$$\mathbf{M}^-, j \Vdash \langle D\rangle\top$$

if and only if there exists an interval i such that

$$(S,\mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x;$$

and

$$\mathbf{M}^-, j \Vdash \langle \overline{F}\rangle\top$$

if and only if there exists an interval i such that

$$(S,\mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models x <_1 y \vee \exists z\,(\downarrow_0(z) \leq_2 x \wedge \downarrow_1(z) \leq_2 y).$$

Proof.

If $\mathbf{M}^-, j \Vdash \langle D\rangle\top$ then there exist i such that $i \sqsubset j$. Let $\delta(i) = (s_i)_{t_i}$ and $\delta(j) = (s_j)_{t_j}$. Since $len(i) < len(j)$ by hypothesis, it follows from the definition of δ that $t_j < t_i$ and, since in the DULSs the layers are numbered from above, $\delta(j)$ lies at least one level above $\delta(i)$. Simplifying the following familiar inequalities

$$2^{l_i - l_j} d_j \leq d_i \leq 2^{l_i - l_j}(d_j + 1) - 1,$$

leads to the same results as in Lemma 66.

Viceversa, let $(S, \mathcal{I}), \sigma[x/\delta(i), y/\delta(j)] \models y <_2 x$. The inverse function δ^{-1} associates the intervals $\langle 2^{-t_i} d_i, 2^{-t_i}(s_i+1)\rangle$ and $[2^{-t_j} s_j, 2^{-t_j}(s_j+1)]$ to the nodes $\delta(i)$ and $\delta(j)$. By hypothesis, $\delta(j)$ and $\delta(i)$ are in the same path, with $\delta(j)$ above $\delta(i)$. So we have:

1. $t_j < t_i$;
2. $2^{t_i - t_j} s_j \leq s_i$;
3. $s_i \leq 2^{t_i - t_j}(s_j + 1) - 1$.

From (1) it follows that $len(i) < len(j)$. From the inequality (2), by multiplying both sides by 2^{-t_i}, we have $2^{-t_j} s_j \leq 2^{-t_i} s_i$. From (3), by summing one to both sides and multiplying by 2^{-t_i} we get $2^{-t_i}(s_i+1) \leq 2^{-t_j}(s_j+1)$. These three facts prove that $i \sqsubset j$; so, $\mathbf{M}^-, j \Vdash \langle D \rangle \top$.

One can easily verify that Lemma 65 is still valid and that the equivalence relative to $\prec$ can be proved as in Lemma 66. ■

We can therefore state the following equi-satisfiability result.

Theorem 75 *If φ is a SL^{du}-formula, then φ is satisfiable if and only if $\tau(\varphi)$ is satisfiable over* 2*-refinable* DULS*s.*

Proof.
The proof, once again, shows that if $\mathbf{M}^-$ is a split model for SL^{du} and i is an interval of $\mathbf{M}^-$, then $\mathbf{M}^-, i \Vdash \varphi$ if and only if $(S, \mathcal{I}), \sigma[x/\delta(i)] \models \tau(\varphi)$, where $(S, \mathcal{I})$ is a 2-refinable DULS that corresponds to $\mathbf{M}^-$. ■

Corollary 76 *The satisfiability problem for SL^{du}-formulas is decidable.*

5.4 On the Inter-Definability of Operators in Split Logic

In classical interval logics, the three operators C, T and D, plus the modal constant π are know to be able to express any other interval modality (interpreted over non-strict structures). In SL the modal constant π is definable, so one can ask wether the same expressivity result hold with the three operators only, or not. Let **split-$\mathcal{CDT}$** be the class of logics of the valid formulas in the language of SL restricted to the modalities C, D and T.

First, we show that the operators $\langle D \rangle$ and $\langle \overline{D} \rangle$ can be defined in split-CDT^n, as it is shown by the following lemma.

Lemma 77 *The operator $\langle D \rangle$ and its converse $\langle \overline{D} \rangle$ are definable in split-*CDT *over n-*LS*s.*

Proof.

The proof is based on the remark that every interval has a finite number of subintervals (since the structure is well-founded) and it is contained in a finite number of intervals (by the hypothesis of maximal intervals). If the dimension n of the n-LS is known, it is possible to reach, with a formula of split-CDT^n, any interval either contained or containing the current interval. Consider the formula which expresses the property that a formula φ holds in an interval "immediately" containing (reachable "in one step") the current one:

$$\mathsf{coarser}(\varphi) \triangleq (\top T\varphi) \vee (\top D\varphi).$$

In split-CDT^n we can express: "the current interval is (properly) contained in an interval where φ holds" with the formula:

$$\bigvee_{1 \leq i < n} \mathsf{coarser}^i(\varphi),$$

where $\mathsf{coarser}^i(\varphi)$ is the ith iteration of the formula $\mathsf{coarser}(\varphi)$ (in particular, we haver that $\mathsf{coarser}^1(\varphi)$ coincides with $\mathsf{coarser}(\varphi)$). From the granular point of view, we may say that the previous formula captures the abstraction primitive over n-LSs.

As for the operator $\langle D\rangle$, consider the formula informally asserting: "φ holds in an immediate subinterval (that is, reachable with one chopping-step) of the current one". Formally:

$$\mathsf{finer}(\varphi) \triangleq (\varphi C\top) \vee (\top C\varphi).$$

In split-CDT^n we may say: "φ holds in a (proper) subinterval of the current interval" with the formula

$$\bigvee_{1 \leq i < n} \mathsf{finer}^i(\varphi).$$

From the granular point of view this formula captures the notion of projection over n-LSs. ■

On the contrary, the operators $\langle F\rangle$ and $\langle\overline{F}\rangle$ are not definable in split-CDT^n. The argument is based on the notion of bisimulation (see [7]) and it is similar to the proof of the undefinability of the past diamond in terms of the future diamond in standard temporal logic.

Lemma 78 *The operator $\langle F\rangle$ and its converse $\langle\overline{F}\rangle$ are not definable in split-*CDT^n.

Proof.

Let us fix a language with one propositional variable p, and let us consider two split models in correspondence (according to Definition 64) with two 2-refinable 2-LSs together with a suitable interpretation, as in Figure 6.3. The two models $\mathbf{M}^-$ and $\mathbf{M}^{-\prime}$ are bisimilar (the bisimulation links each element x in $\mathbf{M}^-$ with the element x' in $\mathbf{M}^{-\prime}$), and then equivalent, as long as the language with C, D and T is considered. However, with respect to the language containing $\langle F\rangle$ and $\langle\overline{F}\rangle$ that is false. If we

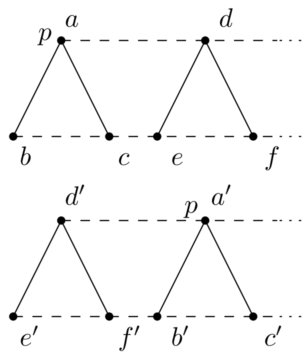


Figure 5.3: Bisimilar 2−refinable 2−LSs.

could define $\langle \overline{F} \rangle$ in split-CDT we could write a formula $\varphi(p)$ holding of an interval i if and only if there exists $j \prec i$ where p holds. Such a formula would be satisfiable in $\mathbf{M}^-$ at d because, in $\mathbf{M}^-$, $a \prec d$, but it would be unsatisfiable at d', which is bisimilar to d, in $\mathbf{M}^{-\prime}$.
A symmetric argument applies for $\langle \overline{F} \rangle$. ■

Now we came to negative results. The operators $\langle D \rangle$ and $\langle \overline{D} \rangle$ cannot be defined in split-CDT^{uu}, as prove in the following theorem. The argument used to prove that is based on the compactness theorem.

Lemma 79 *The operator $\langle D \rangle$ and its converse $\langle \overline{D} \rangle$ are not definable in split-*CDT^{uu}*.*

Proof.
The proof is by contradiction. Let Ψ be a finite set of first-order formulae of split-CDT^{uu} defining $\langle D \rangle p$. For every integer $n > 0$, let $\gamma(n)$ be the formula that states that p does not hold at an interval j that is a subinterval of the current interval i of "depth" less than n. Such a formula can be written as follows (remember the proof of Lemma 77):

$$\gamma(n) = \neg\Big(\bigvee_{1 \le i < n} \mathsf{finer}^i(p) \Big).$$

Let $\Gamma = \{ \gamma(n) \mid n > 0 \}$. For every finite $\Gamma' \subset \Gamma$, $\Psi \cup \Gamma'$ can be shown by induction to be satisfiable. By the compactness theorem, $\Psi \cup \Gamma$ is satisfiable too. But that means that the two intervals are infinitely far apart in an inclusion chain, which is a contradiction. A similar argument applies to $\langle \overline{D} \rangle p$, the only difference being in the definition of $\gamma(n)$ (which in this case uses $\mathsf{coarser}$). ■

The operators $\langle F \rangle$ and $\langle \overline{F} \rangle$ cannot be defined either. However, the bisimulation based proof used in Section 5.3.1 cannot be applied here, since every node is reachable in some way from any other with respect to the (closure of the) accessibility relations of C, D and T. Instead, a compactness argument can serve the purpose.

Lemma 80 *The operator $\langle F \rangle$ and its converse $\langle \overline{F} \rangle$ are not definable in split-*CDT^{uu}*.*

logic	$\langle D \rangle$	$\langle \overline{D} \rangle$	$\langle F \rangle$	$\langle \overline{F} \rangle$
split-CDT^n	yes	yes	no	no
split-CDT^{uu}	no	no	no	no
split-CDT^{du}	no	no	no	no

Table 5.1: Definability results for split logics.

Proof.
The proof is by contradiction. Suppose that Φ is a finite set of formulae of split-CDT defining $\langle \overline{F} \rangle q$. For every integer i we define a *set* of formulae

$$\Delta_i = \left\{ \neg\big(\mathsf{coarser}^i(\neg\gamma(n)D\top)\big) \mid n \in \mathbb{N} \right\}$$

where $\mathsf{coarser}^i(\varphi)$ was defined in the proof of Lemma 77 and the $\gamma(n)$'s are as defined in the proof of Lemma 79. Intuitively, each Δ_i asserts that p does not hold in a subtree of intervals "i steps before" the current interval. Let $\Delta = \bigcup_{i=0}^{\infty} \Delta_i$. For every finite subset $\Delta' \subset \Delta$ we have that $\Delta' \cup \Phi$ is satisfiable: it is sufficient to take an interval far enough in the future with respect to an interval where p holds. Then, by the compactness theorem, $\Delta \cup \Phi$ is satisfiable too. But this is a contradiction, because it means that the the two intervals are infinitely far in time. The proof for $\langle F \rangle$ is similar. ■

On the other hand, we notice that adding $\langle D \rangle$ and $\langle \overline{D} \rangle$ to split-CDT^{uu} allows us to define $\langle F \rangle$ and $\langle \overline{F} \rangle$, because the latter are definable by the following formulae:

$$\langle F \rangle \varphi \triangleq (\varphi \vee \langle D \rangle \varphi) D\top \vee \langle \overline{D} \rangle \big((\varphi \vee \langle D \rangle \varphi) D\top\big);$$
$$\langle \overline{F} \rangle \varphi \triangleq (\varphi \vee \langle D \rangle \varphi) T\top \vee \langle \overline{D} \rangle \big((\varphi \vee \langle D \rangle \varphi) T\top\big).$$

Finally, we have that none of $\langle F \rangle$, $\langle \overline{F} \rangle$, $\langle D \rangle$ and $\langle \overline{D} \rangle$ are definable in split-CDT^{du}. To prove this, arguments similar to those give above are adequate.

In Table 6.1 we listed the definability results over the various classes of structures.

5.5 Correspondence Results for SL

In the previous sections we have proved that logics in SL can be considered as fragments of the theory MFO$[<_1, <_2, \downarrow_0, \downarrow_1]$ when split-frames are either discrete with maximal intervals or dense and it is a fragment of MFO$[<_2, \downarrow_0, \downarrow_1]$ when frames are discrete and with arbitrarily long intervals. It is not hard to see that the given translations can be accomplished, by reusing variables, using at most three variables. We have also shown (see the proof of Theorem 67) that every such formula contains exactly one free variable, which intuitively marks the current interval. Moreover, such a translation actually maps split logic formulas into particular restrictions of the previous theories, namely the *guarded fragment* [7] GF_3 of the theory MFO$[<_1, <_2, \downarrow_0, \downarrow_1]$ (of MFO$[<_2, \downarrow_0, \downarrow_1]$ for UULSs) with three variables and predicates at most ternary.

It is possible to show that also every GF_3-formula has an equivalent translation into split logic.

Let us first of all consider the case of n-LSs and DULSs (whose first-order language is the same). We start by considering the GF_3 of $MFO[<_1, <_2, \downarrow_0, \downarrow_1]$ restricted to three variables one of which is free. We give a translation t of GF_3-formulae into the language of SL ($\varphi(y)$ denotes a formula in which y occurs free):

- $t(x = x) = \top$;
- $t(x <_1 x) = \bot$;
- $t(x <_2 x) = \bot$;
- $t\big(P(x)\big) = p$;
- $t\big(\downarrow_i (x, x)\big) = \bot$, with $i \in \{0, 1\}$;
- $t\big(\exists y\,(x <_1 y \wedge \varphi(y))\big) = \langle\overline{D}\rangle(\neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))) \vee \neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))$;
- $t\big(\exists y\,(y <_1 x \wedge \varphi(y))\big) = \langle\overline{D}\rangle(\neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))) \vee \neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))$;
- $t\big(\exists y\,(x <_2 y \wedge \varphi(y))\big) = \langle D\rangle t(\varphi)$;
- $t\big(\exists y\,(y <_2 x \wedge \varphi(y))\big) = \langle\overline{D}\rangle t(\varphi)$;
- $t\big(\exists y\,(\downarrow_0 (x, y) \wedge \varphi(y))\big) = t(\varphi) C \top$;
- $t\big(\exists y\,(\downarrow_1 (x, y) \wedge \varphi(y))\big) = \top C t(\varphi)$;
- $t\big(\exists y\,(\downarrow_0 (y, x) \wedge \varphi(y))\big) = \top T t(\varphi)$;
- $t\big(\exists y\,(\downarrow_1 (y, x) \wedge \varphi(y))\big) = \top D t(\varphi)$;
- $t\big(\exists y\,(x = y \wedge \varphi(y))\big) = t(\varphi(x))$;
- $t(\neg\varphi) = \neg t(\varphi)$;
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$.

The main result of this section is the following:

Lemma 81 *Let $\varphi(x)$ be a formula* GF_3 *containing one free variable x. For every* n-LS *or* DULS *that corresponds to some split model* $\mathbf{M}^-$ *and for every assignment σ we have:*

$$S, \sigma[x/d] \models \varphi(x) \text{ if and only if } M, \delta^{-1}(d) \Vdash t(\varphi).$$

Proof.
By induction on the complexity of the formulae.

Atomic cases are all straightforward. The one for unary predicates is: $S, \sigma[x/d] \models P(x)$ if and only if (Definition 64) $\mathbf{M}^-, \delta^{-1}(d) \Vdash p$.

Let $S, \sigma[x/d] \models \exists y\,(x <_1 y \wedge \psi(y))$; then there exists d' such that we have $S, \sigma[x/d, y/d'] \models x <_1 y$ and $S, \sigma[y/d'] \models \psi(y)$. Applying the inductive hypothesis, the last truth relation implies $\mathbf{M}^-, \delta^{-1}(d') \Vdash t(\psi)$. On the other hand, the first truth relation implies $\delta^{-1}(d) \prec \delta^{-1}(d')$ and, since $\delta^{-1}(d)$ and $\delta^{-1}(d')$ are mapped onto different trees, even for the maximal intervals $k \sqsupseteq \delta^{-1}(d)$ and $k' \sqsupseteq \delta^{-1}(d')$ the relation $k \prec k'$ holds. Combining these semantic relations leads to the modal formula $\langle\overline{D}\rangle(\neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))) \vee \neg\langle\overline{D}\rangle\top \wedge \langle\overline{F}\rangle(t(\varphi) \vee \langle D\rangle t(\varphi))$. The opposite implication is left to the reader. The case of $\exists y\,(y <_1 x \wedge \psi(y))$ is similar.

The formulae with $<_2$ are immediate, since they directly correspond to the translation of $\langle D\rangle$ and $\langle\overline{D}\rangle$. We prove one of the cases relative to $\downarrow_i$, the remaining cases being analogous: $S, \sigma[x/d] \models \exists y\,(\downarrow_0 (x, y) \wedge \psi(y))$ if and only if there exists an element d' in the domain such that $S, \sigma[x/d, y/d'] \models \downarrow_0 (x, y)$ and $S, \sigma[y/d'] \models \psi(y)$. The latter relation, by inductive hypothesis, holds if and only if $\mathbf{M}^-, \delta^{-1}(d') \Vdash t(\psi)$. Using the bijection δ and remembering the proof of Lemma 65 it is easy to show that for some d'' we have $S, \sigma[x/d, y/d', z/d''] \models \downarrow_0 (x, y) \wedge \downarrow_1 (x, z)$ and this holds if and only if $A(\delta^{-1}(d'), \delta^{-1}(d''), \delta^{-1}(d))$ if and only if $\mathbf{M}^-, \delta^{-1}(d) \Vdash t(\psi)C\top$.

Finally, the cases for boolean connectives are straightforward by applying the inductive hypothesis. ■

Lemma 82 *Every* SL^n *and* SL^{du}*-formula is equivalent to a* GF_3*-formula.*

Proof.
Given an SL^n or a SL^{du}-formula φ, it is sufficient to prove that its translation $\tau(\varphi)$ is equivalent to a GF_3-formula. The only cases that require some thought are those relative to $\langle F\rangle$ and $\langle\overline{F}\rangle$. In n-LSs and DULSs it is sufficient to note that $\tau(\langle\overline{F}\rangle\psi)$ is equivalent to the following guarded formula: $\exists y\,(y <_1 x \wedge \tau(\psi)\{x/y\}) \vee \exists z\,(\downarrow_1 (z) \leq_2 x \wedge \exists y\,(\downarrow_0 (z) \leq_2 y \wedge \tau(\psi)\{x/y\}))$. $\langle\overline{F}\rangle$ has a similar translation. ■

Theorem 83 SL^n *and* SL^{du} *are expressively complete with respect to* GF_3 *restricted to formulae with one free variable*[3].

As for the split logic interpreted over structures corresponding to UULSs, one can repeat the same reasoning using the (simpler) language $\mathrm{MFO}[<_2, \downarrow_0, \downarrow_1]$, and draw the same conclusions.

[3]Such a restriction can be easily generalized to formulae with *at most* one free variable. Sentences, from the modal point of view, are always evaluated to either $\top$ or $\bot$.

5.6 On Decidable Extensions of SL

The language of SL can be extended with other (well-known) interval operators. We now list a few and give their translation into $\mathrm{MFO}[(<_1), <_2, \downarrow_0, \downarrow_1]$. We remark that these translations, given the results for time granularity, imply the decidability of the extended logic (which can be proved as we did for basic SL). Besides, the following formulae are not GF_3-formulae: we conjecture that they are not equivalent to any GF_3-formula and that the logics obtained by adding these operators are non-conservative (decidability-preserving) extensions of SL.

Given any split model $\mathbf{M}^-$ and an interval i, we rescall the semantics of the *staring subinterval* and *finishing subinterval* operators $\langle B\rangle$ and $\langle E\rangle$, and of the *meeting interval*, operator $\langle A\rangle$ (leaving to the reader the dual definitions) as follows:

- $\mathbf{M}^-, i \Vdash \langle B\rangle\varphi$ if for some j we have j *starts* i and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \langle E\rangle\varphi$ if for some j we have j *finishes* i and $\mathbf{M}^-, j \Vdash \varphi$;
- $\mathbf{M}^-, i \Vdash \langle A\rangle\varphi$ if for some j we have i *meets* and $\mathbf{M}^-, j \Vdash \varphi$.

We use the following definition as abbreviations:

- $y \text{ starts } x \triangleq x <_2 y \wedge \neg\exists z\,(x <_2 z \wedge \exists w\,(\downarrow_0(w) \leq_2 z \wedge \downarrow_1(w) \leq_2 y))$;
- $y \text{ ends } x \triangleq x <_2 y \wedge \neg\exists z\,(x <_2 z \wedge \exists w\,(\downarrow_0(w) \leq_2 y \wedge \downarrow_1(w) \leq_2 z))$;
- $\text{maximal}(x) \triangleq \neg\exists y\, y <_2 x$.

The previous operators can be translated as follows:

- $\tau(\langle B\rangle\varphi) = \exists y\,(y \text{ starts } x \wedge \tau(\varphi\{x/y\}))$;
- $\tau(\langle E\rangle\varphi) = \exists y\,(y \text{ ends } x \wedge \tau(\varphi\{x/y\}))$;
- $\tau(\langle A\rangle\varphi) = \exists y\exists v\exists w\exists z\,\big(\downarrow_0(z) = v \wedge \downarrow_1(z) = w \wedge (x \text{ ends } v \vee x = v) \wedge (y \text{ starts } w \vee y = w) \wedge \tau(\varphi\{x/y\})\big) \vee \exists y\exists t\exists u\,\big(\text{maximal}(t) \wedge \text{maximal}(u) \wedge t <_1 u \wedge \neg\exists z\,(t <_1 z <_1 u) \wedge (x \text{ ends } t \vee x = t) \wedge (y \text{ starts } u \vee y = u) \wedge \tau(\varphi\{x/y\})\big)$.

In order to show the usefulness of such operators, we give some example, recalling similar examples of Chapter 4. First we define a binary strong *until* operator as follows:

- $\varphi\langle U\rangle\psi \triangleq \langle A\rangle\big(\varphi \wedge [D]\varphi \wedge \langle A\rangle\psi\big)$.

This operator is "strong" in the sense that we require φ to hold at every subinterval of an interval next to the current interval. A weak version can be obtained by only requiring φ to hold at a consecutive subinterval (but not necessarily over all subintervals of it). The until operator is extensively used in temporal logics.

We can use the above operator to model sentences as "each flight from Milan to Moscow is followed by a period of time during which the traveller is in Moscow":

$$[F][\overline{F}](\mathsf{Milan\text{-}to\text{-}Moscow}\langle U\rangle\mathsf{stay\text{-}in\text{-}Moscow}).$$

Then, we borrow from [42] a couple of examples in the domain of qualitative physics and automatic planning, respectively.

Consider the sentence "if you open the tap, then, unless someone punctures the canteen, the canteen will eventually be filled". This statement can be written pretty in the same way as in HS:

$$\mathsf{open\text{-}tap} \rightarrow \langle A\rangle\big(\neg\langle D\rangle\mathsf{punctures} \rightarrow \langle E\rangle[E]\mathsf{filled}\big).$$

There are some differences with respect to the original formulation of the above statement in HS. In particular, we intend it to be interpreted on (interval structures corresponding to) DULSs, so we do not take into account intervals' endpoints (which could be identified with atomic intervals), as Halpern and Shoham do. In this context, even the shortest occurring events, such as filled or open-tap, have some duration. However, being able to refine every interval, such a duration can be as small as one likes.

Finally, consider the following statement: "if the robot executes the charge-battery routine, then, at the beginning of the following execution of the navigate routine, its batteries will be fully charged". We say that a proposition is *solid* if no two distinct overlapping intervals ever satisfy it. Since we do not have partially overlapping intervals, this definition reduces to the formula

- $\mathsf{solid}(\varphi) \triangleq \varphi \rightarrow \neg\langle D\rangle\varphi$.

The "next time that" statement can be written as

- $[\mathsf{NTT}](\varphi, \psi) \triangleq [A]\big((\neg\varphi \wedge [D]\neg\varphi \rightarrow [A](\varphi \rightarrow \langle B\rangle[B]\psi)\big) \vee (\varphi \wedge \langle B\rangle[B]\psi)\big)$,

where φ is a solid proposition. The robot assertion becomes

$$\mathsf{charge\text{-}battery} \rightarrow [\mathsf{NTT}](\mathsf{navigate}, \mathsf{battery\text{-}full}).$$

Concluding Remarks and Future Work

"640K ought to be enough for anybody."
Bill Gates

One of the main goals of this thesis was that of giving a complete picture of the world of interval temporal logics. We quite soon realized that there were important unexplored natural questions, and we focused our attention on some of them. In particular, we have concentrated our attention on:

1. Abstract characterizations and representation theorems for a number of interval structures;
2. (Relative) expressive power of various interval logics;
3. Sound and complete axiomatic systems;
4. Decidability and undecidability of the satisfiability/validity problem;
5. Tableaux systems.

Clearly, for any one of the above points there are still many open questions. In this final chapter we give a short list of open problems that, in our opinion, deserve to be further investigated.

Abstract characterizations and representation theorems. The first-order theory of the *meets* relation for dense, unbounded, linear orderings has been originally studied by Allen and Hayes, and refined by Ladkin (see Section 1.3.1). Representation theorems for non-strict partial orderings with linear interval property for the relations *starts* and *finishes* have been given by Venema (see Section 1.3.2). Representation theorems for both strict and non-strict interval neighborhood structures have been proved by Goranko, Montanari, and Sciavicco (cfr. Section 1.3.3). Abstract characterizations and representation theorems are not explored yet for many other propositional interval logics of Allen's relations, such as D, $\overline{\mathrm{D}}$, D$\overline{\mathrm{D}}$, $\overline{\mathsf{B}}\overline{\mathsf{E}}$ (i.e., the propositional interval logic whose language contains only the modalities $\overline{\mathrm{B}}$ and $\overline{\mathrm{E}}$), $\mathsf{BE}\overline{\mathsf{B}}\overline{\mathsf{E}}$, $\mathsf{B}\overline{\mathsf{E}}$, and $\overline{\mathsf{B}}\mathsf{E}$. Moreover, one can consider the relation $A(i, j, k)$ and the binary modalities that are based on it, and try to find out representation theorems for C only, in either the strict or the non-strict semantics, and including/excluding the modal constant π. The other two modalities, and the same logic CDT, either with or without π, have not been studied in the case of strict semantics.

Axiomatic systems. The logic of all Allen's relations has been axiomatized by Venema (we presented this problem in Section 2.3.5), in the case of non-strict semantics for the class of all partial orderings with the linear interval property, for the one of all discrete linear orderings, and for the one of all dense linear orderings. The axiomatic system include a special rule of inference, and it is still an open problem if it can be omitted or replaced by an axiom. Venema also gives sound and complete axiomatic systems for CDT interpreted over the classes of all linear, all discrete linear, all unbounded linear, and all dense linear orderings (cfr. Section 2.4.2), again with a non-standard rule of inference, and, once more, it is still an open problem if it can be omitted or replaced by an axiom. Propositional neighborhood logics have been axiomatized in a number of cases by Goranko, Montanari and Sciavicco (cfr. Chapter 3). Finally, the local version of PITL (on discrete linear orderings) has been axiomatized by Moszkowski, and by Bowman and Thompson in various cases (see Section 2.4.1). Axiomatic systems for first-order interval logics and for duration calculi, in different cases, have been studied by Dutertre, Hansen, Zhou, Barua, Roy, Hoare, Ravn, and Guelev (cfr. Section 2.5). As before, for many interesting logics featuring only a small set of unary modalities, over the various semantics, the problem of finding sound and complete axiomatic systems (or non-axiomatizability results) is still unexplored. To cite a few: D, $\overline{\mathrm{D}}$, $\mathrm{D}\overline{\mathrm{D}}$, $\overline{\mathsf{BE}}$, $\mathsf{BE}\overline{\mathsf{BE}}$, and C with/without π, T with/without π, PITL without locality, with/without projection.

Decidability and complexity issues. The decidability of the satisfiability problem turns out to be a hard problem in the field of interval logics. Known decidable interval logics at the propositional level are $\mathrm{B}\overline{\mathrm{B}}$, $\mathrm{E}\overline{\mathrm{E}}$ over linear orderings in various cases (cfr. Section 2.3.3), and, by the Moszkowski, Bowman, and Thompson's results, the Local PITL for discrete linear orderings, also in various cases (see Section 2.4.1). Moreover, Montanari, Sciavicco and Vitacolonna have shown that at least three non-trivial logics in the class $\mathcal{SL}$ are also decidable (this is the focus of Chapter 5). At the first-order level, Duration Calculus has been shown to be decidable under certain syntactic restrictions. Relevant undecidability results are those for HS over almost all the interesting classes of partial and linear orderings (see Section 2.3.5), for its fragment BE (cfr. Section 2.3.2) over dense and over all linear orderings, and for Propositional ITL over discrete linear orderings, and over the class of all linear orderings. It is worth to remark that most decidability result (apart those for $\mathcal{SL}$) have been obtained by some kind of reduction to point-based logics. An interesting open problem is the decidability of the satisfiability problem of logics in the class $\mathcal{PNL}$. Moreover, the decision problem for most fragments of HS over different classes of orderings, either in the strict or non-strict semantics, is still open. To cite a few: D, $\overline{\mathrm{D}}$, $\mathrm{D}\overline{\mathrm{D}}$, $\overline{\mathsf{BE}}$, $\mathsf{BE}\overline{\mathsf{BE}}$, plus C with/without π, T with/without π, PITL without locality, with/without projection. Finally, the work on split logics can be extended in several directions. First, one can try to deal with (more) general split-frames as well as to augment the domain, e.g., by including *partially overlapping* intervals. Such attempts would allow one to properly determine the boundaries beyond which undecidability is attained. Second, more powerful operators can be added to capture full first-order and second-order decidable theories for time granularity. Third, it would be useful to devise a specific decision procedure for split logics.

General questions. We conclude by listing some natural general questions that, at the best of our knowledge, have not been answered in a satisfactory way:

- When and why is the linearity of the intervals essential?
- When and why is the linearity of the ordering essential?
- Are there interval logics with the finite model property w.r.t. non-standard models for interval logics?

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